The generalized resolvents for a certain class of perturbed symmetric operators with equal and finite deficiency indices are investigated. Using the Weinstein-Aronszajn formula, we give a classification of the spectrum.

1. Introduction

The present paper is concerned with the study of spectral properties for a certain class of linear symmetric operator $T$, defined in the Hilbert space $H$ of the form $T = A + B$, where $A$ is a closed linear symmetric operator, with nondensely defined domain in general, $D(A) \subset H$, and $B$ is a finite-rank operator of the form

$$Bf = \sum_{k=1}^{n} a_k (f, y_k) y_k,$$

where $y_1, y_2, \ldots, y_n$ is a linearly independant system in $H$, $a_1, a_2, \ldots, a_n \in \mathbb{R}$. We remark that the operator $T$ can be considered as a perturbation of the operator $A$ by the finite-rank operator $B$.

The case when $A$ is a first-order or second-order differential operator in the spaces $L^2(0, 2\pi)$, $L^2(0, \infty)$ or in the Hilbert space of vector-valued functions, and $B$ is a one-dimensional perturbation ($n = 1$), has been studied by many authors (see, e.g., [9, 20, 24]).

In particular, certain integrodiifferential equations of the above type occur in quantum mechanical scattering theory [8].

In this paper, the generalized resolvents of perturbed symmetric operator $T$ with equal and finite deficiency indices are investigated. Using the Weinstein-Aronszajn formula (see, e.g., [18]), we give a classification of the spectrum. Finally, the obtained results are applied to the study of two classes of first-order and second-order differential operators.

We note that the spectral theory of perturbed symmetric and selfadjoint operators have been investigated using various methods by many authors [3, 4, 5, 6, 11, 12, 13, 14, 15, 16, 17, 21, 22].
2. Preliminaries

Let $A$ be a closed symmetric operator with nondensely defined domain in a separable Hilbert space $H$ with equal deficiency indices $(m, m)$, and $m < \infty$. We denote by $\rho(A)$ the resolvent set of the operator $A$, the resolvent operator $R_\lambda(A)$ of $A$ is defined as $R_\lambda(A) = (A - \lambda I)^{-1}$. The complement of $\rho(A)$ in the complex plane is called the spectrum of $A$ and denoted by $\sigma(A)$. There is a decomposition of the spectrum $\sigma(A)$ into three disjoint subsets, at least one of which is not empty [1, 2, 10]:

\[ \sigma(A) = P\sigma(A) \cup C\sigma(A) \cup PC\sigma(A), \]  

(2.1)

$P\sigma(A)$ is called the point spectrum, $C\sigma(A)$ the continuous spectrum, and $PC\sigma(A)$ the point-continuous spectrum. We denote the essential spectrum of the operator $A$ by $\sigma_e(A) = C\sigma(A) \cup PC\sigma(A)$.

For arbitrary $\lambda \in \mathbb{C}$, we denote $P_\lambda = N_\lambda \cap (D(A) \oplus N_\lambda)$, where $N_\lambda = H\Theta(A - \lambda I)D(A)$ is the deficiency subspace of the operator $A$ [1, 2].

It is known [23] that $P_\lambda = \{0\}$ if and only if $\overline{D(A)} = H$, and if $\overline{D(A)} \neq H$, then the subset

\[ G_\lambda = \{ [\varphi, \psi] \in N_\lambda \times N_\lambda : \varphi - \psi \in D(A) \} \]  

(2.2)

is a graph of the isometric operator $X_\lambda$ with domain $P_\lambda$ and values in $P_\lambda$.

We denote by $\mathfrak{S}$ the set of linear operators $F$ defined from $N_i$ to $N_{-i}$, such that $\|F\| \leq 1$. For each analytic operator-valued function $F(\lambda)$ in $\mathbb{C}^+$, with $\mathbb{C}^+ = \{ \lambda : \text{Im}\lambda > 0 \}$, and values in $\mathfrak{S}$, we introduce the set $\Omega_F(\infty)$ consisting of elements $h \in N_i$ such that

\[ \lim_{\lambda \to \infty, \lambda \in C_i^+} |\lambda| \left( \|h\| - \|F(\lambda)h\| \right) < \infty, \]  

(2.3)

where $C_i^+ = \{ \lambda \in \mathbb{C}^+ : \varepsilon < \arg\lambda < \pi - \varepsilon \}, 0 < \varepsilon < \pi/2$.

