We prove the generalized Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation
\[ f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x) + f(2y) + f(-2y) - 4f(y) + 4f(-y) \]
in various complete random normed spaces.

1. Introduction


The functional equation
\[ f(x + y) + f(x - y) = 2f(x) + 2f(y) \]  
(1.1)
is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. A generalized Hyers-Ulam stability problem for
the quadratic functional equation was proved by Cholewa [6] for mappings $f : X \to Y$, where $X$ is a normed space and $Y$ is a Banach space. Czerwik [7] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [8–12]).

In [13], Jun and Kim consider the following cubic functional equation:

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x). \quad (1.2)$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.2), which is called a cubic functional equation, and every solution of the cubic functional equation is said to be a cubic mapping.

Considered the following quartic functional equation

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y). \quad (1.3)$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation, which is called a quartic functional equation, and every solution of the quartic functional equation is said to be a quartic mapping. One can easily show that an odd mapping $f : X \to Y$ satisfies the additive-quadratic-cubic-quadratic functional equation

$$f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x) + f(2y) + f(-2y) - 4f(y) - 4f(-y) \quad (1.4)$$

if and only if it is an additive-cubic mapping, that is,

$$f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x). \quad (1.5)$$

It was shown in Lemma 2.2 of [14] that $g(x) := f(2x) - 2f(x)$ and $h(x) := f(2x) - 8f(x)$ are cubic and additive, respectively, and that $f(x) = (1/6)g(x) - (1/6)h(x)$.

One can easily show that an even mapping $f : X \to Y$ satisfies (1.4) if and only if it is a quadratic-quartic mapping, that is,

$$f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x) + 2f(2y) - 8f(y). \quad (1.6)$$

Also $g(x) := f(2x) - 4f(x)$ and $h(x) := f(2x) - 16f(x)$ are quartic and quadratic, respectively, and $f(x) = (1/12)g(x) - (1/12)h(x)$.

For a given mapping $f : X \to Y$, we define

$$Df(x, y) := f(x + 2y) + f(x - 2y) - 4f(x + y) - 4f(x - y) + 6f(x) - f(2y) - f(-2y) + 4f(y) + 4f(-y) \quad (1.7)$$

for all $x, y \in X$. 
Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$,
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall the fixed-point alternative of Diaz and Margolis.

**Theorem 1.1** (see [15, 16]). Let $(X,d)$ be a complete generalized metric space and let $F : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$, then for each given element $x \in X$, either

$$d(F^n x, F^{n+1} x) = \infty$$

for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that

1. $d(F^n x, F^{n+1} x) < \infty$ for all $n \geq n_0$,
2. the sequence $\{F^n x\}$ converges to a fixed point $y^*$ of $F$,
3. $y^*$ is the unique fixed point of $F$ in the set $Y = \{y \in X \mid d(F^n x, y) < \infty\}$,
4. $d(y, y^*) \leq (1/(1 - L))d(y, Fy)$ for all $y \in Y$.

In 1996, Isac and Rassias [17] were the first to provide applications of stability theory of functional equations for the proof of new fixed-point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [18–21]).

### 2. Preliminaries

In the sequel, we adopt the usual terminology, notations, and conventions of the theory of random normed spaces, as in [22–26]. Throughout this paper, $\Delta^+$ is the space of all probability distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0, 1]$, such that $F$ is left continuous, nondecreasing on $\mathbb{R}$, $F(0) = 0$ and $\{F(+\infty) = 1\}$. $D^+$ is a subset of $\Delta^+$ consisting of all functions $F \in \Delta^+$ for which $\lim F(\infty) = 1$, where $\lim F(x)$ denotes the left limit of the function $F$ at the point $x$, that is, $\lim_{t \to -\infty} F(t)$. The space $\Delta^+$ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t$ in $\mathbb{R}$. The maximal element for $\Delta^+$ in this order is the distribution function $\varepsilon_0$ given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

A triangular norm (shortly t-norm) is a binary operation on the unit interval $[0, 1]$, that is, a function $T : [0, 1] \times [0, 1] \to [0, 1]$, such that for all $a, b, c \in [0, 1]$ the following four axioms satisfied:

- $(T1)$ $T(a, b) = T(b, a)$ (commutativity),
- $(T2)$ $T(a, (T(b, c))) = T(T(a, b), c)$ (associativity),
(T3) \( T(a, 1) = a \) (boundary condition),
(T4) \( T(a, b) \leq T(a, c) \) whenever \( b \leq c \) (monotonicity).

