NEW CHARACTERIZATIONS FOR HANKEL TRANSFORMABLE SPACES OF ZEMANIAN

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ABSTRACT. In this paper we obtain new characterizations of the Zemanian spaces \( H_\mu \) and \( H'_\mu \).

KEY WORDS AND PHRASES. Hankel transform, distribution, Zemanian spaces

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A. H. Zemanian [7, Ch 5] introduced the space \( H_\mu (\mu \in \mathbb{R}) \) of functions as follows: a complex valued smooth function \( \phi (x), \ x \in I = (0, \infty) \), is in \( H_\mu \), if, and only if, the quantity

\[
\gamma^\mu_{n,k}(\phi) = \sup_{x \in I} |x^n(x^{-1}D)^k(x^{-\mu-1/2}\phi(x))| < \infty
\]

is finite, for every \( n, k \in \mathbb{N} \). This space endowed the topology generated by \( \{\gamma^\mu_{n,k}\}_{n,k \in \mathbb{N}} \) is a Fréchet space. In the sequel we will refer to the above topology as the usual topology of \( H_\mu \). Zemanian introduced the space \( H'_\mu \) to extend the Hankel integral transformation defined by

\[
(h_\mu f)(x) = \int_0^\infty (xt)^{1/2} J_\mu(\sqrt{x}t) f(t) \, dt,
\]

where \( J_\mu \) denotes the Bessel function of the first kind and order \( \mu \), to generalized functions. He proved that \( h_\mu \) is an automorphism of \( H_\mu \) provided that \( \mu \geq -\frac{1}{2} \). The generalized Hankel transform \( h_\mu \) of \( f \in H'_\mu \), the dual space of \( H_\mu \), is defined as the transposed of \( H'_\mu \) through

\[
\langle h_\mu^* f, \phi \rangle = \langle f, h_\mu \phi \rangle \quad \text{for} \quad \phi \in H_\mu.
\]

Thus if \( \mu \geq -\frac{1}{2} \), \( h_\mu^* \) is an automorphism of \( H'_\mu \) when this space is equipped with the weak* topology or with the strong topology.

In [2] J. J. Betancor and I. Marrero have studied the main topological properties of the spaces \( H_\mu \) and \( H'_\mu \). Amongst other results, it is established (Theorem 3.3) that the space \( H_\mu, \mu \geq -\frac{1}{2} \), is constituted by all those complex valued smooth functions \( \phi (x), \ x \in I, \) such that

\[
\tau^\mu_{n,k}(\phi) = \sup_{x \in I} |x^nN_{\mu+k-1}...N_\mu \phi(x)| < \infty
\]

for every \( n, k \in \mathbb{N} \). Moreover, the system of seminorms \( \{\tau^\mu_{n,k}\}_{n,k \in \mathbb{N}} \) generates of \( H_\mu \) its usual topology. Moreover in [4] they gave new descriptions for the usual topology of \( H_\mu \) through \( L_2 \)-norms.

A. H. Zemanian [7, p. 134] defined the space \( O \) formed by all those complex valued smooth functions \( \psi(x), \ x \in I, \) satisfying that for every \( k \in \mathbb{N} \) there exists \( n_k \in \mathbb{N} \) such that \( (1 + x^2)^n(x^{-1}D)^k\psi(x) \) is a bounded function on \( I \). He proved that \( O \) is a space of multiplier of \( H_\mu \). Recently J. J. Betancor and I. Marrero [2, Theorems 2.3 and 4.9] have characterized \( O \) as the space of multipliers of \( H_\mu \) and \( H'_\mu \).

In this paper we characterize the smooth complex valued functions in \( H_\mu, \mu \geq -\frac{1}{2} \), as the ones satisfying

\[
Z_n(\phi) = \sup_{x \in I} |x^n\phi(x)| < \infty
\]

(1)
\[
\gamma_n^\mu (\phi) = \sup_{x \in \mathbb{I}} |N_{\mu+n-1} \cdots N_{\mu} \phi(x)| < \infty \tag{2}
\]

for every \( n \in \mathbb{N} \). Moreover we prove that the usual topology of \( H_\mu \) can be defined by the family of seminorms \( \{ Z_n, \gamma_n^\mu \}_{n \in \mathbb{N}} \) and a new characterization for the elements of \( H_\mu \) is obtained. In the sequel we will assume that \( \mu \geq -\frac{1}{2} \).

**PROPOSITION 1.** A complex valued smooth function \( \phi(x), \ x \in \mathbb{I} \), is in \( H_\mu \) if, and only if, \( \phi \) satisfies (1) and (2) for every \( n \in \mathbb{N} \).

