SHORT PROOFS OF THEOREMS OF LEKKERKERKER AND BALLIEU

MAX RIEDERLE
Eberhardstr. 14
79 Ulm/Donau
West Germany

(Received October 16, 1981)

ABSTRACT. For any irrational number ξ let A(ξ) denote the set of all accumulation points of \{z: z=q(ξ-p), p ∈ ℤ, q ∈ ℤ - {0}, p and q relatively prime\}. In this paper the following theorem of Lekkerkerker is proved in a short and elementary way: The set A(ξ) is discrete and does not contain zero if and only if ξ is a quadratic irrational. The procedure used for this proof simultaneously takes care of a theorem of Ballieu.

KEY WORDS AND PHRASES. Lekkerkerker's Theorem, Approximation of numbers, Quadratic Irrationals.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 10F05, 10F35.

1. INTRODUCTION.

This paper is easily readable by anyone familiar with the elements of continued fractions, as far as Lagrange's theorem on the periodic representation of quadratic irrationals. Throughout this paper, ξ denotes an irrational number which is represented by the regular continued fraction ξ = [b_0, b_1, b_2, ...] = [b_0, b_1, ..., b_{n-1}, \xi_n] and A(ξ) stands for the set of all the real accumulation points of \{z: z = q(ξ-p), p ∈ ℤ, q ∈ ℤ - {0}, p and q relatively prime\}. Obviously, A(ξ) describes those Dirichlet approximation qualities which occur infinitely often. Furthermore, for any sequence (a_n^) let H(a_n^) denote the set of all its limit points and for x ∈ ℜ and ε > 0 set B(x, ε) = (x - ε, x + ε). The main purpose of this paper is to give a simple proof of the following theorem of Lekkerkerker [1] (cf. also [2]).
The set \( A(\xi) \) is discrete and does not contain zero if and only if \( \xi \) is a quadratic irrational.

The proof of the sufficient part of the theorem mainly depends on the irreducible polynomial of \( \xi \), whereas the necessary part is a consequence of the relation between \( A(\xi) \) and the sequence \( \left( \frac{A_n}{B_n} \right) \) and simultaneously establishes the following theorem of Ballieu [3]:

The set \( H(\xi) \) is finite and \( \left( \frac{A_n}{B_n} \right) \) is bounded if and only if \( \xi \) is a quadratic irrational.

Finally for any quadratic irrational \( \xi \) we shall show how to evaluate \( A(\xi) \) in an easy way.

2. BASIC FORMULAS.

In this section we state the formulas used in the sequel. Let \( \frac{A_n}{B_n} \) denote the \( n \)-th convergent of \( [b_0, b_1, b_2, \ldots] \) where \( A_n \) and \( B_n \) are relatively prime. Set \( \rho_n = B_n/B_{n-1} \) and put \( \delta_n = B_n(B_n \xi - A_n) \). Then the following formulas hold for all \( n \in \mathbb{N}_0 \):

\[
\delta_n = \frac{(-1)^n}{\xi \rho_n} \quad (2.1)
\]

\[
\delta_{n-1} = \frac{(-1)^{n-1}}{\rho_n + 1/\xi} \quad (2.2)
\]

\[
\xi_{n+1} = \frac{1 + \sqrt{1 + 4\delta_n \delta_{n-1}}}{2\delta_n} (-1)^n. \quad (2.3)
\]

PROOF. Equation (2.1) is an easy consequence of the well known identity

\[\xi - A_n/B_n = (-1)^n/(B_nB_{n+1} + B_{n+1}B_n),\]

formula (2.2) can be derived from (2.1) when using the identities \( \xi_n = b_n + 1/\xi \) and \( \rho_n = b_n + 1/\rho_{n-1} \) and, finally, (2.3) can be obtained when combining (2.1) and (2.2).

3. PROOF OF LEKKERKERKER'S THEOREM.

(i) Suppose \( \xi \) is a quadratic irrational. There exists an indefinite quadratic form \( f(x,y) = ax^2 + bxy + cy^2 \) with \( a, b, c \in \mathbb{Z} \) and \( f(\xi, 1) = 0 \). If \( \xi \) denotes the algebraic conjugate of \( \xi \) then if follows by Vieta's theorem that

\[ f(p, q) = a(p - q\xi)(p - q\xi) = aq(q\xi - p)(\xi - p/q) \quad (3.1)\]

for all \( (p, q) \in \mathbb{Z} \times (\mathbb{Z} - \{0\}) \) which implies that
When using the notation \( \mathbb{Z} = \{ (x,y) : x,y \in \mathbb{Z}, y \neq 0, x \text{ and } y \text{ relatively prime} \} \) equation (3.2) implies that

\[
A(\xi) = \frac{\tilde{f}(\xi)}{a(\xi - \xi)} \tag{3.3}
\]

since if \((p_n,q_n) \in \mathbb{Z} \) with \( \lim_{n \to \infty} q_n \left( q_n \xi - p \right) = A(\xi) \) then \( \lim_{n \to \infty} p_n/q_n = \xi \). Clearly, \( f(\xi) \neq 0 \) for all \((p,q) \in \mathbb{Z} \) and hence it follows by (3.3) that \( 0 \notin A(\xi) \).