Basic examples are the Łukasiewicz \( t \)-norm \( T_L, T_L(a, b) = \max(a + b - 1, 0) \) for all \( a, b \in [0, 1] \) and the \( t \)-norms \( T_P, T_M, T_D, \) where \( T_P(a, b) := ab, T_M(a, b) := \min\{a, b\}, \)

\[
T_D(a, b) := \begin{cases} 
\min(a, b), & \text{if } \max(a, b) = 1, \\
0, & \text{otherwise.}
\end{cases}
\]

If \( T \) is a \( t \)-norm, then \( x_T^{(n)} \) is defined for every \( x \in [0, 1] \) and \( n \in N \cup \{0\} \) by \( 1, \) if \( n = 0 \) and \( T(x_T^{(n-1)}, x) \) if \( n \geq 1. \) A \( t \)-norm \( T \) is said to be of Hadžić type (we denote by \( T \in H \)) if the family \( (x_T^{(n)})_{n\in N} \) is equicontinuous at \( x = 1 \) (cf. [27]).

Other important triangular norms are the following (see [28]):

1. The Sugeno-Weber family \( \{T_{SW}^{(n)}\}_{n \in (-1, \infty)} \) is defined by \( T_{SW}^{(n)} = T_D, T_{SW}^{(\infty)} = T_P \) and

\[
T_{SW}^{(n)}(x, y) = \max \left( 0, \frac{x + y - 1 + \lambda xy}{1 + \lambda} \right)
\]

if \( \lambda \in (-1, \infty). \)

2. The Domby family \( \{T_{D}^{(n)}\}_{n \in [0, \infty]} \) is defined by \( T_D \) if \( \lambda = 0, T_M \) if \( \lambda = \infty, \) and

\[
T_{D}^{(n)}(x, y) = \frac{1}{1 + \left( ((1 - x)/x)^{\lambda} + ((1 - y)/y)^{\lambda} \right)^{1/\lambda}}
\]

if \( \lambda \in (0, \infty). \)

3. The Aczel-Alsina family \( \{T_{AA}^{(n)}\}_{n \in [0, \infty]} \) is defined by \( T_D \) if \( \lambda = 0, T_M \) if \( \lambda = \infty \) and

\[
T_{AA}^{(n)}(x, y) = e^{-((\log x)^{\lambda} + (\log y)^{\lambda})^{1/\lambda}}
\]

if \( \lambda \in (0, \infty). \)

A \( t \)-norm \( T \) can be extended (by associativity) in a unique way to an \( n \)-array operation taking for \( (x_1, \ldots, x_n) \in [0, 1]^n \) the value \( T(x_1, \ldots, x_n) \) defined by

\[
T_0^{(n)}x_i = 1, \quad T_{i=1}^{(n)}x_i = T^{(i-1)}(T_{i=1}^{(n)}x_i, x_n) = T(x_1, \ldots, x_n).
\]

\( T \) can also be extended to a countable operation taking for any sequence \( (x_n)_{n \in N} \) in \([0, 1]\) the value

\[
T_{i=1}^{(\infty)}x_i = \lim_{n \to \infty} T_{i=1}^{(n)}x_i.
\]

The limit on the right side of (6.4) exists since the sequence \( (T_{i=1}^{(n)}x_i)_{n \in N} \) is nonincreasing and bounded from below.
Proposition 2.1 (see [28]). We have the following.

1. For $T \geq T_L$, the following implication holds:

$$\lim_{n \to \infty} T^{\infty}_{i=1} x_{n+i} = 1 \iff \sum_{n=1}^{\infty} (1 - x_n) < \infty. \quad (2.8)$$

2. If $T$ is of Hadžić type, then

$$\lim_{n \to \infty} T^{\infty}_{i=1} x_{n+i} = 1 \quad (2.9)$$

for every sequence $(x_n)_{n \in \mathbb{N}}$ in $[0,1]$ such that $\lim_{n \to \infty} x_n = 1$.