**PROOF.** It is clear that if \( \phi \in H_\mu \) then \( \phi \) satisfies (1) and (2) for every \( n \in \mathbb{N} \).

Let now \( \phi \) be a complex valued smooth function defined on \( \mathbb{I} \). To see that (1) and (2) \((n \in \mathbb{N})\) are sufficient conditions for \( \phi \) belongs to \( H_\mu \) we proceed by induction. Suppose, as induction hypothesis, that

\[
\sup_{x \in \mathbb{I}} [x^m N_{\mu+n-1} \cdots N_{\mu} \phi(x)] < \infty, \quad m \in \mathbb{N} \quad \text{and} \quad n \in \mathbb{N}, \quad 0 \leq n < \ell
\]

for certain \( \ell \in \mathbb{N}, \ell \geq 1 \).

By using partial integration we can obtain

\[
\|x^m N_{\mu+\ell-1} \cdots N_{\mu} \phi(x)\|_2^2 = \int_0^\infty [x^m N_{\mu+\ell-1} \cdots N_{\mu} \phi(x)]^2 dx
\]

\[
= \int_0^\infty x^{2m} N_{\mu+\ell-1} \cdots N_{\mu} (\phi(x)) N_{\mu+\ell-1} \cdots N_{\mu} (\overline{\phi}(x)) dx
\]

\[
= \int_0^\infty (Dx^{-1})^\ell (x^{2m+\mu+\ell+1/2} N_{\mu+\ell-1} \cdots N_{\mu} (\phi(x))) x^{-\mu-1/2} \overline{\phi}(x) dx
\]

for every \( m \in \mathbb{N}, \ell < 2m + 2 \), because

\[
[(Dx^{-1})^\ell (x^{2m+\mu+\ell+1/2} N_{\mu+\ell-1} \cdots N_{\mu} (\phi(x))) (x^{-1} D)^{\ell-i-1} (x^{-\mu-1/2} \overline{\phi}(x))]_0^\infty = 0 \quad (3)
\]

for each \( i, m \in \mathbb{N}, 0 \leq i < \ell < 2m + 2 \). In effect, if \( m, i \in \mathbb{N}, \ 0 \leq i < \ell < 2m + 2 \) then Leibniz's rule leads to

\[
(Dx^{-1})^\ell (x^{2m+\mu+\ell+1/2} N_{\mu+\ell-1} \cdots N_{\mu} (\phi(x))) (x^{-1} D)^{\ell-i-1} (x^{-\mu-1/2} \overline{\phi}(x))
\]

\[
= \sum_{j=0}^{\ell} a_j x^{2m+2\ell+2\mu+1-2i} (x^{-1} D)^{\ell+i-j} (x^{-\mu-1/2} \phi(x)) (x^{-1} D)^{\ell-i-1} (x^{-\mu-1/2} \overline{\phi}(x))
\]

\[
= \sum_{j=0}^{\ell} a_j x^{2m+1-j} N_{\mu+\ell+i-1} \cdots N_{\mu} (\phi(x)) N_{\mu+\ell-i-2} \cdots N_{\mu} (\phi(x))
\]

where \( a_j, \ j \in \mathbb{N}, \ 0 \leq j \leq i \), are suitable real numbers, and by virtue of induction hypothesis (3) follows.

Most straightforward manipulations allow us to write

\[
(Dx^{-1})^\ell (x^{2m+\mu+\ell+1/2} N_{\mu+\ell-1} \cdots N_{\mu} (\phi(x))) x^{-\mu-1/2} \overline{\phi}(x) = \sum_{j=0}^{\ell} a_j x^{2m-j} \overline{\phi}(x) N_{\mu+2\ell-j-1} \cdots N_{\mu} \phi(x)
\]

with \( m \in \mathbb{N} \) and \( a_j \in \mathbb{R}, \ j \in \mathbb{N}, \ 0 \leq j \leq \ell \).

Hence we can establish

\[
\|x^m N_{\mu+\ell-1} \cdots N_{\mu} \phi(x)\|_2^2 \leq C_1 \sum_{j=0}^{\ell} \int_0^\infty |x^{2m-j} \overline{\phi}(x)| \ |N_{\mu+2\ell-j-1} \cdots N_{\mu} \phi(x)| dx
\]

\[
\leq C_2 \sum_{j=0}^{\ell} \sup_{x \in \mathbb{I}} |(1 + x^2) x^{2m-j} \phi(x)| \sup_{x \in \mathbb{I}} |N_{\mu+2\ell-j-1} \cdots N_{\mu} \phi(x)| < \infty, \quad (4)
\]
provided that \( m \in \mathbb{N}, \ 2m \geq \ell \). Here \( C_i, \ i = 1, 2 \), denotes suitable positive constants.

