ON HORIZONTAL AND COMPLETE LIFTS FROM A MANIFOLD WITH $f_{\lambda}(7,1)$-STRUCTURE TO ITS COTANGENT BUNDLE

LOVEJOY S. DAS, RAM NIVAS, AND VIRENDRA NATH PATHAK

Received 1 August 2003

The horizontal and complete lifts from a manifold $M^n$ to its cotangent bundles $T^*(M^n)$ were studied by Yano and Ishihara, Yano and Patterson, Nivas and Gupta, Dambrowski, and many others. The purpose of this paper is to use certain methods by which $f_{\lambda}(7,1)$-structure in $M^n$ can be extended to $T^*(M^n)$. In particular, we have studied horizontal and complete lifts of $f_{\lambda}(7,1)$-structure from a manifold to its cotangent bundle.

1. Introduction

Let $M$ be a differentiable manifold of class $c^\infty$ and of dimension $n$ and let $C_{TM}$ denote the cotangent bundle of $M$. Then $C_{TM}$ is also a differentiable manifold of class $c^\infty$ and dimension $2n$.

The following are notations and conventions that will be used in this paper.

(1) $\mathcal{S}_r^s(M)$ denotes the set of tensor fields of class $c^\infty$ and of type $(r,s)$ on $M$. Similarly, $\mathcal{S}_r^s(C_{TM})$ denotes the set of such tensor fields in $C_{TM}$.

(2) The map $\Pi$ is the projection map of $C_{TM}$ onto $M$.

(3) Vector fields in $M$ are denoted by $X, Y, Z, \ldots$ and Lie differentiation by $L_X$. The Lie product of vector fields $X$ and $Y$ is denoted by $[X, Y]$.

(4) Suffixes $a, b, c, \ldots, h, i, j, \ldots$ take the values 1 to $n$ and $\tilde{i} = i + n$. Suffixes $A, B, C, \ldots$ take the values 1 to $2n$.

If $A$ is a point in $M$, then $\Pi^{-1}(A)$ is fiber over $A$. Any point $p \in \Pi^{-1}(A)$ is denoted by the ordered pair $(A, p_A)$, where $p$ is 1-form in $M$ and $p_A$ is the value of $p$ at $A$. Let $U$ be a coordinate neighborhood in $M$ such that $A \in U$. Then $U$ induces a coordinate neighborhood $\Pi^{-1}(U)$ in $C_{TM}$ and $p \in \Pi^{-1}(U)$.

2. Complete lift of $f_{\lambda}(7,1)$-structure

Let $f(\neq 0)$ be a tensor field of type $(1,1)$ and class $c^\infty$ on $M$ such that

$$f^2 + \lambda^2 f = 0, \quad (2.1)$$
Horizontal and complete lifts of $f_\lambda(7,1)$-structure

where $\lambda$ is any complex number not equal to zero. We call the manifold $M$ satisfying (2.1) as $f_\lambda(7,1)$-structure manifold. Let $f^i_j$ be components of $f$ at $A$ in the coordinate neighborhood $U$ of $M$. Then the complete lift $f^C$ of $f$ is also a tensor field of type $(1,1)$ in $\mathcal{C}_TM$ whose components $\tilde{f}^A_B$ in $\Pi^{-1}(U)$ are given by [2]

\[
\tilde{f}^i_j = f^i_j, \quad \tilde{f}^\lambda_i = 0,
\]

where

\[
\frac{\partial f^a_j}{\partial x^i} \frac{\partial f^a_i}{\partial x^h} = \tilde{f}^\lambda_i = f^i_h,
\]

where $(x^1, x^2, \ldots, x^n)$ are coordinates of $A$ relative to $U$ and $p_A$ has a component $(p_1, p_2, \ldots, p_n)$.

Thus we can write

\[
f^C = \begin{pmatrix} f^i_j & 0 \\ p_a(\partial_i f^a_j - \partial_j f^a_i) & f^i_h \end{pmatrix},
\]

where $\partial_i = \partial/\partial x^i$.

