We introduce and study a new class of generalized nonlinear variational-like inequalities. Under suitable conditions, we prove the existence of solutions for the class of generalized nonlinear variational-like inequalities. A new iterative algorithm for finding the approximate solutions of the generalized nonlinear variational-like inequality is given and the convergence of the algorithm is also proved. The results presented in this paper improve and generalize some results in recent literature.

1. Introduction

Variational-like inequalities are a useful and important generalization of variational inequalities [3, 8, 26]. They have potential and significant applications in optimization theory, structural analysis, and economics, see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. Some mixed variational-like inequalities have been studied by Parida and Sen [26], Tian [27], and Yao [29] by using the Berge maximum theorem in finite- and infinite-dimensional spaces. Huang and Deng [10] extended the auxiliary principle technique to study the existence of solutions for a class of generalized strongly nonlinear mixed variational-like inequalities. By using the minimax inequality technique, Ding [5, 6] studied some classes of nonlinear variational-like inequalities in reflexive Banach spaces.

The purpose of this paper is to introduce and study a new class of generalized nonlinear variational-like inequalities, which includes several kinds of variational-like inequalities as special cases. A few existence results of solutions for the generalized nonlinear variational-like inequality are established. We construct an iterative algorithm for finding the approximate solutions of the generalized nonlinear variational-like inequality and obtain the convergence of the algorithm under certain conditions.

2. Preliminaries

Let $H$ be a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let $K$ be a nonempty closed convex subset of $H$, let $A, C, F : K \rightarrow H$, $N : H \times H \rightarrow H$, and $\eta : K \times K \rightarrow H$ be mappings, and let $f : K \rightarrow (-\infty, \infty]$ be a real functional.
Suppose that \( a : H \times H \to (-\infty, \infty) \) is a coercive continuous bilinear form, that is, there exist positive constants \( c \) and \( d \) such that

(C1) \( a(v, v) \geq c\|v\|^2 \), for all \( v \in H \);

(C2) \( a(u, v) \leq d\|u\|\|v\| \), for all \( u, v \in H \).

Clearly, \( c \leq d \).

We consider the following generalized nonlinear variational-like inequality problem. Find \( u \in K \) such that

\[
a(u,v-u) + f(v) - f(u) \geq \langle (Nu,Cu) + Fu, \eta(v,u) \rangle, \quad \forall v \in K. \tag{2.1}
\]

**Special cases.**

(A) If \( N(Au,Cu) = Au - Cu, a(u,v) = 0 \) and \( Fu = 0 \) for all \( u, v \in K \), then the generalized nonlinear variational-like inequality problem (2.1) is equivalent to finding \( u \in K \) such that

\[
\langle Cu - Au, \eta(v,u) \rangle \geq f(u) - f(v), \quad \forall v \in K, \tag{2.2}
\]

which was introduced and studied by Ding [5].

(B) If \( N(Au,Cu) = Au - Cu, a(u,v) = gu - gv \) for all \( u, v \in K \), then the generalized nonlinear variational-like inequality problem (2.1) is equivalent to finding \( u \in K \) such that

\[
\langle Cu - Au, gv - gu \rangle \geq f(u) - f(v), \quad \forall v \in K, \tag{2.3}
\]

which was studied by Yao [29].

**Definition 2.1.** Let \( A, C : K \to H, N : H \times H \to H \) and \( \eta : K \times K \to H \) be mappings.

(1) \( A \) is said to be **Lipschitz continuous** with constant \( \alpha \) if there exists a constant \( \alpha > 0 \) such that

\[
\|Au - Av\| \leq \alpha\|u - v\|, \quad \forall u, v \in K. \tag{2.4}
\]

(2) \( N \) is said to be **Lipschitz continuous** with constant \( \beta \) in the first argument if there exists a constant \( \beta > 0 \) such that

\[
\|N(u,w) - N(v,w)\| \leq \beta\|u - v\|, \quad \forall u, v, w \in H. \tag{2.5}
\]