Assume now that \( m \in \mathbb{N}, \ 2m < \ell \). We have

\[
\|x^m N_{\mu+\ell-1} \ldots N_{\mu} \phi(x)\|_2^2 = \left( \int_0^{1} + \int_{1}^{\infty} \right) |x^m N_{\mu+\ell-1} \ldots N_{\mu} \phi(x)|^2 \, dx
\]

\[
\leq \int_0^{1} |N_{\mu+\ell-1} \ldots N_{\mu} \phi(x)|^2 \, dx + \int_0^{\infty} |x^\ell N_{\mu+\ell-1} \ldots N_{\mu} \phi(x)|^2 \, dx.
\]

Therefore, by invoking (4) and the induction hypothesis we infer that

\[
\|x^m N_{\mu+\ell-1} \ldots N_{\mu} \phi(x)\|_2 < \infty, \quad \text{when} \quad m \in \mathbb{N}, \ 2m \leq \ell.
\]

Thus it is concluded that \( \|x^m N_{\mu+\ell-1} \ldots N_{\mu} \phi(x)\|_2 < \infty, \ m \in \mathbb{N} \).

Also, for every \( m \in \mathbb{N}, \ m \geq 1, \) and \( x \in I \),

\[
(x^m N_{\mu+\ell-1} \ldots N_{\mu} \phi(x))^2 = \int_0^{1} D_t(t^m N_{\mu+\ell-1} \ldots N_{\mu} \phi(t))^2 \, dt
\]

\[
= \int_0^{1} 2t^m N_{\mu+\ell-1} \ldots N_{\mu} \phi(t) ([m + \mu + \frac{1}{2} + \ell] t^{m-1} N_{\mu+\ell-1} \ldots N_{\mu} \phi(t) + t^m N_{\mu+\ell} \ldots N_{\mu} \phi(t)) \, dt.
\]

Hence if \( m \in \mathbb{N}, \ m \geq 1, \) and \( x \in I \) by using Holder's inequality we can find \( C \geq 0 \) such that

\[
|x^m N_{\mu+\ell-1} \ldots N_{\mu} \phi(x)|^2 \leq C \left( \|x^m N_{\mu+\ell-1} \ldots N_{\mu} \phi(x)\|_2 \|x^{m-1} N_{\mu+\ell-1} \ldots N_{\mu} \phi(x)\|_2 \right.
\]

\[
+ \sup_{x \in I} |N_{\mu+\ell-1} \ldots N_{\mu} \phi(x)| \left( \|x^m N_{\mu+\ell-1} \ldots N_{\mu} \phi(x)\|_2 + \|x^{m+1} N_{\mu+\ell} \ldots N_{\mu} \phi(x)\|_2 \right)^2
\]

and then \( \sup_{x \in I} |x^m N_{\mu+\ell-1} \ldots N_{\mu} \phi(x)| < \infty, \ m \in \mathbb{N} \).

Thus the proof is finished.

The last proposition allows us to define the usual topology of \( H_\mu \) through a family of seminorms simpler than \( \{\gamma_{m,k}^\mu\}_{m,k \in \mathbb{N}} \).

**PROPOSITION 2.** The usual topology of \( H_\mu \) is defined by the system of seminorms \( \{Z_n, y_n^\mu\}_{n \in \mathbb{N}} \).

**PROOF.** It is clear that the topology generated by \( \{\gamma_{m,k}^\mu\}_{m,k \in \mathbb{N}} \) is finer than the one defined by \( \{Z_n, y_n^\mu\}_{n \in \mathbb{N}} \) on \( H_\mu \). Moreover by proceeding in a way similar to A. H. Zemanian [7, Lemma 5.2-2] we can prove that \( H_\mu \) endowed with the topology generated by \( \{Z_n, y_n^\mu\}_{n \in \mathbb{N}} \) is a Fréchet space. Hence the desired result is an immediate consequence of the Open Mapping Theorem [6, Corollary 2.12].

We now prove a new characterization for the elements of \( H'_\mu \), the dual space of \( H_\mu \). The procedure employed is analogous to the one used by the author [1] and by J. J. Betancor and I. Marrero [2].