It is known [27] that $\Omega_F(\infty)$ is a vector space and for each $h \in \Omega_F(\infty)$,

\[ \lim_{\lambda \to \infty, \lambda \in C_i^+} F(\lambda)h = F_0(\infty)h \]  

(2.4)

exists in the sense of the strong topology, and $F_0(\infty)$ is an isometric operator.

According to the theory of Štraus [28], the generalized resolvents of $A$ are given by the formula

\[ R_\lambda(A) = R_\lambda = (A_{F(\lambda)} - \lambda I)^{-1}, \quad R_\lambda = R_\lambda^*, \quad \lambda \in \mathbb{C}^+, \]  

(2.5)

where $A_{F(\lambda)}$ is an extension of $A$ which is determined by the function $F(\lambda)$, whose values are operators from the deficiency subspace $N_i$ to the deficiency subspace $N_{-i}$ such that $\|F(\lambda)\| \leq 1$ and $F(\lambda)$ satisfy the condition

\[ F_0(\infty)\psi = X_i\psi, \quad \text{for } \psi = 0 \text{ only}, \]  

(2.6)
then $A_{F(\lambda)}$ is a restriction on $H$ of a selfadjoint operator defined in a certain extended Hilbert space and is called quasiselfadjoint extension of the operator $A$ [28] defined on $D(A_{F(\lambda)}) = D(A) + (F(\lambda) - I)N_i$ by

$$A_{F(\lambda)}(f + F(\lambda)\varphi - \varphi) = Af + iF(\lambda)\varphi + i\varphi, \quad f \in D(A), \varphi \in N_i.$$  

For selfadjoint extensions with exit in the space in which acts the considered operators, see, for example, [12, 21] and the references therein.

We denote by $\mathcal{A}$ the set of analytic operator functions $F(\lambda)$ in $\mathbb{C}^+$ with values in $\mathcal{S}$ satisfying the condition (2.6).

Remark 2.1. To each selfadjoint extension of the operator $A$ corresponds a certain constant operator function $F(\lambda) = V$, where $V$ is an isometric operator defined from $N_i$ over $N_{-i}$ satisfying the condition $V\psi = X_i\psi$ for $\psi = 0$ only, and reciprocally.

We denote by $\hat{A}$ a selfadjoint extension of $A$ and we introduce the operator

$$\hat{U}_{\lambda_0} = (\hat{A} - \lambda_0I)(\hat{A} - \lambda I)^{-1}, \quad \text{Im}\lambda > 0. \quad (2.8)$$

We note that (see [19])

$$\hat{U}_{\lambda_0}N_{\lambda_0} = N_{\lambda}, \quad (\text{Im}\lambda)(\text{Im}\lambda_0) \neq 0. \quad (2.9)$$

We denote by

$$\varphi_i^{(1)}, \varphi_i^{(2)}, \ldots, \varphi_i^{(m)} \quad (2.10)$$

a basis of $N_{-i}$. From (2.9), $\varphi_i^{(k)} = U_{\lambda_0}\varphi_i^{(k)}$, $k = 1,2,\ldots,m$ form a basis for $N_{\lambda}$. In particular, the vectors

$$\varphi_{-i}^{(k)} = U\varphi_i^{(k)}, \quad k = 1,2,\ldots,m, \quad (2.11)$$

where $U = U_{-ii}$ is the Cayley transform [1, 2] of $\hat{A}$, form an orthogonal basis of $N_i$.

To get a convenient formula of the generalized resolvents of $A$, we will need the following notation:

$$\Phi_{\lambda\mu} = (\lambda - \mu)[(\varphi_{\lambda}^{(k)} \varphi_{\mu}^{(i)})_{k,s=1}^m]^t, \quad C(\lambda) = \Phi_{\lambda i}^{-1}\Phi_{\lambda(-i)}, \quad (2.12)$$

where $E$ is the identity matrix of order $m$, $\Omega(\lambda)$ is an analytic matrix function in $\mathbb{C}^+$ corresponding, in the bases (2.10) and (2.11), to the operator function $F(\lambda) \in \mathcal{A}$ and $\varphi_\lambda = (\varphi_\lambda^{(1)}, \ldots, \varphi_\lambda^{(m)})^t$, $(f, \varphi\lambda^t) = ((f, \varphi_\lambda^{(1)}), \ldots, (f, \varphi_\lambda^{(m)}))$, $t$ denotes the transpose, and $(\varphi_\lambda, g)$ is defined analogously.
In what follows, we denote by $\Phi$ the set of matrices $\Omega(\lambda)$, $\lambda \in \mathbb{C}^+$, associated in the bases (2.10) and (2.11) to the operator functions $F(\lambda) \in \mathbb{N}$.