3. If $T \in \{ T_{1A}^{\lambda(A)} \}_{\lambda \in (0,\infty)} \cup \{ T_{1A}^{\lambda(D)} \}_{\lambda \in (0,\infty)}$, then

$$\lim_{n \to \infty} T^{\infty}_{i=1} x_{n+i} = 1 \iff \sum_{n=1}^{\infty} (1 - x_n)^{\lambda} < \infty. \quad (2.10)$$

4. If $T \in \{ T_{1A}^{SW} \}_{\lambda \in [-1,\infty)}$, then

$$\lim_{n \to \infty} T^{\infty}_{i=1} x_{n+i} = 1 \iff \sum_{n=1}^{\infty} (1 - x_n) < \infty. \quad (2.11)$$

Definition 2.2 (see [26]). A Random normed space (briefly, RN-space) is a triple $(X, \mu, T)$, where $X$ is a vector space, $T$ is a continuous $t$-norm, and $\mu$ is a mapping from $X$ into $D^+$ such that, the following conditions hold:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$,

(RN2) $\mu_{ax}(t) = \mu_x(t/|a|)$ for all $x \in X$, and $a \neq 0$,

(RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Definition 2.3. Let $(X, \mu, T)$ be an RN-space.

1. A sequence $\{x_n\}$ in $X$ is said to be convergent to $x$ in $X$ if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists positive integer $N$ such that $\mu_{x-x_n}(\varepsilon) > 1 - \lambda$ whenever $n \geq N$.

2. A sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists positive integer $N$ such that $\mu_{x-x_n}(\varepsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.

3. An RN-space $(X, \mu, T)$ is said to be complete if and only if every Cauchy sequence in $X$ is convergent to a point in $X$. A complete RN-space is said to be random Banach space.

Theorem 2.4 (see [25]). If $(X, \mu, T)$ is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n \to \infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.
The theory of random normed spaces (RN-spaces) is important as a generalization of deterministic result of linear normed spaces and also in the study of random operator equations. The RN-spaces may also provide us with the appropriate tools to study the geometry of nuclear physics and have important application in quantum particle physics. The generalized Hyers-Ulam stability of different functional equations in random normed spaces, RN-spaces, and fuzzy normed spaces has been recently studied [20, 24, 29–39].

3. Non-Archimedean Random Normed Space

By a non-Archimedean field, we mean a field $\mathcal{K}$ equipped with a function (valuation) $| \cdot |$ from $K$ into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0, |rs| = |r||s|$, and $|r + s| \leq \max \{|r|, |s|\}$ for all $r, s \in \mathcal{K}$. Clearly, $|1| = |−1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. By the trivial valuation, we mean the mapping $| \cdot |$ taking everything but 0 into 1 and $|0| = 0$. Let $X$ be a vector space over a field $\mathcal{K}$ with a non-Archimedean nontrivial valuation $| \cdot |$. A function $\| \cdot \|: X \rightarrow [0, \infty)$ is called a non-Archimedean norm if it satisfies the following conditions:

(NAN1) $\|x\| = 0$ if and only if $x = 0$,
(NAN2) for any $r \in \mathcal{K}$ and $x \in X$, $\|rx\| = |r|\|x\|$,
(NAN3) the strong triangle inequality (ultrametric), namely,

$$
\|x + y\| \leq \max\{|\|x\|, \|y\|\} \quad (x, y \in X),
$$

then $(X, \| \cdot \|)$ is called a non-Archimedean normed space. Due to the fact that

$$
\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j < n - 1\} \quad (n > m),
$$

a sequence $\{x_n\}$ is a Cauchy sequence if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space, we mean one in which every Cauchy sequence is convergent.

In 1897, Hensel [40] discovered the $p$-adic numbers of as a number theoretical analogues of power series in complex analysis. Fix a prime number $p$. For any nonzero rational number $x$, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = (a/b)p^{n_x}$, where $a$ and $b$ are integers not divisible by $p$. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, which is called the $p$-adic number field.

Throughout the paper, we assume that $X$ is a vector space and $Y$ is a complete non-Archimedean normed space.

**Definition 3.1.** A non-Archimedean random normed space (briefly, non-Archimedean RN-space) is a triple $(X, \mu, T)$, where $X$ is a linear space over a non-Archimedean field $\mathcal{K}$, $T$ is a continuous $t$-norm, and $\mu$ is a mapping from $X$ into $D^*$ such that the following conditions hold:

(NA-RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$,
(NA-RN2) $\mu_{ax}(t) = \mu_x(t/|a|)$ for all $x \in X, t > 0$, and $a \neq 0$,
(NA-RN3) $\mu_{x+y}(\max\{t, s\}) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y, z \in X$ and $t, s \geq 0$. 