If we put

\[
\partial_i f^a_j - \partial_j f^a_i = 2\partial[if^a_j],
\]

then we can write (2.4) in the form

\[
f^C = \begin{pmatrix} f^i_j & 0 \\ 2p_a\partial[if^a_j] & f^i_h \end{pmatrix}.
\]

Thus we have

\[
(f^C)^2 = \begin{pmatrix} f^i_j & 0 \\ 2p_a\partial[if^a_j] & f^i_h \end{pmatrix} \begin{pmatrix} f^i_j & 0 \\ 2p_a\partial[if^a_j] & f^i_h \end{pmatrix},
\]

or

\[
(f^C)^2 = \begin{pmatrix} f^i_j f^j_i & 0 \\ 2p_a f^i_j \partial[if^a_j] + 2p_j f^i_j \partial[jf^i_j] & f^i_h f^i_h \end{pmatrix}.
\]

If we put

\[
2p_a f^i_j \partial[if^a_j] + 2p_j f^i_j \partial[jf^i_j] = L_{ij},
\]

then (2.8) takes the form

\[
(f^C)^2 = \begin{pmatrix} f^i_j f^j_i & 0 \\ L_{ij} & f^i_h f^i_h \end{pmatrix}.
\]
Thus we have

\[(f^C)^4 = \begin{pmatrix} f_i^h f_j^l & 0 \\ L_{hj} & f_i^l f_j^k \end{pmatrix} \begin{pmatrix} f_k^l f_i^h \\ L_{jl} & f_k^l f_j^k \end{pmatrix},\] (2.11)

or

\[(f^C)^4 = \begin{pmatrix} f_i^h f_j^l f_k^l f_i^k \\ f_k^l f_i^k L_{hj} + f_i^l f_j^l L_{jl} & f_k^l f_j^l f_i^l f_i^k \end{pmatrix} \begin{pmatrix} f_i^h f_j^l f_k^l f_i^k \\ f_k^l f_i^k L_{hj} + f_i^l f_j^l L_{jl} & f_k^l f_j^l f_i^l f_i^k \end{pmatrix},\] (2.12)

Putting again

\[f_k^l f_j^l L_{hj} + f_i^l f_j^l L_{jl} = P_{hl},\] (2.13)

then we can put (2.12) in the form

\[(f^C)^4 = \begin{pmatrix} f_i^h f_j^l f_k^l f_i^k \\ P_{hl} & f_i^l f_j^l f_i^k f_i^h \end{pmatrix}.\] (2.14)

Thus,

\[(f^C)^6 = \begin{pmatrix} f_i^h f_j^l f_k^l f_i^k \\ P_{hl} & f_k^l f_j^l f_i^l f_i^h \end{pmatrix} \begin{pmatrix} f_m f_n^m & 0 \\ L_{ln} & f_m f_l^n \end{pmatrix},\] (2.15)

\[(f^C)^6 = \begin{pmatrix} f_i^h f_j^l f_k^l f_i^k \\ P_{hl} f_m f_n^m + L_{ln} f_k^l f_j^l f_i^l f_i^h & f_m f_l^n \end{pmatrix} \begin{pmatrix} f_i^l f_j^l f_k^l f_i^k \\ f_m f_l^n \end{pmatrix}.\] (2.16)

Putting again

\[P_{hl} f_m f_n^m + L_{ln} f_k^l f_j^l f_i^l f_i^h = Q_{hn},\] (2.17)

then (2.16) takes the form

\[(f^C)^6 = \begin{pmatrix} f_i^h f_j^l f_k^l f_i^k \\ Q_{hn} & f_m f_l^n \end{pmatrix} \begin{pmatrix} f_i^l f_j^l f_k^l f_i^k \\ f_m f_l^n \end{pmatrix}.\] (2.18)
Thus, 
\[ (f_C)^7 = \left( \begin{array}{cc} f_i^h f_j^l f_k^j f_m^n f_n^m f_p^m f_p^n \\ Q_hn \\
 f_m^m f_l^m f_j^i f_i^j f_i^h \\
 f_i^h f_j^l f_k^j f_m^m f_p^n \\
 f_p^n f_m^m f_l^m f_j^i f_i^j f_i^h \end{array} \right) \left( \begin{array}{cc} 0 \\ 2p_r \partial[p f_n^r] \\
 f_p^p \\
 0 \\
 f_p^p f_m^m f_l^m f_j^i f_i^j f_i^h \end{array} \right) \], (2.19)

