ABSTRACT. A study of prolongations of F-structure to the tangent bundle of order 2 has been presented.

KEY WORDS AND PHRASES. Prolongations, tangent bundle, integrable, lift, F-structure.

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1. INTRODUCTION.

Let $F$ be a nonzero tensor field of type $(1,1)$ and of class $c^\infty$ on an $n$-dimensional manifold $V_n$ such that [1]

$$F^K + (-)^{K+1}F = 0 \quad \text{and} \quad F^W + (-)^{W+1}F \neq 0 \quad \text{for} \quad 1 < W < K$$  \(1.1\)

where $K$ is a fixed positive integer greater than 2. Such a structure on $V_n$ is called an $F$-structure of rank '$r$' and degree $K$. If the rank of $F$ is constant and $r = r(F)$, then $V_n$ is called an $F$-structure manifold of degree $K(\geq 3)$. The case when $K$ is odd has been considered in this paper.

Let the operators on $V_n$ be defined as follows [1]:

$$I = (-)^{K}F^{-1} \quad \text{and} \quad m = I + (-)^{K+1}F^{-1}$$  \(1.2\)

where $I$ denotes the identity operator on $V_n$.

From the operators defined by (1.2) we have

$$l + m = I \quad \text{and} \quad l^2 = l; \quad \text{and} \quad m^2 = m$$  \(1.3\)

For $F$ satisfying (1.1), there exist complementary distributions $L$ and $M$ corresponding to the projection operators $l$ and $m$ respectively.

If rank $(F) = \text{constant on } V_n$ then $\dim L = r$ and $\dim M = (n - r)$. We have the following results [1]

$$Fl = lF = F \quad \text{and} \quad Fm = mF = 0$$  \(1.4a\)

$$F^{K-1}l = -l \quad \text{and} \quad F^{K-1}m = 0$$  \(1.4b\)

2. PROLONGATIONS OF F-STRUCTURE IN THE TANGENT BUNDLE OF ORDER 2.

Let $V_n$ be an $n$-dimensional differentiable manifold of class $c^\infty$ and $T_p(V_n) = \bigcup_{p \in V_n} T_p(V_n)$ is the tangent bundle over the manifold $V_n$.

Let us denote $T_p^\prime(V_n)$, the set of all tensor fields of class $c^\infty$ and of the type $(r,s)$ in $V_n$ and $T(V_n)$ be the tangent bundle over $V_n$. 

Let us introduce an equivalence relation \( \sim \) in the set of all differentiable mappings \( F: R \rightarrow V_n \) where \( R \) is the real line. Let \( r \geq 1 \) be a fixed integer. If two mappings \( F: R \rightarrow V_n \) and \( G: R \rightarrow V_n \) satisfy the conditions

\[
F^h(0) = G^h(0), \quad \frac{dF^h(0)}{dt} = \frac{dG^h(0)}{dt}, \ldots, \quad \frac{d^rF^h(0)}{dt^r} = \frac{d^rG^h(0)}{dt^r},
\]

the mapping \( F \) and \( G \) being represented respectively by \( X^h = F^h(t) \) and \( X^h = G^h(t) \), \( t \in R \) with respect to local coordinates \( X^h \) in a coordinate neighborhood \( (U, X^h) \) containing the point \( P = F(0) = G(0) \), then we say that the mapping \( F \) is equivalent to \( G \). Each equivalence class determined by the equivalence relation \( \sim \) is called an \( r \)-jet of \( V_n \) and denoted by \( J^r(F) \). The set of all \( r \)-jets of \( V_n \) is called the tangent bundle of order \( r \) and denoted by \( T^r(V_n) \). The tangent bundle \( T_2^2(V_n) \) of order 2 has the natural bundle structure over \( V_n \), its bundle projection \( \pi^2: T_2^2(V_n) \rightarrow V_n \) being defined by \( \pi^2(J^2(F)) = P \). If we introduce a mapping such that \( P = F(0) \), then \( T_2^2(V_n) \) has a bundle structure over \( T(V_n) \) with projection \( \pi_{12} \).