It is easy to see that if (NA-RN3) holds, then so is
\[(RN3) \mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s)).\]

As a classical example, if \((X, \| \cdot \|)\) is a non-Archimedean normed linear space, then the triple \((X, \mu, T_M)\), where
\[
\mu_x(t) = \begin{cases} 
0, & t \leq \|x\|, \\
1, & t > \|x\|,
\end{cases}
\]
is a non-Archimedean RN-space.

**Example 3.2.** Let \((X, \| \cdot \|)\) be a non-Archimedean normed linear space. Define
\[
\mu_x(t) = \frac{t}{t + \|x\|} \quad (x \in X, \ t > 0),
\]
then \((X, \mu, T_M)\) is a non-Archimedean RN-space.

**Definition 3.3.** Let \((X, \mu, T)\) be a non-Archimedean RN-space. Let \(\{x_n\}\) be a sequence in \(X\), then \(\{x_n\}\) is said to be **convergent** if there exists \(x \in X\) such that
\[
\lim_{n \to \infty} \mu_{x_n-x}(t) = 1
\]
for all \(t > 0\). In that case, \(x\) is called the **limit** of the sequence \(\{x_n\}\).

A sequence \(\{x_n\}\) in \(X\) is called a **Cauchy sequence** if for each \(\varepsilon > 0\) and each \(t > 0\) there exists \(n_0\) such that for all \(n \geq n_0\) and all \(p > 0\), we have \(\mu_{x_n-x_p}(t) > 1 - \varepsilon\).

If each Cauchy sequence is convergent, then the random norm is said to be **complete** and the non-Archimedean RN-space is called a non-Archimedean **random Banach space**.

**Remark 3.4** (see [41]). Let \((X, \mu, T_M)\) be a non-Archimedean RN-space, then
\[
\mu_{x_{n+p}-x_n}(t) \geq \min \left\{ \mu_{x_{n+j}-x_n}(t) : j = 0, 1, 2, \ldots, p - 1 \right\}.
\]
(3.6)

So, the sequence \(\{x_n\}\) is a Cauchy sequence if for each \(\varepsilon > 0\) and \(t > 0\) there exists \(n_0\) such that for all \(n \geq n_0\),
\[
\mu_{x_{n+1}-x_n}(t) > 1 - \varepsilon.
\]
(3.7)


Let \(K\) be a non-Archimedean field, let \(X\) be a vector space over \(K\), and let \((Y, \mu, T)\) be a non-Archimedean random Banach space over \(K\).
Next, we define a random approximately AQCQ mapping. Let $\Psi$ be a distribution function on $X \times X \times [0, \infty)$ such that $\Psi(x, y, \cdot)$ is nondecreasing and

$$
\Psi(cx, cx, t) \geq \Psi(x, x, \frac{t}{|c|}) \quad (x \in X, \ c \neq 0). \tag{4.1}
$$

**Definition 4.1.** A mapping $f : X \to Y$ is said to be $\Psi$-approximately AQCQ if

$$
\mu_{Df(x,y)}(t) \geq \Psi(x, y, t) \quad (x, y \in X, \ t > 0). \tag{4.2}
$$

In this section, we assume that $2 \neq 0$ in $\mathcal{K}$ (i.e., characteristic of $\mathcal{K}$ is not 2). Our main result, in this section, is the following.

We prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in non-Archimedean random spaces, an odd case.

**Theorem 4.2.** Let $\mathcal{K}$ be a non-Archimedean field, let $X$ be a vector space over $\mathcal{K}$ and let $(Y, \mu, T)$ be a non-Archimedean random Banach space over $\mathcal{K}$. Let $f : X \to Y$ be an odd mapping and $\Psi$-approximately AQCQ mapping. If for some $\alpha \in \mathbb{R}$, $\alpha > 0$, and some integer $k$, $k > 3$ with $|2^k| < \alpha$,

$$
\Psi\left(2^{-k}x, 2^{-k}y, t\right) \geq \Psi(x, y, \alpha t) \quad (x \in X, \ t > 0), \tag{4.3}
$$

$$
\lim_{n \to \infty} T_{\frac{1}{n}}^{\infty} M\left(2x, \frac{\alpha^i t}{[8]^k}\right) = 1 \quad (x \in X, \ t > 0), \tag{4.4}
$$

then there exists a unique cubic mapping $C : X \to Y$ such that

$$
\mu_{f(x)-2f(x/2)-C(x/2)}(t) \geq T_{i=1}^{\infty} M\left(x, \frac{\alpha^i t}{[8]^k}\right) \tag{4.5}
$$

for all $x \in X$ and $t > 0$, where

$$
M(x, t) := T_{k-1}\left[\Psi\left(x, x, t, \frac{t}{4}\right), \Psi\left(x, x, t, \frac{t}{2}\right), \ldots, \Psi\left(2^{-k}x, 2^{-k}x, t, \frac{t}{4}\right), \Psi\left(2^{-k}x, 2^{-k}x, t, \frac{t}{2}\right)\right] \quad (x \in X, \ t > 0). \tag{4.6}
$$

