ON THE NON-EXISTENCE OF SOME INTERPOLATORY POLYNOMIALS

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ABSTRACT. Here we prove that if \( x_k, k=1,2,\ldots,n+2 \) are the zeros of \( (1-x^2)T_n(x) \) where \( T_n(x) \) is the Tchebycheff polynomial of first kind of degree \( n \), \( \alpha_j, \beta_j, j=1,2,\ldots,n+2 \) and \( \gamma_j, j=2,3,\ldots,n+1 \) are any real numbers there does not exist a unique polynomial \( Q_{3n+3}(x) \) of degree \( \leq 3n+3 \) satisfying the conditions:

\[ Q_{3n+3}(x_j) = \alpha_j, \quad Q''_{3n+3}(x_j) = \beta_j, \quad j=1,2,\ldots,n+2 \text{ and } Q_{3n+3}(x_j) = \gamma_j, \quad j=2,3,\ldots,n+1. \]

Similar result is also obtained by choosing the roots of \( (1-x^2)P_n(x) \) as the nodes of interpolation where \( P_n(x) \) is the Legendre polynomial of degree \( n \).

KEY WORDS AND PHRASES. Roots, interpolatory polynomials, non-existence, nodes.

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1. INTRODUCTION.

In [1] R.B. Saxena considered an interesting problem of \((0,1,3)\) interpolation by taking the roots of \( (1-x^2)P_{n-2}(x) \), where \( P_{n-2}(x) \) is the Legendre polynomial of degree \( n-2 \), as the nodes of interpolation. By \((0,1,3)\) interpolation, Saxena meant that for the collections \( \{\alpha_j\}_{j=1}^n, \{\beta_j\}_{j=2}^n \text{ and } \{\gamma_j\}_{j=1}^{n-1} \) of real numbers and the zeros \( x_j \) of \( (1-x^2)P_{n-2}(x) \) arranged so that

\[-1 = x_n < x_{n-1} < \ldots < x_2 < x_1 = 1\]

a polynomial \( R_n(x) \) of degree \( \leq 3n-3 \) can be constructed so that

\[ R_n(x_j) = \alpha_j; \quad j=1,2,\ldots,n, \]

\[ R_n'(x_j) = \beta_j; \quad j=2,3,\ldots,n-1, \]

and

\[ R_n''(x_j) = \gamma_j; \quad j=1,2,\ldots,n. \]

Saxena proved that such a polynomial exists uniquely if \( n \) is even and for \( n \) odd there does not exist a unique polynomial \( R_n(x) \) satisfying the above conditions.

Later Varma [2] obtained the following result in this direction:

THEOREM 1 (VARMA). Given a positive integer \( n \) and real numbers \( \alpha_k(k=1,2,\ldots,n+2), \beta_k, \gamma_k(k=2,3,\ldots,n+1) \) there is, in general no polynomial \( F_{3n+1}(x) \) of degree \( \leq 3n+1 \) such that \( F_{3n+1}(x_k) = \alpha_k; \quad k=1,2,\ldots,n+2, \quad F_{3n+1}'(x_k) = \beta_k; \)
2. MAIN RESULTS.

In connection with the above results we shall prove the following.

THEOREM 2. For any positive integer \( n \), with \( 1 = \xi_1 > \xi_2 > \ldots > \xi_{n+1} > \xi_{n+2} = -1 \) the zeros of \((1 - x^2)T_n(x)\) where \( T_n(x) \) is the Chebyshev polynomial of first kind and if there exists such a polynomial then there is an infinity of them.

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\[
R_{3n+1}(\xi_j) = a_j; \quad j = 1, 2, \ldots, n + 1, n + 2, \quad (2.1)
\]

\[
R_{3n+1}(\xi_j) = \beta_j; \quad j = 2, 3, \ldots, n + 1 \quad (2.2)
\]

and

\[
R_{3n+1}(\xi_j) = \gamma_j; \quad j = 2, 3, \ldots, n + 1 \quad (2.3)
\]

are satisfied. If there does exist such a polynomial then there are infinitely many of them.

We also prove the following result for Chebyshev nodes:

THEOREM 3. For any positive integer \( n \), with \( 1 = x_1 > x_2 > \ldots > x_n > x_{n+1} > x_{n+2} = -1 \) the zeros of \( \omega_n(x) = (1 - x^2)T_n(x) \), there is in general no polynomial \( Q_{3n+3}(x) \) of degree \( \leq 3n + 3 \) such that, for arbitrary real numbers \( \{a_j\}_{n+2}^{j=1}, \{\beta_j\}_{n+2}^{j=1} \) and \( \{\gamma_j\}_{n+2}^{j=1} \) the conditions:

\[
Q_{3n+3}(x_j) = a_j; \quad j = 1, 2, \ldots, n + 1, n + 2, \quad (2.4)
\]

\[
Q_{3n+3}(x_j) = \beta_j; \quad j = 1, 2, \ldots, n + 1, n + 2 \quad (2.5)
\]

and

\[
Q_{3n+3}(x_j) = \gamma_j; \quad j = 2, 3, \ldots, n + 1 \quad (2.6)
\]

are satisfied. If there does exist such a polynomial then there are infinitely many of them.