(3) \( N \) is said to be **\( \eta \)-antimonotone** with respect to \( A \) in the first argument if

\[
\langle N(Au,w) - N(Av,w), \eta(u,v) \rangle \leq 0, \quad \forall u, v, w \in K, w \in H. \tag{2.6}
\]

(4) \( N \) is said to be **\( \eta \)-relaxed Lipschitz** with constant \( \gamma \) with respect to \( C \) in the second argument if there exists a constant \( \gamma > 0 \) such that

\[
\langle N(w,Cu) - N(w,Cv), \eta(u,v) \rangle \leq -\gamma\|u - v\|^2, \quad \forall u, v \in K, w \in H. \tag{2.7}
\]

(5) \( \eta \) is said to be **Lipschitz continuous** with constant \( \delta \) if there exists a constant \( \delta > 0 \) such that

\[
\|\eta(u,v)\| \leq \delta\|u - v\|, \quad \forall u, v \in K. \tag{2.8}
\]
Similarly, we can define the Lipschitz continuity of $N$ in the second argument.

**Definition 2.2.** Let $K$ be a nonempty closed convex subset of a Hilbert space $H$ and $f : K \to (-\infty, \infty]$ be a real functional.

1. $f$ is said to be convex if for any $u, v \in K$ and for any $\alpha \in [0, 1]$,

$$f(\alpha u + (1 - \alpha)v) \leq \alpha f(u) + (1 - \alpha)f(v). \quad (2.9)$$

2. $f$ is said to be lower semicontinuous on $K$ if for each $\alpha \in (-\infty, \infty]$, the set $\{u \in K : f(u) \leq \alpha\}$ is closed in $K$.

**Lemma 2.3 [1, 2].** Let $X$ be a nonempty closed convex subset of a Hausdorff linear topological space $E$, and let $\phi, \psi : K \times K \to R$ be mappings satisfying the following conditions:

(a) $\psi(x, y) \leq \phi(x, y)$, for all $x, y \in X$, and $\psi(x, x) \geq 0$, for all $x \in X$;
(b) for each $x \in X$, $\phi(x, y)$ is upper semicontinuous with respect to $y$;
(c) for each $y \in X$, the set $\{x \in X : \psi(x, y) < 0\}$ is a convex set;
(d) there exists a nonempty compact set $K \subset X$ and $x_0 \in K$ such that $\psi(x_0, y) < 0$, for all $y \in X \setminus K$.

Then there exists $\hat{y} \in K$ such that $\phi(x, \hat{y}) \geq 0$, for all $x \in X$.

### 3. Existence theorems

In this section, we give four existence theorems of solutions for the generalized nonlinear variational-like inequality (2.1).

**Theorem 3.1.** Let $K$ be a nonempty closed convex subset of a Hilbert space $H$. Let $a : H \times H \to (-\infty, \infty)$ be a coercive continuous bilinear form with (C1) and (C2) and let $f : K \to (-\infty, \infty]$ be a proper convex lower semicontinuous functional with $\text{int}(\text{dom } f) \cap K \neq \emptyset$.

Suppose that $A, C : K \to H$ and $N : H \times H \to H$ are continuous mappings, $\eta : K \times K \to H$ is Lipschitz continuous with constant $\delta$, for each $v \in K$, $\eta(\cdot, v)$ is continuous and $\eta(v, u) = -\eta(u, v)$ for all $u, v \in K$. Assume that $N$ is $\eta$-antimonotone with respect to $A$ in the first argument and $\eta$-relaxed Lipschitz with constant $\xi$ with respect to $C$ in the second argument. Suppose that for given $x, y \in H$ and $v \in K$, the mapping $u \mapsto \langle N(x, y), \eta(u, v) \rangle$ is concave and upper semicontinuous. If $F : K \to H$ is completely continuous, then the generalized nonlinear variational-like inequality (2.1) has a solution $u \in K$.