**Proof.** Letting $x = y$ in (4.2), we get

$$
\mu_{f(3y)-4f(2y)+5f(y)}(t) \geq \Psi(y, y, t) \tag{4.7}
$$

for all $y \in X$ and $t > 0$. Replacing $x$ by $2y$ in (4.2), we get

$$
\mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t) \geq \Psi(2y, y, t) \tag{4.8}
$$
for all \( y \in X \) and \( t > 0 \). By (4.7) and (4.8), we have

\[
\mu_{f(4y)-10f(2y)+16f(y)}(t) \geq T \left( \mu_{f(4y)-4f(2y)+5f(y)}(t), \mu_{f(4y)-4f(2y)+6f(2y)-4f(y)}(t) \right) \\
= T \left( \mu_{f(3y)-4f(2y)+5f(y)}\left( \frac{t}{4} \right), \mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t) \right) \\
\geq T \left( \Psi(y, y, \frac{t}{4}), \Psi(2y, y, t) \right) \tag{4.9}
\]

for all \( y \in X \) and \( t > 0 \). Letting \( y := x/2 \) and \( g(x) := f(2x) - 2f(x) \) for all \( x \in X \) in (4.9), we get

\[
\mu_{g(x)-8g(x/2)}(t) \geq T \left( \Psi\left( \frac{x}{2}, \frac{x}{2}, \frac{t}{4} \right), \Psi\left( \frac{2^{j-1}x}{2}, \frac{2^{j-1}x}{2}, \frac{t}{4} \right) \right) \tag{4.10}
\]

for all \( x \in X \) and \( t > 0 \). Now, we show by induction on \( j \) that for all \( x \in X, t > 0 \) and \( j \geq 1 \),

\[
\mu_{g(2^{-j}x)-8g(2^{-j}x/2)}(t) \geq M_{j}(x, t) \\
:= T^{2j-1} \left[ \Psi\left( \frac{x}{2}, \frac{x}{2}, \frac{t}{4} \right), \Psi\left( \frac{2^{j-1}x}{2}, \frac{2^{j-1}x}{2}, \frac{t}{4} \right) \right] . \tag{4.11}
\]

Putting \( j = 1 \) in (4.11), we obtain (4.10). Assume that (4.11) holds for some \( j \geq 1 \). Replacing \( x \) by \( 2^{-j}x \) in (4.10), we get

\[
\mu_{g(2^{-j}x)-8g(2^{-j}x/2)}(t) \geq T \left( \Psi\left( 2^{j-1}x, 2^{j-1}x, \frac{t}{4} \right), M_{j}(x, t) \right) . \tag{4.12}
\]

Since \( |8| \leq 1 \),

\[
\mu_{g(2^{-j}x)-8^{j+1}g(x/2)}(t) \geq T \left( \mu_{g(2^{-j}x)-8g(2^{-j}x/2)}(t), \mu_{8g(2^{-j}x)-8^{j+1}g(x/2)}(t) \right) \\
= T \left( \mu_{g(2^{-j}x)-8g(2^{-j}x)}(t), \mu_{g(2^{-j}x)-8g(x/2)}\left( \frac{t}{|8|} \right) \right) \\
\geq T^{2} \left( \Psi\left( 2^{j-1}x, 2^{j-1}x, \frac{t}{4} \right), M_{j+1}(x, t) \right) \\
= M_{j+1}(x, t) \tag{4.13}
\]

for all \( x \in X \) and \( t > 0 \). Thus, (4.11) holds for all \( j \geq 2 \). In particular,

\[
\mu_{g(2^{-j}x)-8g(x/2)}(t) \geq M(x, t) \quad (x \in X, t > 0) . \tag{4.14}
\]
Replacing $x$ by $2^{-(k\pi + k - 1)}x$ in (4.14) and using inequality (4.3), we obtain

$$\mu g(x/2^{kn} - 8^kg(x/2^{kn+1}))(t) \geq M\left(\frac{2x}{2^{k(n+1)}}, t\right) \quad (x \in X, \ t > 0, \ n = 0, 1, 2, \ldots).$$  \hspace{1cm} (4.15)