According to the notation used in [7], the generalized resolvents of $A$ are given by

\[ R_\lambda f = R_\lambda f = \hat{R}_\lambda f + (f, \varphi_\lambda)[E - \Omega(\lambda)][C(\lambda)\Omega(\lambda) - E]^{-1}\Phi^{-1}\varphi_\lambda, \quad R_\lambda = R_\lambda^*, \quad \lambda \in \mathbb{C}^+, \quad (2.13) \]

where $\hat{R}_\lambda$ is the resolvent of $\hat{A}$ and $\Omega(\lambda) \in \Phi$.

**Remark 2.2.** The formula (2.13) defines a resolvent of a selfadjoint extension of $A$ if and only if $\Omega(\lambda)$ is a unitary constant matrix.

### 3. Resolvent and spectrum of a symmetric perturbed operator

Let $T = A + B$ be defined on $D(T) = D(A)$, where $A$ is a linear closed symmetric operator in $H$ and $B$ is a finite-rank operator.

**Lemma 3.1.** For $\lambda \in \rho(A) \cap \rho(T)$, the resolvent $R_\lambda(T)$ of the operator $T$ is given by

\[ R_\lambda(T) = R_\lambda(A) - R_\lambda(A)[I + BR_\lambda(A)]^{-1}BR_\lambda(A). \quad (3.1) \]

**Proof.** For $\lambda \in \rho(A) \cap \rho(T)$, the operator

\[ R_\lambda(A)[I + BR_\lambda(A)]^{-1} = R_\lambda(T) \quad (3.2) \]

exists and is bounded. Then, we get

\[
(T - \lambda I)[R_\lambda(A) - R_\lambda(A)(I + BR_\lambda(A))^{-1}BR_\lambda(A)] \\
= (A - \lambda I + B)[R_\lambda(A) - R_\lambda(A)(I + BR_\lambda(A))^{-1}BR_\lambda(A)] \\
= I + BR_\lambda(A) - (I + BR_\lambda(A))(I + BR_\lambda(A))^{-1}BR_\lambda(A) = I
\]

as required. \( \square \)

**Remark 3.2.** If $\|BR_\lambda(A)\| < 1$, then from (3.1), we obtain

\[ R_\lambda(T) = R_\lambda(A)(I + BR_\lambda(A))^{-1} = R_\lambda(A)\sum_{k=0}^{\infty}(-1)^k[BR_\lambda(A)]^k. \quad (3.4) \]

Now, the aim is to give a convenient expression of $(I + BR_\lambda(A))^{-1}$ in a more specific case.

So, we study in detail the case when $B$ is a finite-rank operator. Then,

\[ Bf = \sum_{k=1}^{n} a_k(f, y_k) y_k, \quad f \in H, \quad (3.5) \]

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$; $\{y_1, y_2, \ldots, y_n\}$ is a linearly independent system in $H$. If we put

\[ (I + BR_\lambda(A))^{-1}BR_\lambda(A)f = y, \quad (3.6) \]
we have
\[ y = BR_\lambda(A)f - BR_\lambda(A)y, \] (3.7)
then, \( y \in \text{Im}\, B \), so that
\[ y = \sum_{k=1}^{n} c_k y_k. \] (3.8)
From (3.7) and (3.8), we get
\[ \sum_{k=1}^{n} c_k y_k = BR_\lambda(A)f - \sum_{k=1}^{n} c_k BR_\lambda(A)y_k, \] (3.9)
with
\[ c_k + a_k \sum_{j=1}^{n} c_j (R_\lambda(A)y_j, y_k) = a_k (R_\lambda(A)f, y_k). \] (3.10)
The determinant \( \Delta(\lambda) \) of the system (3.10) is given by
\[ \Delta(\lambda) = \det \left\{ \delta_{kj} + a_k (R_\lambda(A)y_j, y_k) \right\}_{k, j=1}^{n}, \] (3.11)
where \( \delta_{kj} \) is the Kronecker symbol. If we suppose that \( \Delta(\lambda) \neq 0 \), the solution of (3.10) is given by
\[ c_k = c_k(\lambda; f) = \frac{(f, \Delta_k(\lambda))}{\Delta(\lambda)}, \quad k = 1, 2, \ldots, n, \] (3.12)
where \( \Delta_k(\lambda) \) is the determinant obtained from \( \overline{\Delta(\lambda)} \) by replacing the \( k \)-th column by \( [a_j R_\lambda(A)y_j]_{j=1}^{n} \). So, from (3.1), we have
\[ R_\lambda(T)f = R_\lambda(A)f - \sum_{k=1}^{n} \frac{(f, \Delta_k(\lambda))}{\Delta(\lambda)} R_\lambda(A)y_k. \] (3.13)
This completes the proof of the following theorem.

**Theorem 3.3.** Let \( \lambda \in \rho(A) \) such that \( \Delta(\lambda) \neq 0 \). Then, \( \lambda \in \rho(T) \) and the resolvent of the operator \( T \) is given by (3.13).

**Remark 3.4.** From (3.13), we note that the resolvent \( R_\lambda(T) \) is a perturbation of \( R_\lambda(A) \) by a finite-rank operator.
Remark 3.5. For the particular case \( n = 1 \) and \( a_1 = 1 \), the formula (3.13) was established in [9].

Remark 3.6. If \( \lambda \in \rho(A) \) such that \( \Delta(\lambda) = 0 \), then \( \lambda \) is an eigenvalue of the operator \( T \).

Proof. We can show that there exists an element

\[
\psi = \sum_{k=1}^{n} \alpha_k y_k
\]

(3.14)

such that \( \Gamma_\lambda(A)\psi \) is an eigenvector of the operator \( T \), corresponding to the eigenvalue \( \lambda \). Consequently, we have

\[
a_k \sum_{j=1}^{n} \alpha_j (\Gamma_\lambda(A)y_j, y_k) + \alpha_k = 0, \quad k = 1, n.
\]

(3.15)

Since the determinant of this system \( \Delta(\lambda) = 0 \), it admits a nontrivial solution, which gives the desired result. \( \square \)

Theorem 3.7. Let \( \mu \) be a fixed complex number. Then, the following holds.

(a) If \( \mu \in \rho(A) \) and \( \Delta(\mu) \neq 0 \), then \( \mu \in \rho(T) \).

(b) If \( \mu \in \rho(A) \) and \( \Delta(\mu) = 0 \), then \( \mu \in P\sigma(T) \) and the multiplicity of \( \mu \) as an eigenvalue of \( T \) is equal to the order of the zero of \( \Delta(\lambda) \) at \( \mu \).

(c) If \( \mu \in P\sigma(A) \) and \( \mu \) of multiplicity \( k > 0 \) and if \( \mu \) is a pole of \( \Delta(\lambda) \) of multiplicity \( p \) \( (k \geq p) \), then
   (1) for \( k > p \), it holds that \( \mu \in P\sigma(T) \) of multiplicity \( (k - p) \),
   (2) for \( k = p \), it holds that \( \mu \in \rho(T) \).

(d) If \( \mu \in P\sigma(A) \) is neither a zero, nor a pole of \( \Delta(\lambda) \), then \( \mu \in P\sigma(T) \).

(e) If \( \mu \in P\sigma(A) \) of multiplicity \( k \) and \( \mu \) is a root of the function \( \Delta(\lambda) \) of order \( p \), then \( \mu \in P\sigma(T) \) of order \( (k + p) \).

(f) The essential spectra \( \sigma_e(A) \) and \( \sigma_e(T) \), respectively of the operators \( A \) and \( T \), coincide.