REMARK 1. The comparison of our Theorem 2 with the above mentioned result of Saxena shows that if we do not prescribe the third derivative at \( \pm 1 \) then there does not exist a unique polynomial regardless whether \( n \) is even or odd. In an earlier work [3] we have shown that along with the conditions (2.1), (2.2) and (2.3) if we also prescribe the first derivative at \( \pm 1 \) a unique polynomial of degree \( \leq 3n + 3 \) still does not exist. It is also evident from Theorem 3 that even if we prescribe the first derivative at \( \pm 1 \) a unique polynomial of degree \( \leq 3n + 3 \) does not exist although the nodes of interpolation are different from that of [3].

REMARK 2. We shall give here the proof of Theorem 3 only. The proof of Theorem 2 can be obtained along the same lines.

PROOF OF THEOREM 3. We will show that if all of

\[
a_j = 0; \quad j = 1, 2, \ldots, n + 1, n + 2, \quad (2.7)
\]

\[
\beta_j = 0; \quad j = 1, 2, \ldots, n + 1, n + 2,
\]

\[
\gamma_j = 0; \quad j = 2, 3, \ldots, n + 1
\]
then there exists a polynomial \( Q_{3n+3}(x) \) of degree \( \leq 3n + 3 \) which is not identically zero, but satisfies (2.4), (2.5) and (2.6). The desired result then follows immediately from the theory of linear equations. From the definition of \( \omega_n(x) \) and conditions (2.4), (2.5) and (2.6), together with the requirements (2.7), it is clear that the desired polynomial must be of the form

\[
Q_{3n+3}(x) = (1 - x^2)^2 T_n(x) h_{n-1}(x)
\]

where \( h_{n-1}(x) \) is an unknown polynomial of degree \( \leq n - 1 \). Since we have also required \( Q_{3n+3}(x_j) = 0 \) for \( j = 2, 3, \ldots, n + 1 \), simple calculation provides

\[
(1 - x^2) h_{n-1}'(x) - 3x h_{n-1}(x) = c T_n(x)
\]

for unknown real constant \( c \). Letting \( x = \cos \theta \) and

\[
h_{n-1}(x) = \sum_{k=0}^{n-1} a_k \cos k\theta
\]

we obtain

\[
(1 - x^2) h_{n-1}'(x) = \sum_{k=1}^{n-1} a_k k \sin k\theta \sin \theta.
\]

Thus (2.9) becomes

\[
c \cos n\theta = \sum_{k=0}^{n-1} a_k [k \sin k\theta \sin \theta - 3 \cos k\theta \cos \theta].
\]

From this, we obtain on simplification

\[
2c \cos n\theta = \sum_{k=0}^{n-1} a_k [(k - 3) \cos(k - 1)\theta - (k + 3) \cos(k + 1)\theta],
\]

from which, by collecting the coefficients of \( \cos k\theta \), for \( k = 0, 1, \ldots, n \), we may write

\[
-2a_1 - (6a_0 + a_2) \cos \theta - 4a_1 \cos 2\theta + \sum_{k=3}^{n-2} [(k - 2)a_{k+1} - (k + 2)a_{k-1}] \cos k\theta
\]

\[
-(n + 1)a_{n-2} \cos(n - 1)\theta - (n + 2)a_{n-1} \cos n\theta = 2c \cos n\theta.
\]

This, in turn, leads to the following system of equations

\[
-2a_1 = 0
\]
\[
-(6a_0 + a_2) = 0,
\]
\[
-4a_1 = 0,
\]
\[
(k - 2)a_{k+1} - (k + 2)a_{k-1} = 0; k = 3, 4, \ldots, n - 2,
\]
\[
-(n + 1)a_{n-2} = 0,
\]
\[
-(n + 2)a_{n-1} = 2c.
\]

If \( n \) is even, then

\[
a_0 = a_2 = a_4 = \ldots = a_{n-2} = 0; a_1 = 0
\]
but
\[ a_{n-1-2j} = \frac{-2c}{n-2} \prod_{k=0}^{j} \left( \frac{n-2-2k}{n+2-2k} \right); \text{ for } j = 0, 1, \ldots, (n-4)/2 \]
is not necessarily zero.

If \( n \) is odd, then
\[ a_1 = a_3 = a_5 = \ldots = a_{n-2} = 0, \]
while
\[ a_{2j} = \frac{-2c}{n-2} \prod_{k=j}^{(n-1)/2} \left( \frac{2k-1}{2k+3} \right); \text{ for } j = 1, 2, \ldots, (n-1)/2 \]
with the special case
\[ a_0 = -a_2/6 \]
which are not necessarily zero. Hence regardless whether \( n \) is even or odd, in general, there does not exist a unique polynomial \( Q_{3n+3}(x) \) of degree \( \leq 3n+3 \) satisfying (2.4), (2.5) and (2.6) and there are infinitely many if they exist.

This completes the proof of Theorem 3. For a complete history on lacunary interpolation we refer to a paper by J. Balázs [4].

REFERENCES

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