Chan (2004) considered a certain continued fraction expansion and the corresponding Gauss-Kuzmin-Lévy problem. A Wirsing-type approach to the Perron-Frobenius operator of the associated transformation under its invariant measure allows us to obtain a near-optimal solution to this problem.

1. Introduction

The Gauss 1812 problem gave rise to an extended literature. In modern times, the so-called Gauss-Kuzmin-Lévy theorem is still one of the most important results in the metrical theory of regular continued fractions (RCFs). A recent survey of this topic is to be found in [10]. From the time of Gauss, a great number of such theorems followed. See, for example, [2, 6, 7, 8, 18].

Apart from the RCF expansion there are many other continued fraction expansions: the continued fraction expansion to the nearest integer, grotesque expansion, Nakada’s α-expansions, Rosen expansions; in fact, there are too many to mention (see [4, 5, 11, 12, 13, 16, 17] for some background information). The Gauss-Kuzmin-Lévy problem has been generalized to the above continued fraction expansions (see [3, 14, 15, 19, 20, 21]).

Taking up a problem raised in [1], we consider another expansion of reals in the unit interval, different from the RCF expansion. In fact, in [1] Chan has studied the transformation related to this new continued fraction expansion and the asymptotic behaviour of its distribution function. Giving a solution to the Gauss-Kuzmin-Lévy problem, he showed in [1, Theorem 1] that the convergence rate involved is $O(q^n)$ as $n \rightarrow \infty$ with $0 < q < 1$. This unsurprising result can be easily obtained from well-known general results (see [9, pages 202 and 262–266] and [10, Section 2.1.2]) concerning the Perron-Frobenius operator of the transformation under the invariant measure induced by the limit distribution function.

Our aim here is to give a better estimation of the convergence rate discussed. First, in Section 2 we introduce equivalent, but much more concise and rigorous expressions than in [1] of the transformation involved and of the related incomplete quotients. Next, in Section 3, our strategy is to derive the Perron-Frobenius operator of this transformation. 
under its invariant measure. In Section 4, we use a Wirsing-type approach (see [22]) to study the optimality of the convergence rate. Actually, in Theorem 4.3 of Section 4 we obtain upper and lower bounds of the convergence rate which provide a near-optimal solution to the Gauss-Kuzmin-Lévy problem.

2. Another expansion of reals in the unit interval

In this section we describe another continued fraction expansion different from the regular continued fraction expansion for a number $x$ in the unit interval $I = [0,1]$, which has been actually considered in [1].

Define for any $x \in I$ the transformation

$$\tau(x) = 2^{[(\log x^{-1})/\log 2]} - 1, \quad x \neq 0; \quad \tau(0) = 0,$$

where $\{u\}$ denotes the fractional part of a real $u$ while log stands for natural logarithm. (Nevertheless, the definition of $\tau$ is independent of the base of the logarithm used.) Putting

$$a_n(x) = a_1(\tau^{n-1}(x)), \quad n \in \mathbb{N}_+ = \{1,2,\ldots\},$$

with $\tau^0(x) = x$ the identity map and

$$a_1(x) = \left[ \frac{(\log x^{-1})}{\log 2} \right],$$

where $[u]$ denotes the integer part of a real $u$, one easily sees that every irrational $x \in (0,1)$ has a unique infinite expansion

$$x = \frac{2^{-a_1}}{2^{-a_2}} = \left[ a_1, a_2, \ldots \right].$$

Here, the incomplete quotients or digits $a_n(x), n \in \mathbb{N}_+$ of $x \in (0,1)$ are natural numbers.

Let $\mathcal{B}_I$ be the $\sigma$-algebra of Borel subsets of $I$. There is a probability measure $\nu$ on $\mathcal{B}_I$ defined by

$$\nu(A) = \frac{1}{\log(4/3)} \int_A \frac{dx}{(x+1)(x+2)}, \quad A \in \mathcal{B}_I,$$

such that $\nu(\tau^{-1}(A)) = \nu(A)$ for any $A \in \mathcal{B}_I$, that is, $\nu$ is $\tau$-invariant.

3. An operator treatment

In the sequel we will derive the Perron-Frobenius operator of $\tau$ under the invariant measure $\nu$.