Proof. It is sufficient to evaluate the function

\[
C(\lambda) = \det \{I + BR_\lambda(A)\}.
\]

(3.16)

To this end, let \( y \in \text{Im} B \). Then,

\[
BR_\lambda(A)y = \sum_{k=1}^{n} a_k (y, R_\lambda^+(A)y_k)y_k,
\]

(3.17)

it is clear that \( C(\lambda) = \Delta(\lambda) \), and the function \( \Delta(\lambda) \) is meromorphic in \( \rho(A) \cup P\sigma(A) \). From the formula of Weinstein and Aronszajn [18], we have

\[
\overline{\Theta}(\lambda; T) = \overline{\Theta}(\lambda; A) + \Theta(\lambda; \Delta),
\]

(3.18)
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where

$$\vartheta(\lambda; A) = \begin{cases} 0 & \text{if } \lambda \in \rho(A), \\ k & \text{if } \lambda \in P\sigma(A) \text{ and of multiplicity } k, \\ +\infty & \text{otherwise}, \\ \end{cases}$$

$$\vartheta(\lambda; \Delta) = \begin{cases} k & \text{if } \lambda \text{ is a zero of } \Delta(\lambda) \text{ of order } k, \\ -k & \text{if } \lambda \text{ is a pole of } \Delta(\lambda) \text{ of order } k, \\ 0 & \text{for other } \lambda \in \Omega, \end{cases}$$

which gives the desired result. \(\square\)

4. Generalized resolvents

Now, we suppose that \(A\) is a symmetric operator with deficiency indices \((m, m), m < \infty.\)

**Lemma 4.1.** Let \(\lambda \in \mathbb{C}\) such that \(\text{Im}\lambda > 0\) and \(\varphi_\lambda(A) \in N_\lambda(A).\) Then, the element \(\varphi_\lambda(T),\) defined by the formula

$$\varphi_\lambda(T) = D(\lambda)\varphi_\lambda(A) = \varphi_\lambda(A) - \sum_{k=1}^{n} \frac{(\varphi_\lambda(A), \hat{g}_k(\lambda))}{\Delta(\lambda)} R_\lambda(\hat{A}) y_k,$$

is an element of the deficiency subspace \(N_\lambda(T),\) where

$$D(\lambda) = I - R_\lambda(\hat{A})[I + BR_\lambda(\hat{A})]^{-1} B = I - R_\lambda(\hat{T})B, \quad \hat{g}_k(\lambda) = (\hat{A} - \lambda I) \hat{\Delta}_k(\lambda),$$

\(\hat{\Delta}(\lambda)\) and \(\hat{\Delta}_k(\lambda)\) are defined similarly as \(\Delta(\lambda)\) and \(\Delta_k(\lambda)\) in the formula (3.13) by putting the operator \(\hat{A}\) instead of the operator \(A.\)

**Proof.** Since the operators \(\hat{A}\) and \(\hat{T} = \hat{A} + B\) are selfadjoint and \(\lambda\) is nonreal, then \(\lambda \in \rho(\hat{A}) \cap \rho(\hat{T}).\) In addition, from Theorem 3.3 we have \(\Delta(\lambda) \neq 0.\) Furthermore, for each \(f \in D(\hat{A}) = D(T),\) we have

$$([\hat{T} - \lambda I] f, D(\lambda)\varphi_\lambda(A)) = (D^*(\lambda) [\hat{T} - \lambda I] f, \varphi_\lambda(A))$$

$$= ([I - BR\hat{T}](\hat{T}) - \lambda I) f, \varphi_\lambda(A))$$

$$= (\hat{A} - \lambda I) f, \varphi_\lambda(A))$$

$$= 0,$$

and the equality

$$\varphi_\lambda(T) = \varphi_\lambda(A) - \sum_{k=1}^{n} \frac{(\varphi_\lambda(A), \hat{g}_k(\lambda))}{\Delta(\lambda)} R_\lambda(\hat{A}) y_k$$

results from (3.13). \(\square\)

**Remark 4.2.** We note that if \(\varphi_\lambda(A) \neq 0,\) then \(\varphi_\lambda(T) \neq 0.\)
Proof. If we suppose the contrary, we obtain $R_λ(\hat{T})B_\phi(A) = \phi(A)$, which gives $A_\phi(\lambda) = \lambda \phi(A)$. This leads to a contradiction, since a selfadjoint operator can not have nonreal eigenvalues.

Remark 4.3. If $D(A)$ is dense in $H$, then $\phi(\lambda)$ and $\phi(T)$ are, respectively, eigenfunctions of the operators $A^*$ and $T^*$, corresponding to the eigenvalues $\lambda$.

