A REMARK ON THE EXTENSION OF THE CONCEPT OF INCIDENCE ALGEBRAS TO NONLOCALY FINITE PARTIALLY ORDERED SETS

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An incidence algebra of a nonlocally finite partially ordered set \( Q \) is a very rare concept, perhaps nonexistent. In this note, we will attempt to construct such an algebra.

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1. Introduction. Let \( P \) be a partially ordered set (poset) and \( K \) a field of characteristic 0. The functions \( f : P \times P \to K \), such that \( x \not\leq y \) implies \( f(x,y) = 0 \), are called the incidence functions of \( P \) over \( K \). The set of such functions is denoted by \( \vartheta(K,P) \).

\( P \) is called locally finite if for every \( x,y \in P \) the interval \([x,y] = \{ t \in P \mid x \leq t \leq y \}\) is finite. When \( P \) is locally finite, \( \vartheta(K,P) \) becomes a \( K \)-algebra under a multiplication \( (\ast) \) defined by convolution:

\[
f \ast g(x,y) = \sum_{x \leq t \leq y} f(x,t)g(t,y),
\]

(1.1)

and the algebra \( \vartheta(K,P) \) is called the incidence algebra of \( P \) over \( K \) [1, 2].

If \( P \) is not locally finite, the expression (1.1) may not make sense. So, one does not often hear of an incidence algebra of a nonlocally finite poset. Our purpose in this note is to show that if \( Q \) is any nonlocally finite poset and \( P \) is a locally finite poset, we can form a nonlocally finite poset \( QS(P) \) for which an incidence algebra \( \vartheta(K,QS(P)) \) can be constructed.

Moreover, the posets \( Q \) and \( P \) are both embeddable in \( QS(P) \), while the set \( \vartheta(K,Q) \) and the algebra \( \vartheta(K,P) \) are both embeddable in \( \vartheta(K,QS(P)) \), and if \(|P| \leq |Q|\), then \(|QS(P)| = |Q|\). Besides, for the fixed posets \( P \) and \( Q \), the incidence algebra \( \vartheta(K,QS(P)) \) is unique up to isomorphism. All these are established in Section 2.

In Section 3, we isolate the auxiliary locally finite poset \( P \) and try to deal directly with \( Q \). However, because of the problem still posed by (1.1), we can only construct a sequence of what are called truncated incidence algebras for the nonlocally finite poset \( Q \). For this purpose, we will need an additional hypothesis that \( Q \) is well ordered.

2. The construction of \( QS(P) \) and \( \vartheta(K,QS(P)) \). We will assume throughout that \( P \) is a locally finite poset, \( Q \) a nonlocally finite poset, and \( K \) is a field of characteristic 0. Let \( QS(P) \) be the Cartesian product \( P \times Q \). We will denote the order relation in \( P \) by \( \leq^{(1)} \) and the order relation in \( Q \) by \( \leq^{(2)} \). Then we define an order relation \( \leq \) in \( QS(P) \) by...
need to define the convolution multiplication shown in (1.1) on \( \sigma(y, \theta(s)) \) will make sense. By the definition of the order relation in \( (\sigma(y), \theta(s)) \), \( QS(P) \) is a partially ordered set. However, \( QS(P) \) is not locally finite. Let \( \eta \) denote that \( \sigma(y) \) and \( \theta(r) \) are any two elements of \( QS(P) \). Moreover, \( P_r \) and \( P_s \) are locally finite subposets of \( QS(P) \). Denote \( (x, r) \) by \( u \) and \( (y, s) \) by \( v \), and let \( T = \{ t \in P \mid x \leq (1) y \} \). Then the set \( T(u, v) = (T \times \{ r \}) \cup (T \times \{ s \}) \subseteq QS(P) \) is finite. Let \( J(u, v) = [u, v] \cap T(u, v) \). We define the operation \((*)\) on \( \theta(K, QS(P)) \) by the following: for all elements \( u \) and \( v \) in \( QS(P) \) and for all \( f, g \) in \( \theta(K, QS(P)) \),

\[
f \ast g(u, v) = \sum_{z \in J(u, v)} f(u, z) g(z, v). \tag{2.1}
\]

Clearly, (2.1) is now well defined. The associativity follows from [1, Proposition 4.1]. Consequently, with (2.1), \( \theta(K, QS(P)) \) is an incidence algebra of \( QS(P) \) over \( K \). 

\( P \) is isomorphic to \( P_r \) for each \( r \in Q \). Similarly, for each \( y \in P \), \( Q \) is isomorphic to \( Q_y = \{ y' \} \times Q_y \). Hence both \( P \) and \( Q \) are embeddable in \( QS(P) \). Moreover, the correspondence \( \mu_r : f \rightarrow f_r \), where \( f_r \) is defined by \( f_r(x_r, y_r) = f(x, y) \), is an isomorphism of \( \theta(K, P) \) onto \( \theta(K, P_r) \). Consequently, \( \theta(K, P) \) is embeddable in \( \theta(K, QS(P)) \). By a similar device, we find that \( \theta(K, Q) \) is also embeddable in \( \theta(K, QS(P)) \). For the uniqueness of \( \theta(K, QS(P)) \), we will prove the following.

