UNIVALENCE FOR CONVOLUTIONS

HERB SILVERMAN

Department of Mathematics
University of Charleston
Charleston, SC 29424

(Received May 10, 1994)

ABSTRACT. The radius of univalence is found for the convolution \( f * g \) of functions \( f \in S \) (normalized univalent functions) and \( g \in C \) (close-to-convex functions). A lower bound for the radius of univalence is also determined when \( f \) and \( g \) range over all of \( S \). Finally, a characterization of \( C \) provides an inclusion relationship.

KEY WORDS AND PHRASES. Univalent, convolution.


1. INTRODUCTION.

Denote by \( S \) the family consisting of functions \( f(z) = z + \cdots \) that are analytic and univalent in \( \Delta = \{ z : |z| < 1 \} \) and by \( K, S^*, \) and \( C \) the subfamilies of functions that are, respectively, convex, starlike, and close-to-convex in \( \Delta \). It is well known that \( K \subset S^* \subset C \subset S \). The convolution of two power series

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n
\]

is defined as the power series

\[
f(z) = \sum_{n=0}^{\infty} a_n b_n z^n.
\]

The Koebe function \( k(z) = z/(1-z)^2 \) often plays an extremal role in the family \( S \). This enables us to show it to be extreme in many convolution problems. For example, the modulus of the \( n \)th coefficient for \( f * g \), \( f \) and \( g \) in \( S \), is \( n^2 \) and is attained when \( f = g = k \). Similarly, \( |f * g| \) takes its maximum and minimum on the circle \( |z| = r \) when \( f = g = k \).

A question was raised in [4] as to whether

\[
\min_{|z| = r} \text{Re}(f,)(z) = \min_{|z| = r} \text{Re}(f,k)(z) = \min_{|z| = r} \text{Re} f'
\]

when \( f \) and \( g \) are taken over all of \( S \). The classical rotation theorem for \( f \in S \) leads to the sharp result that \( \text{Re} f'(z) \geq 0 \) when \( |z| \leq \sin(\pi/8) \). This was generalized in [4] to \( \text{Re}(f,g)(z) \geq 0 \) for \( |z| \leq \sin(\pi/8) \) when \( f \in S \) and \( g \in S^* \), but could not be extended to \( g \in S \) or even to \( g \in C \). In particular, functions \( f, g \in C \) were found for which \( \text{Re}(f,g)(z) < 0 \) at some point \( z_0 \), \( |z_0| < \sin(\pi/8) \).
In this note, we investigate the radius of univalence for \( f \ast g \), \( f \) and \( g \) in \( S \). For \( f \in S \) and \( g = k \), the Koebe function, \( f \ast g \) is univalent in the disk \( |z| < 2 - \sqrt{3} \). We prove that \( g = k \) can be replaced by any \( g \in C \), but we cannot settle if this extends to arbitrary \( g \in S \). We do show, however, that \( f \ast g \) is univalent for at least \( |z| < 0.8(2 - \sqrt{3}) \).

2. MAIN RESULTS.

**THEOREM 1.** If \( f \in S \) and \( g \in C \), then \( f \ast g \) is univalent in \( |z| < 2 - \sqrt{3} \). The result is sharp.

**PROOF.** It is well known that \( f \) is convex in \( |z| < r \) if and only if \( zf' \) is starlike in \( |z| < r \) and that the radius of convexity of \( S \) is \( 2 - \sqrt{3} \). Thus, \( f \ast k = zf' \) has radius of starlikeness (and hence radius of univalence) at least \( 2 - \sqrt{3} \), the radius of convexity for \( f \in S \). Since

\[
(k \ast k)' = (zk)' = \frac{1+4z+z^2}{(1-z)^3} = 0 \text{ at } z = -(2 - \sqrt{3}),
\]

the radius of univalence of \( f \ast g \) for \( f, g \in S \) can be no greater than \( r = 2 - \sqrt{3} \).

When \( f \in S \), we have \( f(az)/a \in K \) for \( a = 2 - \sqrt{3} \). Hence, by a theorem of Ruscheweyh and Sheil-Small [3], if \( f \in S \) and \( g \in C \) then

\[
\frac{f(az)}{a} \ast g(z) \in C \subset S.
\]

Thus, \( f \ast g \) is univalent for \( |z| < 2 - \sqrt{3} \), and the proof is complete.