Let us denote \( T_2^2(V_n) \), the second order tangent bundle over \( V_n \) and let \( F^{II} \) be the second lift of \( F \) in \( T_2^2(V_n) \). The second lift \( F^{II} \) which belong to \( T_2^2(T_2^2(V_n)) \) has component of the form [3]

\[
F^{II} = \begin{bmatrix}
F^h_t & 0 & 0 \\
0 & F^h_t & 0 \\
(y^t_t \delta_s F^h_t + (1/2)y^t_t \delta_s x^h_t) & F^h_t & F^h_t
\end{bmatrix}
\]

with respect to the induced coordinates in \( T_2^2(V_n) \), \( F^h_t \) being local components of \( F \) in \( V_n \).

Now we obtain the following results on the second lift of \( F \) satisfying (1.1).

For any \( F, G \in T^1_1(V_n) \), the following holds [3]:

\[
(G^{II} F^{II}) X^{II} = G^{II} (FX^{II}),
\]

\[
= G^{II} (FX)^{II}
\]

\[
= G(FX)^{II}
\]

\[
= (GF)^{II} X^{II} \quad \text{for every } X \in T_0^2(V_n),
\]

therefore we have

\[
G^{II} F^{II} = (GF)^{II}
\]

If \( P(s) \) denote a polynomial of variable \( s \), then we have

\[
(P(F))^{II} = P(F^{II}), \quad \text{where } F \in T^1_1(V_n)
\]

We have the following theorem:

**THEOREM 2.1.** The second lift \( F^{II} \) defines a \( F \)-structure in \( T_2^2(V_n) \) iff \( F \) defines a \( F \)-structure in \( V_n \).

**PROOF.** Let \( F \) satisfy (1.1) then \( F \) defines \( F \)-structure in \( V_n \) satisfying

\[
F^K + (-)^{K+1} F = 0,
\]

which in view of equation (2.3) yields
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\[(F^{II})^K + (-)^K + 1F^{II} = 0.\]  \hfill (2.4)

Therefore \(F^{II}\) defines a \(F\)-structure in \(T_2(V_n)\). The converse can be proved in a similar manner.

**THEOREM 2.2.** The second lift \(F^{II}\) is integrable in \(T_2(V_n)\), iff \(F\) is integrable in \(V_n\).

**PROOF.** Let us denote \(N_{II}\) and \(N\), the Nijenhuis tensors of \(F^{II}\) and \(F\) respectively. Then we have \[N_{II}(X,Y) = (N(X,Y))^{II}\] \hfill (2.5)

We know that \(F\)-structure is integrable in \(V_n\), iff

\[N(X,Y) = 0,
\]

which in view of (2.5) is equivalent to

\[N_{II}(X,Y) = 0.\] \hfill (2.6)

Thus \(F^{II}\) is integrable, iff \(F\) is integrable in \(V_n\).

**THEOREM 2.3.** The second lift \(F^{II}\) of \(F\) is partially integrable in \(T_2(V_n)\), iff \(F\) is integrable in \(V_n\).

**PROOF.** We know that for \(F\) to be partially integrable in \(V_n\), the following holds \[N(IX, lV) = 0\]

and

\[N(mX, mY) = 0,
\]

which, in view of equation (2.5), takes the form

\[N_{II}(l^{II}X^{II}, l^{II}Y^{II}) = 0\]

and

\[N_{II}(m^{II}X^{II}, m^{II}Y^{II}) = 0.\] \hfill (2.7)

where \(l^{II}, m^{II}\) are operators in \(T_2(V_n)\) which define the distribution \(L^{II}\) and \(M^{II}\) respectively. Thus equation (2.7) gives the condition for \(F^{II}\) to be partially integrable.

The converse follows in a similar manner.

**REFERENCES**


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