Let $\mu$ be a probability measure on $\mathcal{B}_I$ such that $\mu(\tau^{-1}(A)) = 0$ whenever $\mu(A) = 0, A \in \mathcal{B}_I$, where $\tau$ is the continued fraction transformation defined in Section 2. In particular,
this condition is satisfied if $\tau$ is $\mu$-preserving, that is, $\mu \tau^{-1} = \mu$. It is known from [10, Section 2.1] that the Perron-Frobenius operator $P_\mu$ of $\tau$ under $\mu$ is defined as the bounded linear operator on $L^1_\mu = \{ f : I \to \mathbb{C} \mid \int_I |f| d\mu < \infty \}$ which takes $f \in L^1_\mu$ into $P_\mu f \in L^1_\mu$ with

$$\int_A P_\mu f d\mu = \int_{\tau^{-1}(A)} f d\mu, \quad A \in \mathcal{B}_I. \quad (3.1)$$

In particular the Perron-Frobenius operator $P_\lambda$ of $\tau$ under the Lebesgue measure $\lambda$ is

$$P_\lambda(x) = \frac{d}{dx} \int_{\tau^{-1}(0,x)} f d\lambda \quad \text{a.e. in } I. \quad (3.2)$$

**Proposition 3.1.** The Perron-Frobenius operator $P_\nu = U$ of $\tau$ under $\nu$ is given a.e. in $I$ by the equation

$$U f(x) = \sum_{k \in \mathbb{N}} p_k(x) f(u_k(x)), \quad f \in L^1_\nu, \quad (3.3)$$

where

$$p_k(x) = \frac{\gamma^{k+1}(x+1)(x+2)}{(\gamma^k x + 1) (\gamma^{k+1} x + 1)}, \quad x \in I, \quad (3.4)$$

$$u_k(x) = \frac{\gamma^k}{x+1}, \quad x \in I,$$

with $\gamma = 1/2$.

The proof is entirely similar to that of [10, Proposition 2.1.2].

An analogous result to [10, Proposition 2.1.5] is shown as follows.

**Proposition 3.2.** Let $\mu$ be a probability measure on $\mathcal{B}_I$. Assume that $\mu \ll \lambda$ and let $h = \frac{d\mu}{d\lambda}$. Then

$$\mu(\tau^{-n}(A)) = \int_A \frac{U^n f(x)}{(x+1)(x+2)} dx \quad (3.5)$$

for any $n \in \mathbb{N}$ and $A \in \mathcal{B}_I$, where $f(x) = (x+1)(x+2) h(x), \ x \in I$.

### 4. A Wirsing-type approach

Let $\mu$ be a probability measure on $\mathcal{B}_I$ such that $\mu \ll \lambda$. For any $n \in \mathbb{N}$, put

$$F_n(x) = \mu(\tau^n < x), \quad x \in I, \quad (4.1)$$

where $\tau^0$ is the identity map. As $(\tau^n < x) = \tau^{-n}((0,x))$, by Proposition 3.2 we have

$$F_n(x) = \int_0^x \frac{U^n f_0(u)}{(u+1)(u+2)} du, \quad n \in \mathbb{N}, \ x \in I, \quad (4.2)$$

with $f_0(x) = (x+1)(x+2) F'_0(x), \ x \in I$, where $F'_0 = \frac{d\mu}{d\lambda}$. 
In this section we will assume that $F_0' \in C^1(I)$. So, we study the behaviour of $U^n$ as $n \to \infty$, assuming that the domain of $U$ is $C^1(I)$, the collection of all functions $f : I \to \mathbb{C}$ which have a continuous derivative.

Let $f \in C^1(I)$. Then the series (3.3) can be differentiated term-by-term, since the series of derivatives is uniformly convergent. Putting $\Delta_k = y^k - y^{2k}$, $k \in \mathbb{N}$ we get

$$p_k(x) = y^{k+1} + \frac{\Delta_k}{y^{k+1} + x + 1} - \frac{\Delta_{k+1}}{y^{k+2} + x + 1},$$

$$(Uf)'(x) = \sum_{k \in \mathbb{N}} \left[ p'_k(x)f\left(\frac{y^k}{x + 1}\right) - p_k(x)\frac{y^k}{(x + 1)^2}f'\left(\frac{y^k}{x + 1}\right) \right]$$

$$= \sum_{k \in \mathbb{N}} \left[ \frac{\Delta_{k+1}}{(y^{k+1} + x + 1)^2} - \frac{\Delta_k}{(y^{k+1} + x + 1)^2} \right] f\left(\frac{y^k}{x + 1}\right) - p_k(x)\frac{y^k}{(x + 1)^2}f'\left(\frac{y^k}{x + 1}\right)$$

$$= -\sum_{k \in \mathbb{N}} \left[ \frac{\Delta_{k+1}}{(y^{k+1} + x + 1)^2} \left( f\left(\frac{y^{k+1}}{x + 1}\right) - f\left(\frac{y^k}{x + 1}\right) \right) + p_k(x)\frac{y^k}{(x + 1)^2}f'\left(\frac{y^k}{x + 1}\right) \right],$$

$x \in I$. Thus, we can write

$$(Uf)' = -Vf', \quad f \in C^1(I),$$

where $V : C(I) \to C(I)$ is defined by

$$Vg(x) = \sum_{k \in \mathbb{N}} \left[ \frac{\Delta_{k+1}}{(y^{k+1} + x + 1)^2} \int_{\gamma_k/(x + 1)}^{y^{k+1}/(x + 1)} g(u)du + p_k(x)\frac{y^k}{(x + 1)^2}g\left(\frac{y^k}{x + 1}\right) \right],$$

$g \in C(I), x \in I$. Clearly,

$$(U^n f)' = (-1)^nV^n f', \quad n \in \mathbb{N}_+, \; f \in C^1(I).$$

We are going to show that $V^n$ takes certain functions into functions with very small values when $n \in \mathbb{N}_+$ is large.

