Research Article

Optimal Inequalities for Power Means

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We present the best possible power mean bounds for the product $M_p(a,b)M_{1-a}(a,b)$ for any $p > 0$, $a \in (0,1)$, and all $a, b > 0$ with $a \neq b$. Here, $M_p(a,b)$ is the $p$th power mean of two positive numbers $a$ and $b$.

1. Introduction

For $p \in \mathbb{R}$, the $p$th power mean $M_p(a,b)$ of two positive numbers $a$ and $b$ is defined by

$$M_p(a,b) = \begin{cases} 
\left( \frac{a^p + b^p}{2} \right)^{1/p}, & p \neq 0, \\
\sqrt{ab}, & p = 0.
\end{cases} \quad (1.1)$$

It is well known that $M_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Many classical means are special cases of the power mean, for example, $M_{-1}(a,b) = H(a,b) = 2ab/(a+b)$, $M_0(a,b) = G(a,b) = \sqrt{ab}$ and $M_1(a,b) = A(a,b) = (a+b)/2$ are the harmonic, geometric and arithmetic means of $a$ and $b$, respectively. Recently, the power mean has been the subject of intensive research. In particular, many remarkable inequalities and properties for the power mean can be found in literature [1–22].
Let \( L(a, b) = (a-b)/(\log a - \log b) \), \( P(a, b) = (a-b)/(4 \arctan(\sqrt{a/b}) - \pi) \) and \( I(a, b) = 1/e(a^\alpha/b^\alpha)^{1/(a-b)} \) be the logarithmic, Seiffert and identric means of two positive numbers \( a \) and \( b \) with \( a \neq b \), respectively. Then it is well known that

\[
\min\{a, b\} < H(a, b) < G(a, b) < L(a, b) < P(a, b) < I(a, b) < A(a, b) < \max\{a, b\},
\]

for all \( a, b > 0 \) with \( a \neq b \).

In [23–29], the authors presented the sharp power mean bounds for \( L, I, (IL)^{1/2} \) and \((L + I)/2\) as follows:

\[
M_0(a, b) < L(a, b) < M_{1/3}(a, b), \quad M_{2/3}(a, b) < I(a, b) < M_{\log 2}(a, b),
\]

\[
M_0(a, b) < \sqrt{L(a, b)I(a, b)} < M_{1/2}(a, b), \quad \frac{1}{2}(L(a, b) + I(a, b)) < M_{1/2}(a, b),
\]

for all \( a, b > 0 \) with \( a \neq b \).

Alzer and Qiu [12] proved that the inequality

\[
\frac{1}{2}(L(a, b) + I(a, b)) > M_p(a, b)
\]

holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( p \leq (\log 2)/(1 + \log 2) = 0.40938 \ldots \).

The following sharp bounds for the sum \( aA(a, b) + (1 - a)L(a, b) \), and the products \( A^\alpha(a, b)L^{1-\alpha}(a, b) \) and \( G^\alpha(a, b)L^{1-\alpha}(a, b) \) in terms of power means were proved in [5, 8]:

\[
M_{\log 2/(\log 2-\log a)}(a, b) < aA(a, b) + (1 - a)L(a, b) < M_{(1+2a)/3}(a, b),
\]

\[
M_0(a, b) < A^\alpha(a, b)L^{1-\alpha}(a, b) < M_{(1+2a)/3}(a, b),
\]

\[
M_0(a, b) < G^\alpha(a, b)L^{1-\alpha}(a, b) < M_{(1-a)/3}(a, b),
\]

for any \( \alpha \in (0, 1) \) and all \( a, b > 0 \) with \( a \neq b \).

In [2, 7] the authors answered the questions: for any \( \alpha \in (0, 1) \), what are the greatest values \( p_1 = p_1(\alpha) \), \( p_2 = p_2(\alpha) \), \( p_3 = p_3(\alpha) \), and \( p_4 = p_4(\alpha) \), and the least values \( q_1 = q_1(\alpha) \), \( q_2 = q_2(\alpha) \), \( q_3 = q_3(\alpha) \), and \( q_4 = q_4(\alpha) \), such that the inequalities

\[
M_{p_1}(a, b) < P^\alpha(a, b)L^{1-\alpha}(a, b) < M_{q_1}(a, b),
\]

\[
M_{p_2}(a, b) < A^\alpha(a, b)G^{1-\alpha}(a, b) < M_{q_2}(a, b),
\]

\[
M_{p_3}(a, b) < G^\alpha(a, b)H^{1-\alpha}(a, b) < M_{q_3}(a, b),
\]

\[
M_{p_4}(a, b) < A^\alpha(a, b)H^{1-\alpha}(a, b) < M_{q_4}(a, b),
\]

hold for all \( a, b > 0 \) with \( a \neq b \)?