\[ (f_C)^7 = \left( \begin{array}{cc} f_i^h f_j^l f_k^j f_m^m f_p^n \\ Q_hn f_p^n + 2p_r \partial[p f_n^r] f_m^m f_l^m f_j^i f_i^j f_i^h \\
 f_p^n f_m^m f_l^m f_j^i f_i^j f_i^h \end{array} \right). \] (2.20)

In view of (2.1), we have
\[ f_i^h f_j^l f_k^j f_m^m f_p^n = -\lambda^2 f_p^h, \] (2.21)

and also putting
\[ Q_hn f_p^n + 2p_r \partial[p f_n^r] f_m^m f_l^m f_j^i f_i^j f_i^h = -\lambda^2 p_s \partial[p f_s^h], \] (2.22)

then (2.20) can be given by
\[ (f_C)^7 = \left( \begin{array}{cc} -\lambda^2 f_p^n \\ -\lambda^2 p_r \partial[p f_n^r] \\
 -\lambda^2 f_p^p \end{array} \right). \] (2.23)

In view of (2.6) and (2.23), it follows that
\[ (f_C)^7 + \lambda^2 (f_C) = 0. \] (2.24)

Hence the complete lift \( f^C \) of \( f \) admits an \( f_\lambda(7,1) \)-structure in the cotangent bundle \( C_{TM} \).

Thus we have the following theorem.

**Theorem 2.1.** In order that the complete lift of \( f^C \) of a (1,1) tensor field \( f \) admitting \( f_\lambda(7,1) \)-structure in \( M \) may have the similar structure in the cotangent bundle \( C_{TM} \), it is necessary and sufficient that
\[ Q_hn f_p^n + 2p_r \partial[p f_n^r] f_m^m f_l^m f_j^i f_i^j f_i^h = -\lambda^2 p_s \partial[p f_s^h]. \] (2.25)

3. Horizontal lift of \( f_\lambda(7,1) \)-structure

Let \( f, g \) be two tensor fields of type (1,1) on the manifold \( M \). If \( f^H \) denotes the horizontal lift of \( f \), we have
\[ f^H g^H + g^H f^H = (f g + g f)^H. \] (3.1)

Taking \( f \) and \( g \) identical, we get
\[ (f^H)^2 = (f^2)^H. \] (3.2)
Multiplying both sides by $f^H$ and making use of the same (3.2), we get
\[
(f^H)^3 = (f^3)^H
\]
and so on. Thus it follows that
\[
(f^H)^4 = (f^4)^H, \quad (f^H)^5 = (f^5)^H,
\]
and so on. Thus,
\[
(f^H)^7 = (f^7)^H.
\]
Since $f$ gives on $M$ the $f_\lambda(7,1)$-structure, we have
\[
f^7 + \lambda^2 f = 0.
\]
Taking horizontal lift, we obtain
\[
(f^7)^H + \lambda^2 (f^H) = 0.
\]
In view of (3.5) and (3.7), we can write
\[
(f^H)^7 + \lambda^2 (f^H) = 0.
\]
Thus the horizontal lift $f^H$ of $f$ also admits a $f_\lambda(7,1)$-structure. Hence we have the following theorem.

**Theorem 3.1.** Let $f$ be a tensor field of type $(1,1)$ admitting $f_\lambda(7,1)$-structure in $M$. Then the horizontal lift $f^H$ of $f$ also admits the similar structure in the cotangent bundle $C_{TM}$.