**Proposition 2.1.** If \( P' \) and \( Q' \) are any posets such that \( P \) is isomorphic to \( P' \) and \( Q \) is isomorphic to \( Q' \), then \( \theta(K, QS(P)) \) is isomorphic to \( \theta(K, QS(P')) \).

**Proof.** Let \( \sigma : P - P' \) be an isomorphism while \( \theta : Q - Q' \) is an isomorphism. Define \( \eta : QS(P) - QS(P') \) by \( \eta(x, r) = (\sigma(x), \theta(r)) \). If \( \eta(x, r) = (\sigma(y), s) \), then \( (\sigma(x), \theta(r)) = (\sigma(y), \theta(s)) \). By the definition of the order relation in \( QS(P') \), we must have \( \sigma(x) = \sigma(y) \) and \( \theta(r) = \theta(s) \). Consequently, \( x = y \) and \( r = s \). Hence, \( (x, r) = (y, s) \). This shows that \( \eta \) is injective. Clearly, also \( \eta \) is surjective. Therefore, \( \eta \) is an isomorphism. For each \( u \in QS(P) \), denote \( \eta(u) \) by \( u' \). Now define \( \beta : \theta(K, QS(P)) - \theta(K, QS(P')) \) by \( \beta(f) = f' \), where \( f' \) is defined by \( f'(u', v') = f(u, v) \) for all \( u', v' \in QS(P') \). One can directly check that \( \beta \) is also an isomorphism. Hence, the proposition holds.

We observe that for the locally finite poset \( P \), one could have chosen any nonempty finite subset of \( Q \) itself. We will call the algebra \( \theta(K, QS(P)) \) the incidence algebra of \( Q \) relative to \( P \).

3. **Truncated incidence algebras.** Our interest now is to see what we can achieve by isolating the locally finite poset \( P \) and dealing directly with \( Q \). However, (1.1) still poses a problem. Nevertheless, following the motivation received from Section 2, what we need is to try to use a finite number of elements of the interval \([r, s]\) at a time, for any two elements \( r \) and \( s \) of the nonlocally finite poset \( Q \). Then arises the question of how to choose the finite number of elements from \([r, s]\). The formula for choosing such elements is outlined below for the case where \( Q \) is well ordered. What makes it possible
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is the property of a well-ordered set whereby not only does every nonempty subset of such a set have a first element, but also such a first element is unique [3, Theorems 64 and 65, page 76]. First, we show the existence of a well-ordered nonlocally finite poset $Q$.

**Example 3.1.** Let $Q = \{1/n \mid n \in \mathbb{N}\} \cup \{0\}$, where $\mathbb{N}$ is the set of natural numbers. $Q$ is a poset subject to the usual relation “$\geq$” (greater than or equal to). Clearly, also $Q$ is well ordered by “$\geq$”. However, for any $a \in Q$, $a \neq 0$, the interval $[0,a]$ is infinite. Hence, $Q$ is not locally finite.

Now let $W$ be any well-ordered poset and let $r \leq s \in W$. Set $W_0 = [r,s]$. Let $W_1 = W_0 - \{r\}$. Then, if $W_1 \neq \emptyset$, $W_1$ has a unique first element $t_1$. Let $W_2 = w_1 - \{t_1\}$. If $W_2 \neq \emptyset$, then $W_2$ has a unique first element $t_2$. In general, $W_i = W_{i-1} - \{t_{i-1}\}$, where $t_{i-1}$ = first element of $W_{i-1}$, and $t_0 = r$.

For any fixed natural number $n$, let $T_n(r,s) = \{r,t_1,\ldots,t_n,s\}$. Let

$$J_n(r,s) = \begin{cases} [r,s] & \text{if } [r,s] \text{ is finite,} \\ T_n(r,s) & \text{otherwise.} \end{cases} \quad (3.1)$$

Then define the convolution multiplication $*$ on $\mathcal{I}(K,W)$ by the following: for all $r,s \in W$ and for all $f,g \in \mathcal{I}(K,W)$,

$$f * g(r,s) = \sum_{t \in J_n(r,s)} f(r,t)g(t,s). \quad (3.2)$$

Subject to (3.2), $\mathcal{I}(K,W)$ is an incidence algebra. We denote this incidence algebra by $\mathcal{I}_n(K,W)$, and $\mathcal{I}_n(K,W)$ is called a truncated incidence algebra of $W$ over $K$.

It is clear that $T_n(r,s) \subseteq T_{n+1}(r,s)$ for all $n \in \mathbb{N}$. We will call the incidence algebra $\mathcal{I}_{n+1}(K,W)$ a refinement of the incidence algebra $\mathcal{I}_n(K,W)$. The sequence $\{\mathcal{I}_n(K,W)\}$ of incidence algebras is finite if and only if $W$ is locally finite.

We now observe that a well-ordered nonlocally finite poset $Q$ is associated with an infinite sequence of truncated incidence algebras, where each is a nontrivial refinement of the one before it. Unifying these algebras to form one incidence algebra of $Q$ over $K$ remains an open problem.

**References**


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