It is the aim of this paper to present the best possible power mean bounds for the product \( M_p^\alpha(a, b)M_{1-p}^{1-\alpha}(a, b) \) for any \( p > 0 \), \( \alpha \in (0, 1) \) and all \( a, b > 0 \) with \( a \neq b \).
2. Main Result

Theorem 2.1. Let $p > 0$, $\alpha \in (0,1)$ and $a, b > 0$ with $a \neq b$. Then

1. $M_{(2\alpha-1)p}(a,b) = M_p^\alpha(a,b) M_{1-p}^{1-\alpha}(a,b) = M_0(a,b)$ for $\alpha = 1/2$,
2. $M_{(2\alpha-1)p}(a,b) > M_p^\alpha(a,b) M_{1-p}^{1-\alpha}(a,b) > M_0(a,b)$ for $\alpha > 1/2$ and $M_{(2\alpha-1)p}(a,b) < M_p^\alpha(a,b) M_{1-p}^{1-\alpha}(a,b) < M_0(a,b)$ for $\alpha < 1/2$, and the bounds $M_{(2\alpha-1)p}(a,b)$ and $M_0(a,b)$ for the product $M_p^\alpha(a,b) M_{1-p}^{1-\alpha}(a,b)$ in either case are best possible.

Proof. From (1.1) we clearly see that $M_p(a,b)$ is symmetric and homogenous of degree 1. Without loss of generality, we assume that $b = 1$, $a = x > 1$.

(1) If $\alpha = 1/2$, then (1.1) leads to

\[
M_p^\alpha(x,1) M_{1-p}^{1-\alpha}(x,1) = \left(\frac{1 + x^p}{2}\right)^{1/p} \left(\frac{1 + x^{-p}}{2}\right)^{-1/p} = \left(\frac{1 + x^p}{2}\right)^{1/p} \left(\frac{2x^p}{1 + x^p}\right)^{1/p} = x = M_p^0(x,1) = M_{(2\alpha-1)p}(x,1).
\]

(2) Firstly, we compare the value of $M_{(2\alpha-1)p}(x,1)$ to the value of $M_p^\alpha(x,1) M_{1-p}^{1-\alpha}(x,1)$ for $\alpha \in (0,1/2) \cup (1/2,1)$. From (1.1) we have

\[
\log[M_p^\alpha(x,1) M_{1-p}^{1-\alpha}(x,1)] - \log M_{(2\alpha-1)p}(x,1) = \frac{\alpha}{p} \log \frac{1 + x^p}{2} - \frac{1 - \alpha}{p} \log \frac{1 + x^{-p}}{2} - \frac{1}{(2\alpha - 1)p} \log \frac{1 + x^{(2\alpha-1)p}}{2}.
\]

Let

\[
f(x) = \frac{\alpha}{p} \log \frac{1 + x^p}{2} - \frac{1 - \alpha}{p} \log \frac{1 + x^{-p}}{2} - \frac{1}{(2\alpha - 1)p} \log \frac{1 + x^{(2\alpha-1)p}}{2},
\]

then simple computations lead to

\[
f(1) = 0,
\]

\[
f'(x) = \frac{g(x)}{x(1 + x^p)(1 + x^{(2\alpha-1)p})},
\]

where

\[
g(x) = (\alpha - 1)x^{2ap} + ax^p - ax^{(2\alpha-1)p} + 1 - \alpha,
\]

\[
g'(1) = 0,
\]

\[
g'(x) = apx^{p-1}h(x),
\]
where

\[
    h(x) = 2(\alpha - 1)x^{(2\alpha - 1)p} - (2\alpha - 1)x^{2(\alpha - 1)p} + 1,
\]

\[
    h(1) = 0,
\]

\[
    h'(x) = -2p(1 - \alpha)(2\alpha - 1)x^{2(\alpha - 1)p - 1}(x^p - 1).
\]  

If \(\alpha \in (1/2, 1)\), then (2.9) implies that \(h(x)\) is strictly decreasing in \([1, +\infty)\). Therefore, 
\(M_{(2\alpha - 1)p}(x, 1) > M_p^\alpha(x, 1)M_1^{1-\alpha}(x, 1)\) follows easily from (2.2)–(2.8) and the monotonicity of 
h(x).