**Proof.** We first prove that for each fixed $\hat{u} \in K$, there exists a unique $\hat{w} \in K$ such that

$$a(\hat{w}, v - \hat{w}) + f(v) - f(\hat{w}) \geq \langle N(A\hat{w}, C\hat{w}) + F\hat{u}, \eta(v, \hat{w}) \rangle, \quad \forall v \in K. \quad (3.1)$$

Let $\hat{u}$ be in $K$. Define the functionals $\phi$ and $\psi : K \times K \to R$ by

$$\phi(v, w) = a(v, v - w) + f(v) - f(w) - \langle N(Av, Cv) + F\hat{u}, \eta(v, w) \rangle,$$

$$\psi(v, w) = a(w, v - w) + f(v) - f(w) - \langle N(Aw, Cw) + F\hat{u}, \eta(v, w) \rangle \quad (3.2)$$

for all $v, w \in K$. 

We check that the functionals \( \phi \) and \( \psi \) satisfy all the conditions of Lemma 2.3 in the weak topology. It follows from the definitions of \( \phi \) and \( \psi \) that for all \( v, w \in K \),

\[
\phi(v, w) - \psi(v, w) = a(v - w, v - w) - \langle N(Av, Cv) - N(Aw, Cv), \eta(v, w) \rangle \\
- \langle N(Aw, Cv) - N(Aw, Cw), \eta(v, w) \rangle \\
\geq (c + \xi)\|v - w\|^2 \geq 0,
\]

which means that \( \phi \) and \( \psi \) satisfy the condition (a) of Lemma 2.3. Notice that \( f \) is a convex lower semicontinuous functional and for each \( x, y \in H, \ v \in K \), the mapping \( u \mapsto \langle N(x, y), \eta(u, v) \rangle \) is concave and upper semicontinuous. It follows that \( \phi(v, w) \) is weakly upper semicontinuous with respect to \( w \) and the set \( \{v \in K : \psi(v, w) < 0\} \) is convex for each \( w \in K \). Therefore, the conditions (b) and (c) of Lemma 2.3 hold. Since \( f \) is proper convex lower semicontinuous, for each \( v \in \text{int}(\text{dom } f) \), \( \partial f(v) \neq \emptyset \), see Ekeland and Temam [9]. Let \( v^* \) be in \( \text{int}(\text{dom } f) \cap K \). It follows that

\[
f(u) \geq f(v^*) + \langle r, u - v^* \rangle, \quad \forall r \in \partial f(v^*), \ u \in K.
\]

Put

\[
D = (c + \xi)^{-1}(\|r\| + \delta\|N(Av^*, Cv^*)\| + \delta\|F\hat{u}\|), \\
T = \{w \in K : \|w - v^*\| \leq D\}.
\]

Obviously, \( T \) is a weakly compact subset of \( K \) and for any \( w \in K \setminus T \),

\[
\psi(v^*, w) = a(w - v^*, v^* - w) + f(v^*) - f(w) - \langle N(Aw, Cw) + F\hat{u}, \eta(v^*, w) \rangle \\
\leq -a(w - v^*, w - v^*) - \langle r, w - v^* \rangle \\
+ \langle N(Aw, Cw) - N(Av^*, Cw), \eta(w, v^*) \rangle \\
+ \langle N(Av^*, Cw) - N(Av^*, Cv^*), \eta(w, v^*) \rangle \\
+ \langle N(Av^*, Cv^*), \eta(w, v^*) \rangle + \langle F\hat{u}, \eta(w, v^*) \rangle \\
\leq -\|w - v^*\|[(c + \xi)\|w - v^*\| - \|r\| - \delta\|N(Av^*, Cv^*)\| - \delta\|F\hat{u}\|] < 0,
\]

which yields that the condition (d) of Lemma 2.3 holds. Thus Lemma 2.3 ensures that there exists a \( \hat{w} \in K \) such that \( \phi(v, \hat{w}) \geq 0 \) for all \( v \in K \), that is,

\[
a(v, v - \hat{w}) + f(v) - f(\hat{w}) \geq \langle N(Av, Cv) + F\hat{u}, \eta(v, \hat{w}) \rangle, \quad \forall v \in K.
\]

