SOME INEQUALITIES IN $B(H)$

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ABSTRACT. Let $H$ denote a separable Hilbert space and let $B(H)$ be the space of bounded and linear operators from $H$ to $H$. We define a subspace $\Delta(A,B)$ of $B(H)$, and prove two inequalities between the distance to $\Delta(A,B)$ of each operator $T$ in $B(H)$, and the value $\sup\{\|A^nTB^n - T\| : n = 1, 2, \ldots\}$.

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1. Notations. Throughout this paper $H$ denotes a separable Hilbert space and $\{e_n\}_{n=1}^{\infty}$ an orthonormal basis. Let $L_A$ and $R_B$ be left and right translation operators on $B(H)$ for $A, B \in B(H)$, satisfying $\|A\| \leq 1$ and $\|B\| \leq 1$. Then the set $\Delta(A,B)$ is defined by

$$\Delta(A,B) = \{T \in B(H) : ATB = T\} = \{T \in B(H) : ST = T\},$$

where $S = L_AR_B$.

An operator $C \in B(H)$ is called positive, if $\langle Cx, x \rangle \geq 0$ for all $x \in H$. Then for any positive operator $C \in B(H)$ we define $\text{tr}C = \sum_{n=1}^{\infty} \langle e_n, Ce_n \rangle$. The number $\text{tr}C$ is called the trace of $C$ and is independent of the orthonormal basis chosen. An operator $C \in B(H)$ is called trace class if and only if $\text{tr}|C| < \infty$ for $|C| = (C^*C)^{1/2}$, where $C^*$ is adjoint of $C$. The family of all trace class operators is denoted by $L_1(H)$. The basic properties of $L_1(H)$ and the functional $\text{tr}(\cdot)$ are the following:

(i) Let $\|\cdot\|$ be defined in $L_1(H)$ by $\|C\|_1 = \text{tr}|C|$. Then $L_1(H)$ is a Banach space with the norm $\|\cdot\|_1$ and $\|C\| \leq \|C\|_1$.

(ii) $L_1(H)$ is $*$- ideal, that is,

(a) $L_1(H)$ is a linear space,

(b) if $C \in L_1(H)$ and $D \in B(H)$, then $CD \in L_1(H)$ and $DC \in L_1(H)$,

(c) if $C \in L_1(H)$, then $C^* \in L_1(H)$.

(iii) $\text{tr}(\cdot)$ is linear.

(iv) $\text{tr}(CD) = \text{tr}(DC)$ if $C \in L_1(H)$ and $D \in B(H)$.

(v) $B(H) = L_1(H)^*$, that is, the map $T \to \text{tr}(T)$ is an isometric isomorphism of $B(H)$ onto $L_1(H)^*$, (see [3]).

Let $X$ be a Banach space. If $M \subset X$, then

$$M^\perp = \{x^* \in X^* : \langle x, x^* \rangle = 0, \ x \in M\}$$

(1.2)

is called the annihilator of $M$. If $N \subset X^*$, then

$$^\perp N = \{x \in X : \langle x, x^* \rangle = 0, \ x^* \in N\}$$

(1.3)
is called the preannihilator of \( N \). Rudin [4] proved for these subspaces:

(i) \( \overset{\perp}{\overline{M}} \) is the norm closure of \( M \) in \( X \).

(ii) \( \overset{\perp}{(\overset{\perp}{N})} \) is the weak-* closure of \( N \) in \( X^* \).

2. Main results

**Lemma 2.1.** Let \( X \) be a Banach space. If \( P \) is a continuous operator in the weak-* topology on the dual space \( X^* \), then there exists an operator \( T \) on \( X \) such that \( P = T^* \).

**Proof.** If \( P : X^* \to X^* \), then \( P^* : X^{**} \to X^{**} \). We know that the continuous functionals in the weak-* topology on \( X^* \) are simply elements of \( X \), (see [4]). Then we must show that \( P^* x \) is continuous in the weak-* topology on \( X^* \) for all \( x \in X \). Let \( (x'_n) \) be a sequence in \( X^* \) such that \( x'_n \to x' \), \( x' \in X^* \). Then we have

\[
\langle P^* x_n, x'_n \rangle = \langle x_n, P x'_n \rangle \to \langle x, P x' \rangle = \langle P^* x, x' \rangle.
\]

Hence \( P^* x \) is continuous in the weak-* topology on \( X^* \) for all \( x \in X \), so \( P^* x \in X \). If \( T \) is the restriction to \( X \) of \( P^* \), then we have

\[
\langle x, T^* x' \rangle = \langle T x, x' \rangle = \langle P^* x, x' \rangle = \langle x, P x' \rangle
\]

for all \( x \in X \) and \( x' \in X^* \). Hence \( P = T^* \). \( \Box \)

**Definition 2.2.** If \( P_* \) is the operator \( T \) in Lemma 2.1, then \( P_* \) is called the preadjoint operator of \( P \).