**Proposition 4.1.** There are positive constants $v > 0.206968896$ and $w < 0.209364308$, and a real-valued function $\varphi \in C(I)$ such that $v\varphi \leq V\varphi \leq w\varphi$.

**Proof.** Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a continuous bounded function such that $\lim_{x \to \infty} h(x) < \infty$. We look for a function $g : (0,1] \to \mathbb{R}$ such that $Ug = h$, assuming that the equation

$$Ug(x) = \sum_{k \in \mathbb{N}} p_k(x)g\left(\frac{y^k}{x + 1}\right) = h(x)$$

holds for $x \in \mathbb{R}_+$. Then (4.7) yields

$$\frac{h(x)}{x + 2} - \frac{h(2x + 1)}{2x + 3} = \frac{x + 1}{(x + 2)(2x + 3)}g\left(\frac{1}{x + 1}\right), \quad x \in \mathbb{R}_+. \quad (4.8)$$
Hence
\[ g(u) = (u+2)h\left(\frac{1}{u}-1\right) - (u+1)h\left(\frac{2}{u}-1\right), \quad u \in (0,1], \quad (4.9) \]
and we indeed have \( U_g = h \) since
\[ U_g(x) = \sum_{k \in \mathbb{N}} p_k(x) \left[ \left( \frac{y^k}{y^k+1} + 2 \right) h\left( \frac{x+1}{y^k} - 1 \right) - \left( \frac{y^k+1}{y^k+1+1} \right) h\left( \frac{x+1}{y^k+1} - 1 \right) \right] \]
\[ = \left( \frac{x+2}{2} \right) \sum_{k \in \mathbb{N}} \frac{y^{2k}}{(y^k+x+1)(y^{k+1}+x+1)} \]
\[ \times \left[ \left( \frac{x+1}{y^k+1} + 1 \right) h\left( \frac{x+1}{y^k} - 1 \right) - \left( \frac{x+1}{y^k+1} \right) h\left( \frac{x+1}{y^k+1} - 1 \right) \right] \]
\[ = h(x), \quad x \in \mathbb{R}_+. \quad (4.10) \]

In particular, for any fixed \( a \in I \) we consider the function \( h_a : \mathbb{R}_+ \to \mathbb{R} \) defined by
\[ h_a(x) = 1/(x+a+1), \quad x \in \mathbb{R}_+. \] By the above, the function \( g_a : (0,1] \to \mathbb{R} \) defined as
\[ g_a(x) = (x+2)h_a\left(\frac{1}{x}-1\right) - (x+1)h_a\left(\frac{2}{x}-1\right) \]
\[ = \frac{x(x+2)}{ax+1} - \frac{x(x+1)}{ax+2}, \quad x \in (0,1], \quad (4.11) \]
satisfies \( U_{g_a}(x) = h_a(x), \quad x \in I. \) Setting
\[ \varphi_a(x) = g'_a(x) = \frac{3ax^2 + 4(a+1)x + 6}{(ax+2)^2(ax+1)^2}, \quad (4.12) \]
we have
\[ V \varphi_a(x) = -(U_{g_a})'(x) = \frac{1}{(x+a+1)^2}, \quad x \in I. \quad (4.13) \]

We choose \( a \) by asking that \( (\varphi_a/V \varphi_a)(0) = (\varphi_a/V \varphi_a)(1). \) This amounts to \( 3a^4 + 12a^3 + 18a^2 - 2a - 17 = 0 \) which yields as unique acceptable solution \( a = 0.794741181… \) For this value of \( a, \) the function \( \varphi_a/V \varphi_a \) attains its maximum equal to \( (3/2)(a+1)^2 = 4.83164386… \) at \( x = 0 \) and \( x = 1, \) and has a minimum \( m(a) \approx (\varphi_a/V \varphi_a)(0.39) = 4.776363306… \) It follows that for \( \varphi = \varphi_a \) with \( a = 0.794741181…, \) we have
\[ \frac{2\varphi}{3(a+1)^2} \leq V \varphi \leq \frac{\varphi}{m(a)}, \quad (4.14) \]
that is, \( v \varphi \leq V \varphi \leq w \varphi, \) where \( v = 2/3(a+1)^2 > 0.206968896, \) and \( w = 1/m(a) < 0.209364308. \)