Let \( t \) be in \((0, 1]\) and \( v \) be in \( K \). Replacing \( v \) by \( v_t = tv + (1 - t)\hat{w} \) in (3.7), we see that

\[
a(v_t, t(v - \hat{w})) + f(v_t) - f(\hat{w}) \geq \langle N(Av_t, Cv_t) + F\hat{u}, \eta(v_t, \hat{w}) \rangle, \quad \forall v \in K.
\]

Note that \( a \) is bilinear and \( f \) is convex. From (3.8) we deduce that

\[
t[a(v_t, v - \hat{w}) + f(v) - f(\hat{w})] \geq t\langle N(Av_t, Cv_t) + F\hat{u}, \eta(v, \hat{w}) \rangle, \quad \forall v \in K,
\]
which implies that
\[ a(v_t, v - \hat{w}) + f(v) - f(\hat{w}) \geq \langle N(Av_t, Cv_t) + F\hat{u}, \eta(v, \hat{w}) \rangle, \quad \forall v \in K. \quad (3.10) \]

Letting \( t \to 0^+ \) in the above inequality, we conclude that
\[ a(\hat{w}, v - \hat{w}) + f(v) - f(\hat{w}) \geq \langle N(A\hat{w}, C\hat{w}) + F\hat{u}, \eta(v, \hat{w}) \rangle, \quad \forall v \in K. \quad (3.11) \]

That is, \( \hat{w} \) is a solution of (3.1). Now we prove the uniqueness. For any two solutions \( w_1, w_2 \in K \) of (3.1), we know that
\[ a(w_1, w_2 - w_1) + f(w_2) - f(w_1) \geq \langle N(Aw_1, Cw_1) + F\hat{u}, \eta(w_2, w_1) \rangle, \]
\[ a(w_2, w_1 - w_2) + f(w_1) - f(w_2) \geq \langle N(Aw_2, Cw_2) + F\hat{u}, \eta(w_1, w_2) \rangle. \quad (3.12) \]

Adding these inequalities, we deduce that
\[ c \|w_1 - w_2\|^2 \leq a(w_1 - w_2, w_1 - w_2) \leq \langle N(Aw_1, Cw_1) - N(Aw_2, Cw_2), \eta(w_1, w_2) \rangle + \langle N(Aw_2, Cw_1) - N(Aw_2, Cw_2), \eta(w_1, w_2) \rangle \leq -\xi \|w_1 - w_2\|^2, \quad (3.13) \]

which yields that \( w_1 = w_2 \). That is, \( \hat{w} \) is a unique solution of (3.1). This means that there exists a mapping \( G : K \to K \) satisfying \( G(\tilde{u}) = \hat{w} \), where \( \hat{w} \) is the unique solution of (3.1) for each \( \tilde{u} \in K \).

Next we show that \( G \) is a completely continuous mapping. Let \( u_1 \) and \( u_2 \) be arbitrary elements in \( K \). Using (3.1), we get that
\[ a(Gu_1, Gu_2 - Gu_1) + f(Gu_2) - f(Gu_1) \geq \langle N(A(Gu_1), C(Gu_1)) + Fu_1, \eta(Gu_2, Gu_1) \rangle, \]
\[ a(Gu_2, Gu_1 - Gu_2) + f(Gu_1) - f(Gu_2) \geq \langle N(A(Gu_2), C(Gu_2)) + Fu_2, \eta(Gu_1, Gu_2) \rangle. \quad (3.14) \]