Then

$$\mu 8^k g(x/2^{kn} - 8^{kn+1} g(x/2^{kn+1}))(t) \geq M\left(2x, \frac{a^{n+1}}{|8^k(n+1)|} t\right) \quad (x \in X, \ t > 0, \ n = 0, 1, 2, \ldots).$$  \hspace{1cm} (4.16)

Hence

$$\mu 8^k g(x/2^{kn} - 8^{kn+1} g(x/2^{kn+1}))(t) \geq T_{j=n}^{n+p} M\left(8^k g(x/2^{kn} - 8^{kn+1} g(x/2^{kn+1}))(t)\right) \geq T_{j=n}^{n+p} M\left(2x, \frac{a^{j+1}}{|8^j(j+1)|} t\right) \quad (x \in X, \ t > 0, \ n = 0, 1, 2, \ldots).$$  \hspace{1cm} (4.17)

Since

$$\lim_{n \to \infty} T_{j=n}^{\infty} M\left(2x, \frac{a^{j+1}}{|8^j(j+1)|} t\right) = 1 \quad (x \in X, \ t > 0),$$  \hspace{1cm} (4.18)

then

$$\left\{8^k g\left(\frac{x}{2^{kn}}\right)\right\}_{n \in \mathbb{N}}$$  \hspace{1cm} (4.19)

is a Cauchy sequence in the non-Archimedean random Banach space ($Y, \mu, T$). Hence we can define a mapping $C : X \to Y$ such that

$$\lim_{n \to \infty} \mu (8^k)^n g(x/2^{kn} - C(x))(t) = 1 \quad (x \in X, \ t > 0).$$  \hspace{1cm} (4.20)

Next for each $n \geq 1$, $x \in X$ and $t > 0$,

$$\mu g(x) - (8^k)^n g(x/2^{kn}))(t) = \mu \sum_{i=0}^{n-1} (8^k)^i g(x/2^{kn}) - (8^k)^{i+1} g(x/2^{kn+1}))(t) \geq T_{i=0}^{n-1} \left(8^k)^i g(x/2^{kn}) - (8^k)^{i+1} g(x/2^{kn+1}))(t) \geq T_{i=0}^{n-1} M\left(2x, \frac{a^{i+1}}{|8^k|^{i+1}} t\right),$$  \hspace{1cm} (4.21)
Therefore,

\[
\mu_{g(x) - C(x)}(t) \geq T \left( \mu_{g(x) - (8^i)g(x/2^i)}(t), \mu_{(8^i)g(x/2^i) - C(x)}(t) \right)
\]

\[
\geq T \left( \frac{2x}{|8^i|^{i+1}}, \frac{2^{i+1}t}{|8^i|^{i+1}} \right), \mu_{(8^i)g(x/2^i) - C(x)}(t) \right).
\]

By letting \( n \to \infty \), we obtain

\[
\mu_{g(x) - C(x)}(t) \geq T_{i=1}^\infty M \left( 2x, \frac{2^{i+1}t}{|8^i|^{i+1}} \right).
\]

So,

\[
\mu_{f(x) - 2f(x/2) - C(x/2)}(t) \geq T_{i=1}^\infty M \left( x, \frac{2^{i+1}t}{|8^i|^{i+1}} \right).
\]

This proves (4.5). From \( Dg(x, y) = Df(2x, 2y) - 2Df(x, y) \), by (4.2), we deduce that

\[
\mu_{Df(2x, 2y)}(t) \geq \Psi(2x, 2y, t),
\]

\[
\mu_{-2Df(x, y)}(t) = \mu_{Df(x, y)} \left( \frac{t}{2} \right) \geq \mu_{Df(x, y)}(t) \geq \Psi(x, y, t),
\]

and so, by (NA-RN3) and (4.2), we obtain

\[
\mu_{Dg(x, y)}(t) \geq T(\mu_{Df(2x, 2y)}(t), \mu_{-2Df(x, y)}(t)) \geq T(\Psi(2x, 2y, t), \Psi(x, y, t)) := N(x, y, t).
\]

