COMMON COINCIDENCE POINTS OF $R$-WEAKLY COMMUTING MAPS

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ABSTRACT. A common coincidence point theorem for $R$-weakly commuting mappings is obtained. Our result extend several ones existing in literature.

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1. Introduction. Throughout this paper, $X$ denotes a metric space with metric $d$. For $x \in X$ and $A \subseteq X$, $d(x,A) = \inf\{d(x,y) : y \in A\}$. We denote by $CB(X)$ the class of all nonempty bounded closed subsets of $X$. Let $H$ be the Hausdorff metric with respect to $d$, that is,

$$H(A,B) = \max\left\{\sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A)\right\}$$

(1.1)

for every $A,B \in CB(X)$. The mappings $T : X \to CB(X)$, $f : X \to X$ are said to be commuting if, $fT X \subseteq T f X$. A point $p \in X$ is said to be a fixed point of $T : X \to CB(X)$ if $p \in T p$. The point $p$ is called a coincidence point of $f$ and $T$ if $fp \in T p$. The mappings $f : X \to X$ and $T : X \to CB(X)$ are called weakly commuting if, for all $x \in X$, $fT x \in CB(X)$ and $H(fT x, T f x) \leq d(fx, Tx)$.


**Theorem 1.1.** Let $X$ be a complete metric space and $T : X \to CB(X)$. If $\alpha$ is a function of $(0, \infty)$ to $(0,1]$ such that $\limsup_{r \to \tau^+} \alpha(r) < 1$ for each $\tau \in [0, \infty)$ and if

$$H(Tx, Ty) \leq \alpha(d(x,y)) d(x,y)$$

(1.2)

for each $x, y \in X$, then $T$ has a fixed point in $X$.

The purpose of this paper is to obtain a coincidence point theorem for $R$-weakly commuting multivalued mappings analogous to **Theorem 1.1**. We follow the same technique used in [2]. The notion of $R$-weak commutativity for single-valued mappings was defined by Pant [7] to generalize the concept of commuting and weakly commuting mappings [9]. Recently, Shahzad and Kamran [10] extended this concept to the setting of single and multivalued mappings, and studied the structure of common fixed points.
**Definition 1.2** (see [10]). The mappings \( f : X \to X \) and \( T : X \to CB(X) \) are called \( R \)-weakly commuting if for all \( x \in X \), \( fTx \in CB(X) \) and there exists a positive real number \( R \) such that

\[
H(Tfx, Tfx) \leq Rd(fx, Tx).
\]

(1.3)

2. Main result. Before giving our main result, we state the following lemmas which are noted in Nadler [6], and Assad and Kirk [1].

**Lemma 2.1.** If \( A, B \in CB(X) \) and \( a \in A \), then for each \( \varepsilon > 0 \), there exists \( b \in B \) such that

\[
d(a, b) \leq H(A, B) + \varepsilon.
\]

(2.1)

**Lemma 2.2.** If \( \{A_n\} \) is a sequence in \( CB(X) \) and \( \lim_{n \to \infty} H(A_n, A) = 0 \) for \( A \in CB(X) \). If \( x_n \in A_n \) and \( \lim_{n \to \infty} d(x_n, x) = 0 \), then \( x \in A \).

Now, we prove our main result.

**Theorem 2.3.** Let \( X \) be a complete metric space, \( f, g : X \to X \) and \( S, T : X \to CB(X) \) are continuous mappings such that \( SX \subseteq gX \) and \( TX \subseteq fX \). Let \( \alpha : (0, \infty) \to (0, 1] \) be such that \( \limsup_{r \to t^+} \alpha(r) < 1 \) for each \( t \in [0, \infty) \) and

\[
H(Sx, Ty) \leq \alpha(d(gx, fy)) d(gx, fy)
\]

(2.2)

for each \( x, y \in X \). If the pairs \( (g, T) \) and \( (f, S) \) are \( R \)-weakly commuting, then \( g, S \) and \( f, T \) have a common coincidence point.

**Proof.** Our method is constructive. We construct sequences \( \{x_n\}, \{y_n\}, \) and \( \{A_n\} \) in \( X \) and \( CB(X) \), respectively as follows. Let \( x_0 \) be an arbitrary point of \( X \) and \( y_0 = fx_0 \). Since \( SX_0 \subseteq gX \), there exists a point \( x_1 \in X \) such that \( y_1 = gx_1 \in SX_0 = A_0 \). Choose a positive integer \( n_1 \) such that

\[
\alpha^{n_1}(d(y_0, y_1)) < [1 - \alpha(d(y_0, y_1)))] d(y_0, y_1).
\]

Now Lemma 2.1 and the fact \( TX \subseteq fX \) guarantee that there is a point \( y_2 = fx_2 \in T, x_1 = A_1 \) such that

\[
d(y_2, y_1) \leq H(A_1, A_0) + \alpha^{n_1}(d(y_0, y_1)).
\]