Adding (3.14), we arrive at
\[ c \|Gu_1 - Gu_2\|^2 \leq a(Gu_1 - Gu_2, Gu_1 - Gu_2) \leq \langle N(A(Gu_1), C(Gu_1)) - N(A(Gu_2), C(Gu_1)), \eta(Gu_1, Gu_2) \rangle + \langle N(A(Gu_2), C(Gu_1)) - N(A(Gu_2), C(Gu_2)), \eta(Gu_1, Gu_2) \rangle + \langle Fu_1 - Fu_2, \eta(Gu_1, Gu_2) \rangle \leq -\xi \|Gu_1 - Gu_2\|^2 + \delta \|Fu_1 - Fu_2\||Gu_1 - Gu_2||, \quad (3.15) \]

that is,
\[ \|Gu_1 - Gu_2\| \leq \frac{\delta}{c + \xi} \|Fu_1 - Fu_2\|. \quad (3.16) \]
Since $F$ is completely continuous, it follows from (3.16) that $G : K \to K$ is a completely continuous mapping. Hence the Schauder fixed point theorem guarantees that $G$ has a fixed point $u \in K$, which means that $u$ is a solution of the generalized nonlinear variational-like inequality (2.1). This completes the proof.

**Theorem 3.2.** Let $a$, $f$, $C$, $N$, $F$, and $\eta$ be as in Theorem 3.1 and let $N$ be Lipschitz continuous with constant $\xi$ in the first argument. Suppose that $A : K \to H$ is Lipschitz continuous with constant $\rho$. If $c + \xi > \delta \xi \rho$, then the generalized nonlinear variational-like inequality (2.1) has a unique solution $u \in K$.

**Proof.** Put

$$D = (c + \xi - \delta \xi \rho)^{-1}\left(\|r\| + \delta \|N(Av^*, Cv^*)\| + \delta \|F\|\right),$$

$$T = \{w \in K : \|w - v^*\| \leq D\}.$$

As in the proof of Theorem 3.1, we conclude that

$$\psi(v^*, w) \leq -a(w - v^*, w - v^*) - \langle r, w - v^* \rangle$$

$$+ \langle N(Aw, Cw) - N(Av^*, Cw), \eta(w, v^*) \rangle$$

$$+ \langle N(Av^*, Cw) - N(Av^*, Cv^*), \eta(w, v^*) \rangle$$

$$+ \langle N(Av^*, Cv^*), \eta(w, v^*) \rangle + \langle F\hat{u}, \eta(w, v^*) \rangle$$

$$\leq -\|w - v^*\|\|c + \xi - \delta \xi \rho\|\|w - v^*\|$$

$$- \|r\| - \delta \|N(Av^*, Cv^*)\| - \delta \|F\hat{u}\| < 0$$

for any $w \in K \setminus T$. The rest of the argument is now essentially the same as in the proof of Theorem 3.1 and therefore is omitted.

**Theorem 3.3.** Let $a$, $f$, $A$, $C$, $N$, and $\eta$ be as in Theorem 3.1. Suppose that $F : K \to H$ is Lipschitz continuous with constant $l$. If $\delta l/(c + \xi) < 1$, then the generalized nonlinear variational-like inequality (2.1) has a unique solution $u \in K$.

**Proof.** Let $u_1$ and $u_2$ be arbitrary elements in $K$. As in the proof of Theorem 3.1, we deduce that

$$\|Gu_1 - Gu_2\| \leq \frac{\delta}{c + \xi} \|Fu_1 - Fu_2\| \leq \frac{\delta l}{c + \xi} \|u_1 - u_2\|, \quad \forall u_1, u_2 \in K,$$  

which yields that $G : K \to K$ is a contraction mapping and hence it has a unique fixed point $u \in K$, which is a unique solution of the generalized nonlinear variational-like inequality (2.1). This completes the proof.

The following theorem follows from the arguments of Theorems 3.1, 3.2 and, 3.3.