(ii) Suppose that \( A(\xi) \) is discrete and \( 0 \notin A(\xi) \). From equation (2.1) we can see that

\[
\| A(\xi) \|_1 \leq 1 \text{ for all } n \in \mathbb{N}_0. \text{ Therefore and since all the numbers } \delta_n \text{ are distinct, } H(\delta_n) \text{ is a compact subset of } A(\xi) \text{ and hence } H(\delta_n) \text{ is finite and } 0 \notin H(\delta_n). \]

Now by (2.3) it is easy to see that \( H(\xi_n) \) is finite and \( (\xi_n) \) is bounded. Therefore, in order to complete the proof, it suffices to prove Ballieu's theorem.

4. PROOF OF BALLIEU'S THEOREM.

(i) Suppose that \( (\xi_n) \) is bounded and \( H(\xi_n) \) is finite, say \( H(\xi_n) = \{ z_1, \ldots, z_m \} \).

It follows from the identity \( \xi_n = [b_n, b_{n+1}, \ldots] \) that there exists a \( k \in \mathbb{N} \) such that \( b_n \leq k \) for all \( n \in \mathbb{N} \), and hence \( b_n + 1/k \leq \xi_n \leq b_n + 1 - 1/(k+1) \). Therefore, the set \( H(\xi_n) \cap \mathbb{Z} \) is empty and we can find a number \( \epsilon > 0 \) such that the sets \( B(z,\epsilon) \) are pairwise disjoint and contained in \( \mathbb{R} - \mathbb{Z} \).

Let \( I(z) \) denote the greatest integer not exceeding \( z \) and for \( z \notin \mathbb{Z} \) put

\[
\tau(z) = (z - I(z))^{-1}. \text{ Clearly, } \xi_{n+1} = \tau(\xi_n) \text{ for all } n \in \mathbb{N}_0. \text{ Also the function } \tau \text{ is continuous on } H(\xi_n) \text{, therefore } \tau(H(\xi_n)) \subset H(\xi_n) \text{ and we can find a } \delta, 0 < \delta < \epsilon, \text{ such that } \tau(B(z,\delta)) \subset B(\tau(z),\epsilon) \text{ for all } z \in [1, \ldots, m]. \text{ There exists a number } n_0 \text{ such that } \xi_{n_0} \in B(z,\delta) \text{ and } \xi_n \in \bigcup_{\nu=1}^{m} B(z,\delta) \text{ for all } n \geq n_0. \text{ Therefore, when writing } \tau(p) \text{ for the } p\text{-th composition map of } \tau, \text{ we obtain by induction that}
\]

\[
\xi_{n_0} + p \in B(\tau(\xi_{n_0} + p),\delta) \text{ for all } p \in \mathbb{N}_0. \text{ Since } \tau(H(\xi_n)) \subset H(\xi_n) \text{ and } H(\xi_n) \text{ is finite, the sequence } (\tau(p)(z_1)), p \in \mathbb{N}_0, \text{ is periodic. From the identities } b_{n_0} = I(\xi_{n_0} + p), \text{ and thus, by Lagrange's theorem, } \xi \text{ is a quadratic irrational.}
\]

(ii) The other direction of Ballieu's theorem is an easy consequence of
Lagrange's theorem.

5. CONCLUDING REMARKS.

The inclusion in (3.3) is actually an equality. In order to prove this, we need the following well known theorem (cf. [4], p. 22-23):

Let $f(x,y)$ be an indefinite quadratic form with integer coefficients and let $\xi$ be one of its roots. Then for any pair $(p,q) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ there are infinitely many relatively prime integers $p_n, q_n$ such that $f(p_n, q_n) = f(p, q)$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} (q_n \xi - p) = 0$.

In fact, this result combined with (5) and (6), leads to the following:

THEOREM. Suppose that $\xi$ is a quadratic irrational, say $f(\xi,1) = 0$ for some indefinite quadratic form $f(x,y)$ with integer coefficients. Moreover, let $\xi$ be the algebraic conjugate of $\xi$. Then

$$A(\xi) = \frac{f(\xi)}{f(1,0)(\xi - \xi)}.$$

REFERENCES


Special Issue on
Time-Dependent Billiards

Call for Papers
This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Event</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>December 1, 2008</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
</tr>
</tbody>
</table>

Guest Editors

Edson Denis Leonel, Departamento de Estatística, Matemática Aplicada e Computação, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob’evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru