NEIGHBORHOODS OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

M. K. AOUF

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By making use of the familiar concept of neighborhoods of analytic functions, the author proves several inclusion relations associated with the \((n, \delta)\)-neighborhoods of various subclasses defined by Salagean operator. Special cases of some of these inclusion relations are shown to yield known results.

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1. Introduction

Let \( T(j) \) denote the class of functions of the form

\[
f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \quad (a_k \geq 0; \ j \in \mathbb{N} = \{1, 2, \ldots\})
\]  

(1.1)

which are analytic in the open unit disc \( U = \{z : |z| < 1\} \).

Following [5, 8], we define the \((j, \delta)\)-neighborhood of a function \( f(z) \in A(j) \) by

\[
N_{j,\delta}(f) = \left\{ g \in T(j) : g(z) = z - \sum_{k=j+1}^{\infty} b_k z^k, \ \sum_{k=j+1}^{\infty} k |a_k - b_k| \leq \delta \right\}.
\]  

(1.2)

In particular, for the identity function \( e(z) = z \), we immediately have

\[
N_{j,\delta}(e) = \left\{ g \in T(j) : g(z) = z - \sum_{k=j+1}^{\infty} b_k z^k, \ \sum_{k=j+1}^{\infty} k |b_k| \leq \delta \right\}.
\]  

(1.3)

The main object of this paper is to investigate the \((j, \delta)\)-neighborhoods of the following subclasses of the class \( T(j) \) of normalized analytic functions in \( U \) with negative coefficients.
2 Neighborhoods of certain classes

For a function \( f(z) \in T(j) \), we define

\[
D^0 f(z) = f(z), \\
D^1 f(z) = Df(z) = zf'(z), \\
D^n f(z) = D(D^{n-1} f(z)) \quad (n \in \mathbb{N}).
\] (1.4)

The differential operator \( D^n \) was introduced by Sălăgean [9]. With the help of the differential operator \( D^n \), we say that a function \( f(z) \in T(j) \) is in the class \( T_j(n,m,\alpha) \) if and only if

\[
\text{Re} \left\{ \frac{D^{n+m} f(z)}{D^n f(z)} \right\} > \alpha \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \ m \in \mathbb{N}).
\] (1.5)

for some \( \alpha \ (0 \leq \alpha < 1) \), and for all \( z \in U \).

The operator \( D^{n+m} \) was studied by Sekine [11], Aouf et al. [2], Aouf et al. [3], and Hossen et al. [6]. We note that \( T_j(0,1,\alpha) = S_j^*(\alpha) \), the class of starlike functions of order \( \alpha \), and \( T_j(1,1,\alpha) = C_j(\alpha) \), the class of convex functions of order \( \alpha \) (Chatterjea [4] and Srivastava et al. [12]).

2. Neighborhood for the class \( T_j(n,m,\alpha) \)

For the class \( T_j(n,m,\alpha) \), we need the following lemma given by Sekine [11].

**Lemma 2.1.** A function \( f(z) \in T(j) \) is in the class \( T_j(n,m,\alpha) \) if and only if

\[
\sum_{k=j+1}^{\infty} k^n (k^m - \alpha) a_k \leq 1 - \alpha
\] (2.1)

for \( n,m \in \mathbb{N}_0 \) and \( 0 \leq \alpha < 1 \). The result is sharp.

Applying the above lemma, we prove the following.

**Theorem 2.2.** \( T_j(n,m,\alpha) \subset N_{j,\delta}(e) \), where

\[
\delta = \frac{(1 - \alpha)}{(j + 1)^{n-1}[((j+1)^{m} - \alpha)].
\] (2.2)

**Proof.** It follows from (2.1) that if \( f(z) \in T_j(n,m,\alpha) \), then

\[
(j + 1)^{n-1}[((j+1)^{m} - \alpha)] \sum_{k=j+1}^{\infty} k a_k \leq 1 - \alpha,
\] (2.3)

that is, that

\[
\sum_{k=j+1}^{\infty} k a_k \leq \frac{1 - \alpha}{(j + 1)^{n}[((j+1)^{m} - 1]} = \delta,
\] (2.4)

which, in view of definition (1.3), proves Theorem 2.2. \( \square \)
Putting \( j = 1 \) in Theorem 2.2, we have the following.

**Corollary 2.3.** \( T_1(n, m, \alpha) \subset N_{1, \delta}(e) \), where \( \delta = (1 - \alpha)/2^{n-1}[2^m - \alpha] \).

**Remark 2.4.** (i) Putting \( n = 0 \) and \( m = 1 \) in Theorem 2.2 and Corollary 2.3, we obtain the results obtained by Altintas and Owa [1].

(ii) Putting \( n = m = 1 \) in Theorem 2.2 and Corollary 2.3, we obtain the results obtained by Altintas and Owa [1].

3. **Neighborhoods for the classes** \( R_j(n, \alpha) \) **and** \( P_j(n, \alpha) \)

A function \( f(z) \in T(j) \) is said to be in the class \( R_j(n, \alpha) \) if it satisfies

\[
\text{Re}(D^n f(z))' > \alpha \quad (z \in U)
\]

for some \( \alpha \) (0 \leq \alpha < 1) and \( n \in \mathbb{N}_0 \). The class \( R_1(n, \alpha) \) was studied by Yaguchi and Aouf [13]. We note that \( R_j(0, \alpha) = R_j(\alpha) \) (Sarangi and Uralegaddi [10]).