It follows that

\[
\mu_{g(x) - 2g(x/2^n)}(t) = \mu_{Dg(x/2^n, y/2^n)} \left( \frac{t}{|8^n|^{kn}} \right)
\]

\[
\geq N \left( \frac{x}{2^{kn}}, \frac{y}{2^{kn}}, \frac{t}{|8^n|^{kn}} \right) \geq \cdots \geq N \left( x, y, \frac{\alpha^{n-1}t}{|8|^{k(n-1)}} \right)
\]

for all \( x, y \in X, t > 0, \) and \( n \in \mathbb{N} \). Since

\[
\lim_{n \to \infty} N \left( x, y, \frac{\alpha^{n-1}t}{|8|^{k(n-1)}} \right) = 1
\]
for all \(x, y \in X\) and \(t > 0\), by Theorem 2.4, we deduce that

\[
\mu_{DC(x,y)}(t) = 1
\]  

(4.29)

for all \(x, y \in X\) and \(t > 0\). Thus, the mapping \(C : X \rightarrow Y\) satisfies (1.4).

Now, we have

\[
C(2x) - 8C(x) = \lim_{n \to \infty} \left[8^n g \left(\frac{x}{2^{n-1}}\right) - 8^{n+1} g \left(\frac{x}{2^n}\right)\right]
\]

(4.30)

\[
= 8 \lim_{n \to \infty} \left[8^{n-1} g \left(\frac{x}{2^{n-1}}\right) - 8^n g \left(\frac{x}{2^n}\right)\right] = 0
\]

for all \(x \in X\). Since the mapping \(x \rightarrow C(2x) - 2C(x)\) is cubic (see Lemma 2.2 of [14]), from the equality \(C(2x) = 8C(x)\), we deduce that the mapping \(C : X \rightarrow Y\) is cubic. \(\square\)

**Corollary 4.3.** Let \(\mathcal{K}\) be a non-Archimedean field, let \(X\) be a vector space over \(\mathcal{K}\), and let \((Y, \mu, T)\) be a non-Archimedean random Banach space over \(\mathcal{K}\) under a \(t\)-norm \(T \in \mathcal{K}\). Let \(f : X \rightarrow Y\) be an odd and \(\Psi\)-approximately AQCQ mapping. If, for some \(\alpha \in \mathbb{R}, \alpha > 0\), and some integer \(k, k > 3\), with \(\|2^k\| < \alpha\),

\[
\Psi \left(2^{-k}x, 2^{-k}y, t\right) \geq \Psi (x, y, \alpha t) \quad (x \in X, \ t > 0),
\]

(4.31)

then there exists a unique cubic mapping \(C : X \rightarrow Y\) such that

\[
\mu_{f(x) - 2f(x/2) - C(x/2)}(t) \geq T^{\infty}_{\|t\|} M \left(x, \frac{\alpha t}{\|t\|^2}\right)
\]

(4.32)

for all \(x \in X\) and \(t > 0\).

**Proof.** Since

\[
\lim_{n \to \infty} M \left(x, \frac{\alpha^n t}{\|t\|^n}\right) = 1 \quad (x \in X, \ t > 0)
\]

(4.33)

and \(T\) is of Hadžić type, from Proposition 2.1, it follows that

\[
\lim_{n \to \infty} T^{\infty}_{\|t\|} M \left(x, \frac{\alpha^n t}{\|t\|^n}\right) = 1 \quad (x \in X, \ t > 0).
\]

(4.34)

Now, we can apply Theorem 4.2 to obtain the result. \(\square\)

**Example 4.4.** Let \((X, \mu, T_M)\) be non-Archimedean random normed space in which

\[
\mu_x(t) = \frac{t}{t + \|x\|} \quad (x \in X, \ t > 0).
\]

(4.35)
And let \((Y, \mu, T_M)\) be a complete non-Archimedean random normed space (see Example 3.2). Define

\[
\Psi(x, y, t) = \frac{t}{1 + t}.
\] (4.36)

It is easy to see that (4.3) holds for \(\alpha = 1\). Also, since

\[
M(x, t) = \frac{t}{1 + t'},
\] (4.37)

we have

\[
\lim_{n \to \infty} T_{M,j=n}^\infty \left( x, \frac{\alpha^j t}{|8|^k} \right) = \lim_{n \to \infty} \left( \lim_{m \to \infty} T_{M,j=n}^m \left( x, \frac{t}{|8|^k} \right) \right)
\]

\[
= \lim_{n \to \infty} \lim_{m \to \infty} \left( \frac{t}{t + |8|^n} \right)
\]

\[
= 1 \quad (x \in X, \ t > 0).
\] (4.38)