The operator \( x \otimes y \in B(H) \) for each \( x, y \in H \) is defined by \( (x \otimes y)z = \langle z, y \rangle x \) for all \( z \in H \). It is easy to see that this operator has the following properties:

(i) \( T(x \otimes y) = Tx \otimes y \).

(ii) \( (x \otimes y^*)^* = x \otimes T^* y \).

(iii) \( \text{tr}(x \otimes y) = \langle y, x \rangle \).

The following lemma is an easy application of some properties of the operator \( x \otimes y \ (x, y \in H) \) and the functional \( \text{tr}(\cdot) \).

**Lemma 2.3.** (i) Suppose \( K \) is a closed subset in the weak-* topology of \( B(H) \). Then \( K \) is closed in the weak-* topology of \( B(H) \).

(ii) \( S = L_A R_B \) is continuous in the weak-* topology of \( B(H) \) for all \( A, B \in B(H) \), satisfying \( \|A\| \leq 1 \) and \( \|B\| \leq 1 \).

**Lemma 2.4.** There exists a linear subspace \( M \) of \( L_1(H) \) such that \( \Delta(H) = M^\perp \) and \( M \) is closed linear span of \( \{S_n X - X : X \in L_1(H)\} \), where \( S_n \) is the preadjoint operator of \( S \).

**Proof.** Note that

\[
\overset{\perp}{\Delta(A,B)} = \{U \in L_1(H) : \langle U, U^* \rangle = 0, U^* \in \Delta(A,B) \}.
\]

It is known that \( \overset{\perp}{\Delta(A,B)} \) is the weak-* closure of \( \Delta(A,B) \) (see [4]). Then we can write \( \overset{\perp}{\overset{\perp}{\Delta(A,B)}} = \Delta(A,B) \), since \( \Delta(A,B) \) is a closed set in the weak-* topology of \( B(H) \). We say \( \overset{\perp}{\Delta(A,B)} = M \). Now we show that \( M \) is the closed linear span of \( \{S_n U - U : U \in L_1(H)\} \). For this, it is sufficient to prove that \( \langle S_n U - U, T \rangle = 0 \) for all \( T \in \Delta(A,B) \).
Indeed since $ST = T$, we have
\[
\langle S^*X - X, T \rangle = \langle (S^* - I)X, T \rangle = \langle X, (S^* - I)^*T \rangle = \langle X, (S - I)T \rangle = 0.
\] (2.4)

**Lemma 2.5.** Let $K(T)$ be the closed convex hull of $\{S^nT : n = 1, 2, \ldots\}$ in the weak operator topology, for a fixed $T \in B(H)$. Then we have
\[
K(T) \cap \Delta(A,B) \neq 0.
\] (2.5)

**Proof.** Assume $K(T) \cap \Delta(A,B) = 0$. By Lemma 2.3, $K(T)$ is closed in the weak-* topology. It is easy to see that $K(T)$ is bounded. Then $K(T)$ is compact in the weak-* topology by Alaoglu, [1]. Since $S$ is continuous in the weak-* topology, if $U_\alpha \to U$ for $(U_\alpha)_{\alpha \in I} \subset \Delta(A,B)$, then $SU_\alpha = U_\alpha \to SU$. Hence $\Delta(A,B)$ is closed in the weak-* topology. This shows that $U \in \Delta(A,B)$.