If \(\alpha \in (0, 1/2)\), then (2.9) leads to the conclusion that \(h(x)\) is strictly increasing in 
\([1, +\infty)\). Therefore, 
\(M_{(2\alpha - 1)p}(x, 1) < M_p^\alpha(x, 1)M_1^{1-\alpha}(x, 1)\) follows easily from (2.2)–(2.8) and the monotonicity of \(h(x)\).

Secondly, we compare the value of \(M_0(x, 1)\) to the value of 
\(M_p^\alpha(x, 1)M_1^{1-\alpha}(x, 1)\). It follows from (1.1) that

\[
    \log \left[ M_p^\alpha(x, 1)M_1^{1-\alpha}(x, 1) \right] - \log M_0(x, 1)
    = \frac{\alpha}{p} \log \frac{1 + x^p}{2} - \frac{1 - \alpha}{p} \log \frac{1 + x^{-p}}{2} - \frac{1}{2} \log x.
\]  

Let

\[
    F(x) = \frac{\alpha}{p} \log \frac{1 + x^p}{2} - \frac{1 - \alpha}{p} \log \frac{1 + x^{-p}}{2} - \frac{1}{2} \log x,
\]

then simple computations lead to

\[
    F(1) = 0,
\]

\[
    F'(x) = \frac{(2\alpha - 1)(x^p - 1)}{x(1 + x^p)(1 + x^{2\alpha - 1})}.
\]

If \(\alpha \in (1/2, 1)\), then (2.13) implies that \(F(x)\) is strictly increasing in \([1, +\infty)\). Therefore, 
\(M_p^\alpha(x, 1)M_1^{1-\alpha}(x, 1) > M_0(x, 1)\) follows easily from (2.10)–(2.13) and the monotonicity of 
\(F(x)\).

If \(\alpha \in (0, 1/2)\), then (2.13) leads to the conclusion that \(F(x)\) is strictly decreasing in 
\([1, +\infty)\). Therefore, 
\(M_p^\alpha(x, 1)M_1^{1-\alpha}(x, 1) < M_0(x, 1)\) follows easily from (2.10)–(2.12) and the monotonicity of \(F(x)\).

Next, we prove that the bound \(M_{(2\alpha - 1)p}(a, b)\) for the product 
\(M_p^\alpha(a, b)M_1^{1-\alpha}(a, b)\) in 
either case is best possible.
If $\alpha \in (0, 1/2)$, then for any $\epsilon \in (0, (1 - 2\alpha)p)$ and $x > 0$ we have

$$M^\alpha_p(1 + x, 1)M^{1-\alpha}_p(1 + x, 1) - M_{(2\alpha-1)p+\epsilon}(1 + x, 1)$$

$$= \left[ \frac{1 + (1 + x)^p}{2} \right]^{\alpha/p} \left[ \frac{1 + (1 + x)^{-p}}{2} \right]^{(\alpha-1)/p}$$

$$- \left[ \frac{1 + (1 + x)^{(2\alpha-1)p+\epsilon}}{2} \right]^{1/[(2\alpha-1)p+\epsilon]}$$

(2.14)

Letting $x \to 0$ and making use of Taylor’s expansion, one has

$$\left[ \frac{1 + (1 + x)^p}{2} \right]^{\alpha/p} \left[ \frac{1 + (1 + x)^{-p}}{2} \right]^{(\alpha-1)/p} - \left[ \frac{1 + (1 + x)^{(2\alpha-1)p+\epsilon}}{2} \right]^{1/[(2\alpha-1)p+\epsilon]}$$

$$= \left[ 1 + \frac{\alpha}{2} x + \frac{\alpha(p + \alpha - 2)}{8} x^2 + o(x^2) \right]$$

$$\times \left[ 1 + \frac{1 - \alpha}{2} x - \frac{(1 - \alpha)(p + \alpha + 1)}{8} x^2 + o(x^2) \right]$$

$$- \left[ 1 + \frac{1}{2} x + \frac{(2\alpha - 1)p + \epsilon - 1}{8} x^2 + o(x^2) \right]$$

$$= \left[ 1 + \frac{1}{2} x + \frac{(2\alpha - 1)p - 1}{8} x^2 + o(x^2) \right]$$

$$- \left[ 1 + \frac{1}{2} x + \frac{(2\alpha - 1)p + \epsilon - 1}{8} x^2 + o(x^2) \right]$$

$$= -\frac{\epsilon}{8} x^2 + o(x^2).$$

(2.15)