In our next theorem, we replace \( C \) with \( S \) in the hypothesis and this leads to a weaker conclusion.

**THEOREM 2.** Denote by \( r_0 \) the largest value for which \( f \ast g \) is univalent in \( |z| < r_0 \) for all \( f, g \in S \). Then \( 0.8(2 - \sqrt{3}) < r_0 < 2 - \sqrt{3} \).

**PROOF.** The upper bound was found in Theorem 1. Krzyz [1] determined the radius of close-to-convexity for \( S \) to be \( t_0 = 0.80 + \). Since \( f(az)/a \in K \), \( a = 2 - \sqrt{3} \), and \( g(t_0 z)/t_0 \in C \), we have from the Ruscheweyh and Sheil-Small theorem [3] that

\[
\frac{f(az)}{a} \ast g(t_0 z)/t_0 \in C,
\]

which shows that \( f \ast g \) is univalent for \( |z| < t_0(2 - \sqrt{3}) \). This furnishes us with the lower bound, and the proof is complete.

Though we are unable to prove that \( r_0 = 2 - \sqrt{3} \) in Theorem 2, the lower bound on \( r_0 \) most certainly can be improved. Ruscheweyh defined the family \( M \) consisting of normalized functions \( f \) by

\[
M = \{ f : f \ast g \neq 0 ; g \in S^* , 0 < |z| < 1 \}.
\]

He proved the proper inclusions \( C \subset M \subset S \) and that \( f \ast g \in M \) for \( f \in K \) and \( g \in M \) [2]. Hence, if \( t_1 \) is the largest value for which \( g(t_1 z)/t_1 \in M \) when \( g = S \), methods identical to those of Theorem 2 show that \( f \ast g \) is univalent in \( |z| < t_1(2 - \sqrt{3}) \) for \( f, g \in S \). Unfortunately the value of \( t_1 \), the radius of "M-ness" for \( S \), is unknown.

3. A CHARACTERIZATION OF \( C \).

The inclusion \( C \subset M \) is not obvious and was proved by Ruscheweyh using his duality principle [2]. Our final result is a characterization of \( C \) that leads to a more elementary proof that \( C \subset M \). We make use of a result found in [3].

**LEMMA 3.** If \( \phi \in K, \Psi \in S^* \), and \( F \) is analytic with \( \text{Re} F > 0 \) for \( z \in \Delta \), then

\[
\text{Re} \frac{e^{\phi F} \Psi}{\phi \Psi} > 0.
\]
THEOREM 3. A function \( f \in C \) if and only if to each \( g \in S^* \) we may associate an \( h \in S^* \) for which \( \text{Re} \frac{L_x}{h} > 0, \ z \in \Delta \).

PROOF. To show that the condition is sufficient for \( f \) to be in \( C \), we choose \( g(z) = z/(1 - z)^2 \in S^* \). Then \( \text{Re} \frac{L_x}{h} = \text{Re} \frac{zL_y}{h} > 0 \), which means that \( f \in C \).

On the other hand, if \( f \in C \) we can find a \( \Psi = S^* \) for which \( \text{Re}zf'/\Psi > 0 \) Set \( F(z) = zf'(z)/\Psi(z) \). Then for \( g \in S^* \) there corresponds \( \phi \in K \) such that \( z\phi' = g \). Note that \( f* g = zf'*\phi = \phi*F\Psi \) and that \( h = \phi*\Psi \in S^* \). By Lemma A,

\[
\text{Re} \frac{\phi*F\Psi}{\phi*\Psi} = \text{Re} \frac{L_x}{h} > 0,
\]

and the proof is complete.

COROLLARY. \( C \subset M \)

PROOF. Since \( \text{Re} \frac{L_x}{h} > 0 \Rightarrow f*g \neq 0 \), the result follows from Theorem 3.

REFERENCES


2. RUSCHEWEYH, ST., Convolutions in geometric function theory, Les Presses De L'Université De Montréal, Montreal, Canada, 1982.


Special Issue on
Time-Dependent Billiards

Call for Papers

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Event</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>December 1, 2008</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
</tr>
</tbody>
</table>

Guest Editors
Edson Denis Leonel, Departamento de Estatística, Matemática Aplicada e Computação, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil; edleonel@rc.unesp.br
Alexander Loskutov, Physics Faculty, Moscow State University, Vorob’evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru

Hindawi Publishing Corporation
http://www.hindawi.com