Since $K(T)$ is compact and convex in the weak-* topology, and $\Delta(A,B)$ is closed in the weak-* topology, and $K(T) \cap \Delta(A,B) = 0$, there exists some $U_0 \in M$ and $\sigma > 0$ such that
\[
|\text{tr}(TU_0)| \geq \sigma
\] (2.6)
for all $T \in \Delta(A,B)$, (see [2]). Now we define the operators $T_n \sum_{k=1}^n S^kT$ for all positive integer $n$. These operators are clearly in $K(T)$. It is easy to show that the operators $T_n$ is bounded. Also by Lemma 2.4, there is a $U \in L_1(H)$ such that $U_0 = S^*U - U$. Then we have
\[
|\langle T_n, U_0 \rangle| = |\langle T_n, S^*U - U \rangle| = |\langle ST_n, U \rangle - \langle T_n, U \rangle|
\]
\[
= \left| \left\langle S \left( \frac{1}{n} \sum_{k=1}^n A^kTB^k \right), U \right\rangle - \left\langle \frac{1}{n} \sum_{k=1}^n A^kTB^k, U \right\rangle \right|
\]
\[
= \left\langle \frac{1}{n} \sum_{k=1}^n A^{k+1}TB^{k+1}, U \right\rangle - \left\langle \frac{1}{n} \sum_{k=1}^n A^kTB^k, U \right\rangle
\]
\[
= \frac{1}{n} |\left\langle A^{n+1}TB^{n+1} - ATB, U \right\rangle|
\]
\[
\leq \frac{1}{n} 2\|T\| \cdot \|U\|.
\] (2.7)

This implies that $|\langle T_n, X_0 \rangle| \to 0$, which is a glaring contradiction to (2.6).

**Theorem 2.6.** Let $H$ be separable Hilbert space and $T \in B(H)$. Then we have
\[
\begin{align*}
(i) & \quad \text{d}(T, \Delta(A,B)) \geq (1/2) \sup_n \|S^nT - T\|, \\
(ii) & \quad \text{d}(T, \Delta(A,B)) \leq \sup_n \|S^nT - T\|.
\end{align*}
\]

**Proof.** (i) We can write
\[
S^nT - T = S^n(T - T_0) - (T - T_0) + S^nT_0 - T_0
\] (2.8)
for each $T_0 \in \Delta(A,B)$. Hence we have
\[
\|S^nT - T\| \leq ||S^n||\|T - T_0\| + ||T - T_0\| \leq 2\|T - T_0\|.
\] (2.9)
This shows that
\[ \frac{1}{2} \sup_n \| S^n T - T \| \leq \inf_{T_0 \in \Delta(A,B)} \| T - T_0 \|. \quad (2.10) \]

The inequality (2.10) gives that
\[ d(T, \Delta(A,B)) \geq \frac{1}{2} \sup_n \| S^n T - T \|. \quad (2.11) \]

(ii) Let \( K(T) \) be as Lemma 2.5. Then we can write
\[ K(T) = \text{co} \{ S^n T : n = 1, 2, \ldots \}. \quad (2.12) \]

Now take any element \( U = \sum_{k=1}^{n} \lambda_k S^k T \) in the set \( \text{co} \{ S^n T : n = 1, 2, \ldots \} \), where \( \sum_{k=1}^{n} \lambda_k = 1, \lambda_k \geq 0 \). Then
\[
\| U - T \| = \left\| \sum_{k=1}^{n} \lambda_k S^k T - T \right\| \leq \left\| \sum_{k=1}^{n} \lambda_k S^k T - \sum_{k=1}^{n} \lambda_k T \right\|
\leq \sum_{k=1}^{n} \lambda_k \| S^k T - T \| \leq \sum_{k=1}^{n} \lambda_k \sigma(T) = \sigma(T),
\]
where \( \sigma(T) = \sup_n \| S^n T - T \| \). That is, for all \( U \in \text{co} \{ S^n T : n = 1, 2, \ldots \} \) is
\[ \| U - T \| \leq \sup_n \| S^n T - T \|. \quad (2.14) \]

Since there is a sequence \( (U_n) \) in \( \text{co} \{ S^n T : n = 1, 2, \ldots \} \) such that \( U_n \to V \) for all \( V \in K(T) \), then we write
\[ \| V - T \| \leq \| V - T_n \| + \| T_n - T \|. \quad (2.15) \]

If we use the inequalities (2.14) and (2.15), we easily see that
\[ \| V - T \| \leq \sup_n \| S^n T - T \|. \quad (2.16) \]

Also since \( K(T) \cap \Delta(A,B) \neq 0 \) by Lemma 2.5, then we obtain
\[ \| T - T_0 \| \leq \sup_n \| S^n T - T \| \quad (2.17) \]

for a \( T_0 \in K(T) \cap \Delta(A,B) \). Hence we can write
\[ d(T, \Delta(A,B)) = \inf_{U \in \Delta(A,B)} \| T - U \| \leq \| T - T_0 \| \leq \sup_n \| S^n T - T \|. \quad (2.18) \]

This completes the proof.

\[ \square \]

**References**


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