Equations (2.14) and (2.15) imply that for any $\alpha \in (0, 1/2)$ and $\epsilon \in (0, (1 - 2\alpha)p)$ there exists $\delta_1 = \delta_1(\epsilon) > 0$, such that $M^\alpha_p(1 + x, 1)M^{1-\alpha}_p(1 + x, 1) < M_{(2\alpha-1)p+\epsilon}(1 + x, 1)$ for $x \in (0, \delta_1)$.

If $\alpha \in (1/2, 1)$, then for any $\epsilon \in (0, (2\alpha - 1)p)$ and $x > 0$ we have

$$M^\alpha_p(1 + x, 1)M^{1-\alpha}_p(1 + x, 1) - M_{(2\alpha-1)p-\epsilon}(1 + x, 1)$$

$$= \left[ \frac{1 + (1 + x)^p}{2} \right]^{\alpha/p} \left[ \frac{1 + (1 + x)^{-p}}{2} \right]^{(\alpha-1)/p}$$

$$- \left[ \frac{1 + (1 + x)^{(2\alpha-1)p-\epsilon}}{2} \right]^{1/[(2\alpha-1)p-\epsilon]}$$

(2.16)
Letting \( x \to 0 \) and making use of Taylor’s expansion, one has

\[
\frac{1 + (1 + x)^p}{2} \left[ \frac{1 + (1 + x)^p}{2} \right]^{(a-1)/p} - \left[ \frac{1 + (1 + x)^{(2a-1)p-e}}{2} \right]^{1/(2a-1)p-e} = \left[ 1 + \frac{\alpha}{2} x + \frac{\alpha(p + \alpha - 2)}{8} x^2 + o(x^2) \right] \\
\times \left[ 1 + \frac{1 - \alpha}{2} x - \frac{(1 - \alpha)(p + \alpha + 1)}{8} x^2 + o(x^2) \right] \\
- \left[ 1 + \frac{1}{2} x + \frac{(2\alpha - 1)p - e - 1}{8} x^2 + o(x^2) \right] \\
= \frac{e}{8} x^2 + o(x^2).
\]

Equations (2.16) and (2.17) imply that for any \( \alpha \in (1/2, 1) \) and \( \epsilon \in (0, (2\alpha - 1)p) \) there exists \( \delta_2 = \delta_2(\epsilon) > 0 \), such that \( M_p^a(1 + x, 1)M_{1-p}^{1-a}(1 + x, 1) > M_{(2\alpha-1)p-e}(1 + x, 1) \) for \( x \in (0, \delta_2) \).

Finally, we prove that the bound \( M_0(a, b) \) for the product \( M_p^a(a, b)M_{1-p}^{1-a}(a, b) \) in either case is best possible.

If \( \alpha \in (0, 1/2) \), then for any \( \epsilon > 0 \) we clearly see that

\[
\lim_{x \to +\infty} \frac{M_p^a(x, 1)M_{1-p}^{1-a}(x, 1)}{M_{-\epsilon}(x, 1)} = +\infty. \tag{2.18}
\]

Equation (2.18) implies that for any \( \alpha \in (0, 1/2) \) and \( \epsilon > 0 \) there exists \( T_1 = T_1(\epsilon) > 1 \), such that \( M_p^a(x, 1)M_{1-p}^{1-a}(x, 1) > M_{-\epsilon}(x, 1) \) for \( x \in (T_1, +\infty) \).

If \( \alpha \in (1/2, 1) \), then for any \( \epsilon > 0 \) we have

\[
\lim_{x \to +\infty} \frac{M_p^a(x, 1)M_{1-p}^{1-a}(x, 1)}{M_{\epsilon}(x, 1)} = 0. \tag{2.19}
\]

Equation (2.19) implies that for any \( \alpha \in (1/2, 1) \) and \( \epsilon > 0 \) there exists \( T_2 = T_2(\epsilon) > 1 \), such that \( M_p^a(x, 1)M_{1-p}^{1-a}(x, 1) < M_{\epsilon}(x, 1) \) for \( x \in (T_2, +\infty) \).
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References