**PROPOSITION 3.** Let \( f \) be a linear functional defined on \( H_\mu \). Then \( f \) is in \( H'_\mu \) if, and only if, there exist \( r \in \mathbb{N} \) and \( f_k, \ g_k \in L_\infty(0,\infty) \) (the space of essentially bounded functions on \( (0,\infty) \)), \( k \in \mathbb{N}, \ 0 \leq k \leq r \), such that

\[
f = \sum_{k=0}^{r} h_k^\mu \left( x^k f_k + x^{-\mu+1/2}(x^{-1}D)^k x^{k+\mu-1/2} g_k \right).
\]

**PROOF.** Let \( f \in H'_\mu \). By virtue of a well-known result ([7, Theorem 1.8-1]) there exist \( r \in \mathbb{N} \) and \( C > 0 \) such that

\[
|\langle f, \phi \rangle| \leq C \max_{0 \leq k \leq r} \{Z_k(\phi), y_k^\mu(\phi)\}, \quad \phi \in H_\mu.
\]

According to [7, Lemma 5.4-1(2), (3) and Theorem 5.4-1] and since \( z^{1/2} J_{\mu}(x) \) is a bounded function on \( I \) for every \( k \in \mathbb{N} \) one has
\[
\sup_{x \in I} |x^k \phi(x)| = \sup_{x \in I} |x^k h_\mu(h_\mu \phi)(x)| \leq C \int_0^\infty |N_{\mu+k-1} \ldots N_{\mu} h_\mu(h_\mu \phi)(t)| \, dt \tag{7}
\]
and
\[
\sup_{x \in I} |N_{\mu+k-1} \ldots N_{\mu} \phi(x)| = \sup_{x \in I} |N_{\mu+k-1} \ldots N_{\mu} h_\mu(h_\mu \phi)(x)| \leq C \int_0^\infty |t^k (h_\mu \phi)(t)| \, dt \tag{8}
\]
for a suitable \( C > 0 \).

The linear mapping
\[
j : H_\mu \to JH_\mu \subset L_1(0, \infty)^{2r+2}
\]
\[
\phi \mapsto (x^k h_\mu \phi, N_{\mu+k-1} \ldots N_{\mu} h_\mu \phi)_{k=0}^r
\]
is one to one because \( h_\mu \) is an automorphism of \( H_\mu \) ([7, Theorem 5.4-1]). Here \( L_1(0, \infty) \) denotes the usual Lebesgue space of order 1.

On the other hand, the inequalities (6), (7) and (8) imply that the linear mapping
\[
L : JH_\mu \subset L_1(0, \infty)^{2r+2} \to \mathbb{C}
\]
\[
(z^k h_\mu \phi, N_{\mu+k-1} \ldots N_{\mu} h_\mu \phi)_{k=0}^r \to \langle f, \phi \rangle
\]
is continuous when \( JH_\mu \) is endowed with the topology induced by \( L_1(0, \infty)^{2r+2} \). Hence, by invoking the Hahn-Banach Theorem \( L \) can be extended to \( L_1(0, \infty)^{2r+2} \) as a member of \( (L_1(0, \infty)^{2r+2})^\prime \), the dual space of \( L_1(0, \infty)^{2r+2} \). Since, as it is well known, \( L_1(0, \infty)^\prime = L_\infty(0, \infty) \) there exist \( f_k, g_k \in L_\infty(0, \infty), \ k \in \mathbb{N}, \ 0 \leq k \leq r \), such that
\[
\langle f, \phi \rangle = \sum_{k=0}^r \left( \langle f_k, x^k h_\mu \phi \rangle + \langle g_k, x^{k+1/2}(x^{-1}D)_k(x^{-1/2} \phi) \rangle \right), \quad \phi \in H_\mu.
\]
Therefore
\[
f = \sum_{k=0}^r h_\mu^k (x^k f_k + (-1)^k x^{-1/2}(x^{-1}D)_k x^{k+1/2} g_k).
\]
Thus the proof of necessity is finished.

Conversely, if \( f \) is a linear functional defined on \( H_\mu \) by (5) for certain \( r \in \mathbb{N} \) and \( f_k, g_k \in L_\infty(0, \infty), \ k \in \mathbb{N}, \ 0 \leq k \leq r \), then
\[
|\langle f, \phi \rangle| \leq C \sum_{k=0}^r \left( \|f_k\|_{\infty} \sup_{x \in I} |(1 + x^2)x^k(h_\mu \phi)(x)| + \|g_k\|_{\infty} \sup_{x \in I} |(1 + x^2)N_{\mu+k-1} \ldots N_{\mu} h_\mu \phi)(x)| \right)
\]
for \( \phi \in H_\mu \), where \( \|\cdot\|_{\infty} \) denotes the usual norm in \( L_\infty(0, \infty) \). Hence, according to [7, Theorem 5.4-1] and [2, Theorem 3.3], \( f \) is in \( H_\mu^r \).

REFERENCES

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