**Corollary 4.2.** Let \( f_0 \in C^1(I) \) such that \( f'_0 > 0. \) Put \( \alpha = \min_{x \in I} \varphi(x)/f'_0(x) \) and \( \beta = \max_{x \in I} \varphi(x)/f'_0(x). \) Then
\[ \frac{\alpha}{\beta} v^n f'_0 \leq V^n f'_0 \leq \frac{\beta}{\alpha} w^n f'_0, \quad n \in \mathbb{N}_+. \]
Since $V$ is a positive operator, we have
\[ v^n \varphi \leq V^n \varphi \leq w^n \varphi, \quad n \in \mathbb{N}. \] (4.16)

Noting that $\alpha f_0' \leq \varphi \leq \beta f_0'$, we can write
\[
\frac{\alpha}{\beta} v^n f_0' \leq \frac{1}{\beta} v^n \varphi \leq \frac{1}{\beta} V^n \varphi \leq \frac{1}{\alpha} V^n \varphi \leq \frac{\beta}{\alpha} w^n f_0',
\] (4.17)
n $\in \mathbb{N}$, which shows that (4.15) holds.

**Theorem 4.3** (near-optimal solution to Gauss-Kuzmin-Lévy problem). Let $f_0 \in C^1(I)$ such that $f_0' > 0$. For any $n \in \mathbb{N}$ and $x \in I$,
\[
\frac{(\log(4/3))^2 \alpha \min_{x \in I} f_0'(x)}{2\beta} v^n F(x) (1 - F(x)) \leq |\mu(\tau^n < x) - F(x)| \leq \frac{(\log(4/3))^2 \beta \max_{x \in I} f_0'(x)}{\alpha} w^n F(x) (1 - F(x)),
\] (4.18)
where $\alpha, \beta, v$ and $w$ are defined in Proposition 4.1 and Corollary 4.2 and $F(x) = (1/\log(4/3)) \log(2(x+1))/x+2$. In particular, for any $n \in \mathbb{N}$ and $x \in I$,
\[
0.01023923 v^n F(x) (1 - F(x)) \leq |\lambda(\tau^n < x) - F(x)| \leq 0.334467468 w^n F(x) (1 - F(x)).
\] (4.19)

**Proof.** For any $n \in \mathbb{N}$ and $x \in I$, set $d_n(F(x)) = \mu(\tau^n < x) - F(x)$. Then by (4.2) we have
\[
d_n(F(x)) = \int_0^x U^n f_0(u) \frac{u}{(u+1)(u+2)} \, du - F(x).
\] (4.20)

Differentiating twice with respect to $x$ yields
\[
d_n''(F(x)) = \frac{1}{(\log(4/3))(x+1)(x+2)} U^n f_0(x) \frac{1}{(x+1)(x+2)} - \frac{1}{(\log(4/3))(x+1)(x+2)},
\] (4.21)

\[
(U^n f_0(x))' = \frac{1}{(\log(4/3))^2} \frac{d_n''(F(x))}{(x+1)(x+2)}, \quad n \in \mathbb{N}, \ x \in I.
\]

Hence by (4.6) we have
\[
d_n''(F(x)) = (-1)^n \left( \log \left( \frac{4}{3} \right) \right)^2 (x+1)(x+2) V^n f_0'(x), \quad n \in \mathbb{N}, \ x \in I.
\] (4.22)

Since $d_n(0) = d_n(1) = 0$, it follows from a well-known interpolation formula that
\[
d_n(x) = - \frac{x(1-x)}{2} d_n''(\theta), \quad n \in \mathbb{N}, \ x \in I
\] (4.23)
for a suitable $\theta = \theta(n,x) \in I$. Therefore
\[
\mu(\tau^n < x) - F(x) = (-1)^{n+1} \left( \log \left( \frac{4}{3} \right) \right)^2 \frac{\theta + 1}{2} V^n f'_0(\theta) F(x) (1 - F(x)) \tag{4.24}
\]
for any $n \in \mathbb{N}$ and $x \in I$, and another suitable $\theta = \theta(n,x) \in I$. The result stated follows now from Corollary 4.2. In the special case $\mu = \lambda$, we have $f_0(x) = (x+1)(x+2)$, $x \in I$. Then with $a = 0.794741181\ldots$, we have
\[
\alpha = \min_{x \in I} \frac{\varphi(x)}{f'_0(x)} = \frac{7a + 10}{5(a+2)^2(a+1)^2} = 0.123720515\ldots, \\
\beta = \max_{x \in I} \frac{\varphi(x)}{f'_0(x)} = 0.5, \tag{4.25}
\]
so that $(\log(4/3))^2 \alpha/2\beta = 0.01023923\ldots$ and $(\log 4/3)^2 \beta/\alpha = 0.334467468\ldots$. The proof is complete. \qed

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**References**


A Wirsing-type approach to some continued fraction


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