Let $\phi^{(k)}_i(T) = D(i)\phi^{(k)}_i(A)$, $k = 1, 2, \ldots, m$, defined by the formula (4.1). If $\phi^{(1)}_i(A)$, $\phi^{(2)}_i(A), \ldots, \phi^{(m)}_i(A)$ is a basis of the deficiency subspace $N_\lambda(A)$ of the operator $A$, then $\phi^{(1)}_i(T), \phi^{(2)}_i(T), \ldots, \phi^{(m)}_i(T)$ is a basis of the deficiency subspace $N_\lambda(T)$ of the operator $T$. Putting

$$\hat{U}_{\lambda,0}(\hat{T}) = (\hat{T} - \lambda_0)R_λ(\hat{T}), \quad \phi^{(k)}_i(T) = \hat{U}_{λ,0}(\hat{T})\phi^{(k)}_i(T), \quad k = 1, 2, \ldots, m,$$

$$\phi_λ(T) = (\phi^{(1)}_λ(T), \ldots, \phi^{(m)}_λ(T))^T, \quad \Phi_{\lambda,\mu}(T) = (\lambda - \bar{\mu}) [(\lambda^{(k)}_λ(T), \phi^{(j)}_\mu(T))]_{k,j=1}^m,$$

$C(\lambda) = \Phi_{\lambda,\lambda}^{-1}(T)\Phi_{\lambda,(-\bar{\mu})}(T)$ denotes the characteristic matrix of the operator $T$, and $\omega(\lambda)$ the corresponding matrix of order $m \times m$, in the bases $\phi^{(1)}_\lambda(T), \phi^{(2)}_\lambda(T), \ldots, \phi^{(m)}_\lambda(T)$ and $\phi^{(-1)}_{\lambda,\lambda}(T), \phi^{(2)}_{\lambda,\lambda}(T), \ldots, \phi^{(m)}_{\lambda,\lambda}(T)$.

Theorem 4.4. The set of all generalized resolvents of the operator $T$ is given by

$$R_\lambda(T)f = R_\lambda(\hat{T})f + (f, \phi_\lambda(T))^T[E - \omega(\lambda)]\left[\frac{C(\lambda)\omega(\lambda) - E}{\Delta(\lambda)}\Phi_{\lambda,\mu}(T)\phi_\lambda(T)\right], \quad \forall f \in H,$$

where

$$R_\lambda(\hat{T})f = R_\lambda(\hat{A})f - \sum_{k=1}^n \left(\frac{\phi_\lambda(A_\omega)}{\Delta(\lambda)}\hat{\Delta}(\lambda)\right)R_\lambda(\hat{A})y_k.$$

Proof. The proof results from Lemma 4.1 and formula (2.13).

We denote, respectively, by $A_\omega$ and $T_\omega$ the quasiselfadjoint extensions of operators $A$ and $T$ corresponding to the operator function $F(\lambda) \in \mathfrak{F}$, defined by the matrix $\omega(\lambda)$.

Remark 4.5. To selfadjoint extensions of these operators correspond the constant unitary matrices $\omega = [\omega_{ij}]$.

Theorem 4.6. Suppose that $y_1, y_2, \ldots, y_n \in \text{Im} A$, $\mu$ is an eigenvalue of the quasiselfadjoint extension $A_\omega$ of the operator $A$, $\mu \in \rho(\hat{A})$ and $\Delta(\mu) \neq 0$, then $\mu$ is an eigenvalue of the operator $T_\omega = A_\omega + B$ and the corresponding eigenfunction $\phi_\mu(T_\omega)$ is given by

$$\phi_\mu(T_\omega) = D(\mu)\phi_\mu(A_\omega) = \phi_\mu(A_\omega) - \sum_{k=1}^n \left(\frac{\phi_\mu(A_\omega)}{\Delta(\mu)}\hat{\Delta}(\mu)\right)R_\mu(\hat{A})y_k,$$

where $\phi_\mu(A_\omega)$ is the eigenfunction of the operator $A_\omega$, corresponding to the eigenvalue $\mu$. 