**Theorem 3.4.** Let $a$, $f$, $A$, $C$, $N$, and $\eta$ be as in Theorem 3.2. Suppose that $F : K \to H$ is Lipschitz continuous with constant $l$. If $0 < \delta l/(c + \xi - \delta \xi \rho) < 1$, then the generalized nonlinear variational-like inequality (2.1) has a unique solution $u \in K$. 
4. Algorithm and convergence theorems

Based on Theorem 3.1, we suggest the following iterative algorithm.

**Algorithm 4.1.** Let \( A, C, F : K \to H, N : H \times H \to H, \) and \( \eta : K \times K \to H \) be mappings, and let \( f : K \to (-\infty, \infty] \) be a real functional. For any given \( u_0 \in K \), computes sequence \( \{u_n\}_{n \geq 0} \) by the iterative scheme

\[
a(u_{n+1}, v - u_{n+1}) + f(v) - f(u_{n+1}) \geq \langle N(Au_{n+1}, Cu_{n+1}) + Fu_n, \eta(v, u_{n+1}) \rangle, \quad (4.1)
\]

for all \( v \in K \) and \( n \geq 0 \).

**Theorem 4.2.** Let \( a, f, F, N, A, C, \) and \( \eta \) be as in Theorem 3.3. If \( \delta l/(c + \xi) < 1 \), then the generalized nonlinear variational-like inequality (2.1) possesses a unique solution and the iterative sequence \( \{u_n\}_{n \geq 0} \) generated by Algorithm 4.1 converges strongly to the unique solution.

**Proof.** Using Algorithm 4.1, we obtain that

\[
a(u_{n+1}, u_n - u_{n+1}) + f(u_n) - f(u_{n+1}) \geq \langle N(Au_{n+1}, Cu_{n+1}) + Fu_n, \eta(u_n, u_{n+1}) \rangle, \quad (4.2)
\]

for all \( n \geq 1 \). Adding (4.2), we get that

\[
c\|u_{n+1} - u_n\|^2 \leq a(u_{n+1} - u_n, u_{n+1} - u_n) \\
\leq \langle N(Au_{n+1}, Cu_{n+1}) - N(Au_n, Cu_n), \eta(u_{n+1}, u_n) \rangle \\
+ \langle N(Au_n, Cu_n) - N(Au_{n-1}, Cu_{n-1}), \eta(u_{n+1}, u_n) \rangle \\
+ \langle Fu_n - Fu_{n-1}, \eta(u_{n+1}, u_n) \rangle \\
\leq -\xi\|u_{n+1} - u_n\|^2 + \delta l\|u_n - u_{n-1}\|\|u_{n+1} - u_n\|,
\]

that is,

\[
\|u_{n+1} - u_n\| \leq \frac{\delta l}{c + \xi} \|u_n - u_{n-1}\|, \quad \forall n \geq 1, \quad (4.4)
\]

which yields that \( \{u_n\}_{n \geq 0} \) is a Cauchy sequence by \( \delta l/(c + \xi) < 1 \). Consequently, \( \{u_n\}_{n \geq 0} \) converges to some element \( u \) in \( K \). Letting \( n \to \infty \) in (4.1), we infer that

\[
a(u, v - u) + f(v) - f(u) \geq \langle N(Au, Cu) + Fu, \eta(v, u) \rangle, \quad \forall v \in K. \quad (4.5)
\]

Hence \( u \) is a solution of the generalized nonlinear variational-like inequality (2.1). It follows from Theorem 3.3 that \( u \) is the unique solution of the generalized nonlinear variational-like inequality (2.1). This completes the proof. \( \square \)
Similarly we have the following result.

**Theorem 4.3.** Let $a, f, F, N, A, C,$ and $\eta$ be as in Theorem 3.4. If $0 < \delta l/(c + \xi - \delta \zeta \rho) < 1$, then the generalized nonlinear variational-like inequality (2.1) possesses a unique solution and the iterative sequence $\{u_n\}_{n \geq 0}$ generated by Algorithm 4.1 converges strongly to the unique solution.

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