Further, a function \( f(z) \in T(j) \) is said to be a member of the class \( P_j(n, \alpha) \) if it satisfies

\[
\text{Re}\left\{\frac{D^n f(z)}{z}\right\} > \alpha \quad (z \in U)
\]

for some \( \alpha \) (0 \leq \alpha < 1) and \( z \in U \). The class \( P_1(n, \alpha) \) was studied by Nunokawa and Aouf [7].

It is easy to see the following.

**Lemma 3.1.** A function \( f(z) \in T(j) \) is in the class \( R_j(n, \alpha) \) if and only if

\[
\sum_{k=j+1}^{\infty} k^{n+1} a_k \leq 1 - \alpha.
\]

The result is sharp.

**Lemma 3.2.** A function \( f(z) \in T(j) \) is in the class \( P_j(n, \alpha) \) if and only if

\[
\sum_{k=j+1}^{\infty} k^n a_k \leq 1 - \alpha.
\]

The result is sharp.

From the above lemmas, we see that \( R_j(n, \alpha) \subset P_j(n, \alpha) \).

**Theorem 3.3.** \( R_j(n, \alpha) \subset N_{j, \delta}(e) \), where

\[
\delta = \frac{1 - \alpha}{(j + 1)^n}.
\]

**Proof.** If \( f(z) \in R_j(n, \alpha) \), we have

\[
(j + 1)^n \sum_{k=j+1}^{\infty} k a_k \leq 1 - \alpha,
\]
which gives
\[
\sum_{k=j+1}^{\infty} ka_k \leq \frac{1 - \alpha}{(j+1)^n} = \delta, \tag{3.7}
\]
which, in view of definition (1.3), proves Theorem 3.3. \qed

Putting \( j = 1 \) in Theorem 2.2, we have the following.

**Corollary 3.4.** \( R_1(n, \alpha) \subset N_{1, \delta}(e), \) where \( \delta = (1 - \alpha)/2^n. \)

**Theorem 3.5.** \( P_j(n, \alpha) \subset N_{j, \delta}(e), \) where
\[
\delta = \frac{1 - \alpha}{(j+1)^{n-1}}. \tag{3.8}
\]

**Proof.** If \( f(z) \in P_j(n, \alpha), \) we have
\[
(j+1)^{n-1} \sum_{k=j+1}^{\infty} ka_k \leq 1 - \alpha, \tag{3.9}
\]
which gives
\[
\sum_{k=j+1}^{\infty} ka_k \leq \frac{1 - \alpha}{(j+1)^{n-1}} = \delta, \tag{3.10}
\]
which, in view of definition (1.3), proves Theorem 3.5. \qed

Putting \( j = 1 \) in Theorem 3.5, we have the following.

**Corollary 3.6.** \( P_1(n, \alpha) \subset N_{1, \delta}(e), \) where \( \delta = (1 - \alpha)/2^{n-1}. \)

### 4 Neighborhoods of certain classes

A function \( f(z) \in T(j) \) is said to be in the class \( K_j(n, m, \alpha, \beta) \) if it satisfies
\[
\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \alpha \quad (z \in U) \tag{4.1}
\]
for some \( \alpha (0 \leq \alpha < 1) \) and \( g(z) \in T_j(n, m, \beta) (0 \leq \beta < 1). \)

**Theorem 4.1.** \( N_{j, \delta}(g) \subset K_j(n, m, \alpha, \beta), \) where \( g(z) \in T_j(n, m, \beta) \) and
\[
\alpha = 1 - \frac{(j+1)^{n-1}[(j+1)^m - \beta]\delta}{(j+1)^n[(j+1)^m - \beta] - 1 + \beta}, \tag{4.2}
\]
where \( \delta \leq (j+1) - (1 - \beta)(j+1)^{-n}[(j+1)^m - \beta]^{-1}. \)
Proof. Let \( f(z) \) be in \( N_{j,\delta}(g) \) for \( g(z) \in T_j(n,m,\beta) \). Then we know that

\[
\sum_{k=\infty}^{\infty} k |a_k - b_k| \leq \delta,
\]

\[ (4.3) \]

\[
\sum_{k=\infty}^{\infty} b_k \leq \frac{1 - \beta}{(j+1)^n[(j+1)^m - \beta]}.
\]

Thus we have

\[
\left| \frac{f(z) - 1}{g(z) - 1} \right| \leq \frac{\sum_{k=\infty}^{\infty} |a_k - b_k|}{1 - \sum_{k=\infty}^{\infty} b_k} \leq \delta \cdot \frac{(j+1)^n[(j+1)^m - \beta]}{(j+1)^n[(j+1)^m - \beta] - 1 + \beta}
\]

\[ (4.4) \]

\[
= \frac{(j+1)^{n-1}[(j+1)^m - \beta] \delta}{(j+1)^n[(j+1)^m - \beta] - 1 + \beta} = 1 - \alpha.
\]

This implies that \( f(z) \in K_j(n,m,\alpha,\beta) \). \( \square \)

Putting \( j = 1 \) in Theorem 4.1, we have the following.

Corollary 4.2. \( N_{1,\delta}(g) \subset K_1(n,m,\alpha,\beta) \), where \( g(z) \in T_1(n,m,\beta) \) and

\[
\alpha = 1 - \frac{2^{m} - \beta}{2^{m} - \beta - 1 + \beta}.
\]

\[ (4.5) \]

Remark 4.3. Putting \( n = 0 \) and \( m = 1 \) in Theorem 4.1 and Corollary 4.2, we obtain the results obtained by Altintas and Owa [1].

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References

6 Neighborhoods of certain classes


M. K. Aouf: Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura 35516, Egypt

*E-mail address*: mkaouf@mans.edu.eg
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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

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<thead>
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<th>Event</th>
<th>Date</th>
</tr>
</thead>
<tbody>
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<td>December 1, 2008</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
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