Proof. Since \( y_1, y_2, \ldots, y_n \in \text{Im} A \), then \( B \varphi_\mu(A) \in \text{Im} A \). We also have

\[
\varphi_\mu(T_\omega) = D(\mu) \varphi_\mu(A_\omega) = \varphi_\mu(A_\omega) - R_\mu(T) B \varphi_\mu(A) = \varphi_\mu(A_\omega) - \psi_\mu,
\]

where

\[
\psi_\mu = R_\mu(T) B \varphi_\mu(A) \in D(A).
\]

Then,

\[
T_\omega \varphi_\mu(T_\omega) = T_\omega (\varphi_\mu(A_\omega) - \psi_\mu) = (A_\omega + B) \varphi_\mu(A_\omega) - T_\omega R_\mu(T) B \varphi_\mu(A) = \mu \varphi_\mu(A_\omega) + B \varphi_\mu(A_\omega) - B \varphi_\mu(A_\omega) + \mu R_\mu(T) B \varphi_\mu(A) = \mu \varphi_\mu(T_\omega).
\]

\[\square\]

5. Applications

5.1. Perturbed first-order differential operator. Consider in \( L^2(0,2\pi) \) the operator \( T = A + B \), where \( A \) is defined by \( Ay = iy' \) with domain \( D(A) = H^1_0(0,2\pi) \) and \( B \) is given by

\[
(By)(x) = \sum_{k=1}^n a_k(y, y_k)y_k(x),
\]

where \( y_1, y_2, \ldots, y_n \in L^2(0,2\pi) \) and \( a_k \in \mathbb{R} \), for all \( k = \overline{1,n} \). From \([1, 2]\), the operator \( A \) is regular symmetric of deficiency indices \((1,1)\) and each selfadjoint extension of \( A \) has a discrete spectrum.

**Theorem 5.1.** The generalized resolvent \( R_\lambda(T_\theta) \) of \( T \), corresponding to the function \( \omega(\lambda) = \theta(\lambda) \), is an integral operator with kernel

\[
K(x,t) = \left[ 1_{[x,2\pi]}(x) + \frac{1}{\theta(\lambda)e^{2\pi i} + 1} \right] e^{i\lambda(t-x)} + \sum_{k=1}^n \theta_k(\lambda, x) \phi_k(\lambda, t),
\]

where \( 1_{[x,2\pi]}(x) \) is the characteristic function of the interval \([x,2\pi]\),

\[
\phi_k(\lambda, t) = (\Delta^\theta_k(\lambda))(t), \quad \theta_k(\lambda, x) = \frac{(R_\lambda(A_\theta) y_k)(x)}{\Delta^\theta(\lambda)}
\]

where \( R_\lambda(A_\theta) \), associated to the function \( \theta(\lambda) \), is given by

\[
(R_\lambda(A_\theta) y)(x) = \int_0^x y(t)e^{i\lambda(t-x)}dt - \frac{1}{\theta(\lambda)e^{2\pi i} + 1} \int_0^{2\pi} y(t)e^{i\lambda(t-x)}dt
\]

with

\[
\Delta^\theta(\lambda) = \{ \delta_{kj} + a_k(R_\lambda(A_\theta) y_j, y_k) \},
\]

and \( \Delta^\theta_k \) is the determinant obtained from \( \Delta^\theta(\lambda) \) replacing the kth column by \([a_k R_\lambda(A_\theta) y_k]_1^n \).
Proof. The proof results from [26] and Theorem 3.3. □

Corollary 5.2. Let $T_\theta$ be a selfadjoint extension of $T$ corresponding to the function $\theta$, $|\theta| = 1$.

(1) The spectrum of $T_\theta$ is simple if and only if the roots of $\Delta^\theta(\lambda)$ are simple and for $k = 0, \pm 1, \pm 2, \ldots, \Delta^\theta(1/2 + k - \varphi_0/2\pi) \neq 0$, where \{1/2 + k - \varphi_0/2\pi\} is the spectrum of $A_\theta$, and $\varphi_0 = \arg \theta$.

(2) $\sigma(T_\theta) = P\sigma(T_\theta) = E_1 \cup E_2$, where $E_1$ is the set of points of $\sigma(A_\theta) = \{1/2 + k - \varphi_0/2\pi, k = 0, \pm 1, \pm 2, \ldots\}$ in which $\Delta^\theta(\lambda)$ is analytic, $E_2$ is the set of roots of $\Delta^\theta(\lambda)$.