(2.4)

The above inequality in view of (2.2) and (2.3) implies that \( d(y_2, y_1) < d(y_0, y_1) \). Now choose a positive integer \( n_2 > n_1 \) such that

\[
\alpha^{n_2}(d(y_2, y_1)) < [1 - \alpha(d(y_2, y_1)))] d(y_2, y_1).
\]

(2.5)

Again using Lemma 2.1 and the fact \( SX \subseteq gX \), we get a point \( y_3 = gx_3 \in SX_2 = A_2 \) such that

\[
d(y_3, y_2) \leq H(A_2, A_1) + \alpha^{n_2}(d(y_2, y_1)).
\]

(2.6)

Now (2.2) and (2.5) further imply that \( d(y_3, y_2) < d(y_2, y_1) \).
By induction we obtain sequences \( \{x_n\}, \{y_n\}, \) and \( \{A_n\} \) in \( X \) and \( CB(X) \), respectively, such that

\[
y_{2k+1} = gx_{2k+1} \in Sx_{2k} = A_{2k}, \quad y_{2k} = fx_{2k} \in Tx_{2k-1} = A_{2k-1}, \tag{2.7}
\]

\[
d(y_{2k+1}, y_{2k}) \leq H(A_{2k}, A_{2k-1}) + \alpha^{k+1}(d(y_{2k}, y_{2k-1})), \tag{2.8}
\]

where

\[
\alpha^{n+1}(d(y_{2k}, y_{2k-1})) < \{1 - \alpha(d(y_{2k}, y_{2k-1}))\} d(y_{2k}, y_{2k-1}) \tag{2.9}
\]

for each \( k \). So we have \( d(y_{2k+1}, y_{2k}) < d(y_{2k}, y_{2k-1}) \). Therefore, the sequence \( \{d(y_{2k+1}, y_{2k})\} \) is monotone nonincreasing. Then, as in the proof of Theorem 2.1 in [2], \( \{y_n\} \) is a Cauchy sequence in \( X \). Further, equation (2.2) ensures that \( \{A_n\} \) is a Cauchy sequence in \( CB(X) \). It is well known that if \( X \) is complete, then so is \( CB(X) \). Therefore, there exist \( z \in X \) and \( A \in CB(X) \) such that \( y_n \to z \) and \( A_n \to A \). Moreover, \( gx_{2k+1} \to z \) and \( fx_{2k} \to z \). Since

\[
d(z, A) = \lim_{n \to \infty} d(y_n, A_n) \leq \lim_{n \to \infty} H(A_{n-1}, A_n) = 0, \tag{2.10}
\]

it follows from Lemma 2.2 that \( z \in A \). Also

\[
\lim_{k \to \infty} fx_{2k} = z \in A = \lim_{k \to \infty} Sx_{2k}, \quad \lim_{k \to \infty} gx_{2k+1} = z \in A = \lim_{k \to \infty} Tx_{2k-1}. \tag{2.11}
\]

Using (2.7) and \( R \)-weak commutativity of the pairs \((g, T)\) and \((f, S)\), we have

\[
d(gfx_{2k+2}, Tgx_{2k+1}) \leq H(gTx_{2k+1}, Tgx_{2k+1}) \leq Rd(gx_{2k+1}, Tx_{2k+1}),
\]

\[
d(fgx_{2k+1}, Sfx_{2k}) \leq H(Sfx_{2k}, Sfx_{2k}) \leq Rd(fx_{2k}, Sx_{2k}). \tag{2.12}
\]

Now it follows from the continuity of \( f, g, T, \) and \( S \) that \( gz \in Tz \) and \( fz \in Sz \). \( \square \)

If we put \( T = S \) and \( f = g \) in Theorem 2.3, we get the following corollary.

**Corollary 2.4.** Let \( X \) be a complete metric space, and let \( f : X \to X \) be a continuous mapping and \( T : X \to CB(X) \) be a mapping such that \( TX \subseteq fX \). Let \( \alpha : (0, \infty) \to (0, 1] \) be such that \( \limsup_{r \to t^+} \alpha(r) < 1 \) for each \( t \in [0, \infty) \) and

\[
H(Tx, Ty) \leq \alpha(H(fx, fy)) d(fx, fy) \tag{2.13}
\]

for each \( x, y \in X \). If the mappings \( f \) and \( T \) are \( R \)-weakly commuting, then \( f \) and \( T \) have coincidence point.

**Remark 2.5.** (1) Theorem 2.3 improves and extends some known results of Hu [3], Kaneko [4], Mizoguchi and Takahashi [5], and Nadler [6].

(2) In Corollary 2.4, \( T \) is not assumed to be continuous. In fact the continuity of \( T \) follows from the continuity of \( f \).

(3) If we put \( f = I \), the identity map, in Corollary 2.4, we obtain Theorem 1.1.
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References