### 4. Nijenhuis tensor of complete lift of $f^7$

The Nijenhuis tensor of a $(1,1)$ tensor field $f$ on $M$ is given by
\[
N_{f,f}(X,Y) = [fX, fY] - f[fX, Y] - f[X, fY] + f^2[X, Y]. \tag{4.1}
\]
Also for the complete lift of $f^7$, we have
\[
N(f^7)^C, (f^7)^C(X^C,Y^C) = \left[ (f^7)^C X^C, (f^7)^C Y^C \right] - (f^7)^C \left[ (f^7)^C X^C, Y^C \right]
- (f^7)^C \left[ X^C, (f^7)^C Y^C \right] + (f^7)^C (f^7)^C [X^C, Y^C]. \tag{4.2}
\]
In view of (2.1), the above (4.2) takes the form
\[
N(f^7)^C, (f^7)^C(X^C,Y^C)
= \left[ (-\lambda^2 f)^C X^C, (-\lambda^2 f)^C Y^C \right] - (-\lambda^2 f)^C \left[ (-\lambda^2 f)^C X^C, Y^C \right]
- (-\lambda^2 f)^C \left[ X^C, (-\lambda^2 f)^C Y^C \right] + (-\lambda^2 f)^C (-\lambda^2 f)^C [X^C, Y^C]. \tag{4.3}
\]
or

\[ N(f^7)_C, (f^7)_C (X_C, Y_C) = \lambda^4 \left\{ (f)_C (f)_C (X_C, Y_C) - (f)_C (f)_C (X_C, Y_C) \right\}. \] (4.4)

We also know that [3]

\[ (f)_C X_C = (f)_C X_C + \nu \mathbb{L}_X f, \] (4.5)

where \( \nu f \) has components

\[ \nu f = \begin{pmatrix} O^a \\ P_{af_i} \end{pmatrix}. \] (4.6)

In view of (4.5), (4.4) takes the form

\[ N(f^7)_C, (f^7)_C X_C, Y_C \]

\[ = \lambda^4 \left\{ \left[ (f)_C (f)_C + \nu (\mathbb{L}_X f), (f)_C \right] + \left[ (f)_C (f)_C, \nu (\mathbb{L}_Y f) \right] \right\}. \] (4.7)

We now suppose that

\[ \mathbb{L}_X f = \mathbb{L}_Y f = 0. \] (4.8)

Then from (4.7), we have

\[ N(f^7)_C, (f^7)_C (X_C, Y_C) = \lambda^4 \left\{ \left[ (f)_C (f)_C - (f)_C (f)_C \right] - \left[ (f)_C (f)_C + (f)_C (f)_C \right] \right\}. \] (4.9)

Further, if \( f \) acts as an identity operator on \( M \) [2], that is,

\[ f X = X \quad \forall X \in \mathfrak{g}_0(M), \] (4.10)

then we have from (4.9)

\[ N(f^7)_C, (f^7)_C (X_C, Y_C) = \lambda^8 \left[ [X_C, Y_C] - [X_C, Y_C] - [X_C, Y_C] + [X_C, Y_C] \right] = 0. \] (4.11)

Hence we have the following theorem.

**Theorem 4.1.** The Nijenhuis tensor of the complete lift of \( f^7 \) vanishes if the Lie derivatives of the tensor field \( f \) with respect to \( X \) and \( Y \) are both zero and \( f \) acts as an identity operator on \( M \).
References


Lovejoy S. Das: Department of Mathematics, Kent State University Tuscarawas, New Philadelphia, OH 44663, USA  
*E-mail address*: ldas@kent.edu

Ram Nivas: Lucknow University, Lucknow, UP 226007, India  
*E-mail address*: rnivas@sify.com

Virendra Nath Pathak: Lucknow University, Lucknow, UP 226007, India
Mathematical Problems in Engineering

Special Issue on
Modeling Experimental Nonlinear Dynamics and Chaotic Scenarios

Call for Papers

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>February 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>May 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>August 1, 2009</td>
</tr>
</tbody>
</table>

Guest Editors

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; elbert@lac.inpe.br

Celso Grebogi, Department of Physics, King’s College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk

Hindawi Publishing Corporation
http://www.hindawi.com