Proof. The proof results from (5.4), Theorem 3.7, and Lemma 4.1. □

5.2. Perturbed second-order differential operator. Consider in $L^2(0, \infty)$ the operator $T = A + B$, where $A$ is defined by

$$Ay = -y'' + x^2 y$$

with domain $D(A)$ consisting of all variables $y$ which satisfy

(i) $y \in L^2(0, \infty)$,
(ii) $y'$ is absolutely continuous on all compact subintervals of $[0, \infty[$,
(iii) $Ay \in L^2(0, \infty)$,
(IV) $y(0) = y(\infty) = \lim_{x \to \infty} y(x) = 0, y'(0) = y'(\infty) = 0$,

and $B$ is given by

$$(By)(x) = \sum_{k=1}^n a_k (y, y_k) y_k(x),$$

where $y_1, y_2, \ldots, y_n \in L^2(0, 2\pi)$ and $a_k \in IR$, for all $k = \overline{1, n}$.

From [1, 2], the operator $A$ is symmetric of deficiency indices $(1, 1)$. Let $u_1, u_2$ be two solutions of (5.6), satisfying the initial conditions

$$u_1(0, \lambda) = 1, \quad u'_1(x, \lambda) \big|_{x=0} = 0,$$
$$u_2(0, \lambda) = 0, \quad u'_2(x, \lambda) \big|_{x=0} = -1.$$  \hspace{0.5cm} (5.8)

There exists a function $m(\lambda)$ [29] analytic in $\mathbb{C} \setminus \mathbb{R}$ such that

$$\psi(x, \lambda) = u_2(x, \lambda) + m(\lambda)u_1(x, \lambda) \in L^2(0, \infty).$$ \hspace{0.5cm} (5.9)

Theorem 5.3. The generalized resolvents $R_\lambda(T_\theta)$ of the operator $T$ are defined by

$$R_\lambda(T_\theta) y = R_\lambda(A_\theta) y - \sum_{k=1}^n \frac{(y, \Delta^\theta_k(\lambda))}{\Delta^\theta(\lambda)} R_\lambda(A_\theta) y_k, \quad \text{Im} \lambda > 0,$$ \hspace{0.5cm} (5.10)
where
\[
R_\lambda(A_\theta)y = \psi(x, \lambda) \int_0^x y(s)u_1(s, \lambda)ds + u_1(x, \lambda)\int_x^\infty y(s)\psi(s, \lambda)ds
- \frac{\psi(x, \lambda)}{\theta(\lambda) + m(\lambda)} \int_0^\infty y(s)\psi(s, \lambda)ds,
\]
\[
\Delta^\theta(\lambda) = \det \{ \sigma_{jk} + a_k(R_\lambda(A_\theta)y_j, y_k) \}, \quad \lambda \in \mathbb{C}^+,
\]
with \( \theta(\lambda) \) an arbitrary function analytic in \( \mathbb{C}^+ \) and such that \( \text{Im} \theta(\lambda) \geq 0 \) or \( \theta(\lambda) \) is an infinite constant.

**Proof.** First, we show that for \( \lambda \in \mathbb{C}^+ \), \( \Delta^\theta(\lambda) \neq 0 \) (then, \( \Delta^\Theta \neq 0 \)). We know (see [1, 2]) that for each quasiselfadjoint extension of a symmetric operator, \( \mathbb{C}^+ \) is contained in the set of regular points of this operator. Then, if \( \lambda \in \mathbb{C}^+ \), we have \( \lambda \in \rho(A_\theta) \) and \( \lambda \in \rho(T_\theta) \). If we suppose that \( \lambda \in \mathbb{C}^+ \) and \( \Delta^\theta(\lambda) = 0 \), from Theorem 3.7, we obtain \( \lambda \in P\sigma(T_\theta) \), which is a contradiction. The formula (5.11) results from [25]. Using Theorem 3.3, we end the proof.

**Corollary 5.4.** Let \( T_\theta \) be a selfadjoint extension associated to \( \theta \in \overline{\mathbb{R}} \), let \( \lambda_1, \lambda_2, \ldots \) be the roots of \( \Delta^\theta(\lambda) \) in \( \rho(A_\theta) \) and let \( z_1, z_2, \ldots \) be the poles of \( \Delta^\theta(\lambda) \). Then,
\[
P\sigma(T_\theta) = (P\sigma(A_\theta) \setminus \{z_j\}_1^\infty) \cup \{\lambda_j\}_1^\infty.
\]
**Proof.** The proof results from (b) and (c) of Theorem 3.7.

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**References**


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