Let \(f : X \to Y\) be an odd and \(\Psi\)-approximately AQCQ mapping. Thus, all the conditions of Theorem 4.2 hold, and so there exists a unique cubic mapping \(C : X \to Y\) such that

\[
\mu f(x) - 2f(x/2) - C(x/2) \geq \frac{t}{1 + t'}.
\] (4.39)

**Theorem 4.5.** Let \(K\) be a non-Archimedean field, let \(X\) be a vector space over \(K\), and let \((Y, \mu, T)\) be a non-Archimedean random Banach space over \(K\). Let \(f : X \to Y\) be an odd mapping and \(\Psi\)-approximately AQCQ mapping. If for some \(\alpha \in \mathbb{R}, \alpha > 0,\) and some integer \(k, k > 1\) with \(|2^k| < \alpha\)

\[
\Psi\left(2^{-k}x, 2^{-k}y, t\right) \geq \Psi(x, y, \alpha t) \quad (x \in X, \ t > 0),
\]

\[
\lim_{n \to \infty} T_{j=n}^\infty M\left(2x, \frac{\alpha^j t}{2|k^j|}\right) = 1 \quad (x \in X, \ t > 0),
\] (4.40)

then there exists a unique additive mapping \(A : X \to Y\) such that

\[
\mu f(x) - 8f(x/2) - A(x/2) \geq T_{i=1}^\infty M\left(x, \frac{\alpha^{i+1} t}{|2|^{k^j}}\right)
\] (4.41)
for all $x \in X$ and $t > 0$, where

$$M(x, t) := T^{k-1} \left[ \Psi \left( \frac{x}{2}, \frac{x}{2}, \frac{t}{|4|} \right), \Psi \left( \frac{x}{2}, \frac{x}{2}, t \right), \ldots, \Psi \left( \frac{2^{k-1}x}{2}, \frac{2^{k-1}x}{2}, \frac{t}{|4|} \right), \Psi \left( \frac{2^{k-1}x}{2}, \frac{2^{k-1}x}{2}, t \right) \right]$$

$(x \in X, t > 0)$  

(4.42)

Proof. Letting $y := x/2$ and $g(x) := f(2x) - 8f(x)$ for all $x \in X$ in (4.9), we get

$$\mu_{g(x)-2g(x/2)}(t) \geq T \left( \Psi \left( \frac{x}{2}, \frac{x}{2}, \frac{t}{|4|} \right), \Psi \left( \frac{x}{2}, \frac{x}{2}, t \right) \right)$$

(4.43)

for all $x \in X$ and $t > 0$.

The rest of the proof is similar to the proof of Theorem 4.2. \qed

**Corollary 4.6.** Let $\mathcal{K}$ be a non-Archimedean field, let $X$ be a vector space over $\mathcal{K}$, and let $(Y, \mu, T)$ be a non-Archimedean random Banach space over $\mathcal{K}$ under a t-norm $T \in \mathcal{H}$. Let $f : X \to Y$ be an odd and $\Psi$-approximately AQCQ mapping. If, for some $\alpha \in \mathbb{R}$, $\alpha > 0$, and some integer $k$, $k > 1$, with $|2^k| < \alpha$,

$$\Psi \left( 2^{-k}x, 2^{-k}y, t \right) \geq \Psi \left( x, y, \alpha t \right) \quad (x \in X, t > 0),$$

(4.44)

then there exists a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-8f(x/2)-A(x/2)}(t) \geq T^{\infty}_{i=1} M \left( x, \frac{\alpha^{i+1}t}{|2|^{k^i}} \right)$$

(4.45)

for all $x \in X$ and $t > 0$.

Proof. Since

$$\lim_{n \to \infty} M \left( x, \frac{\alpha^i t}{|2|^{k^i}} \right) = 1 \quad (x \in X, t > 0)$$

(4.46)

and $T$ is of Hadžić type, from Proposition 2.1, it follows that

$$\lim_{n \to \infty} T^{\infty}_{i=n} M \left( x, \frac{\alpha^i t}{|2|^{k^i}} \right) = 1 \quad (x \in X, t > 0).$$

(4.47)

Now, we can apply Theorem 4.5 to obtain the result. \qed
Example 4.7. Let \((X, \mu, T_M)\) non-Archimedean random normed space in which

\[
\mu_x(t) = \frac{t}{t + \|x\|} \quad (x \in X, t > 0),
\]

and let \((Y, \mu, T_M)\) be a complete non-Archimedean random normed space (see Example 3.2). Define

\[
\Psi(x, y, t) = \frac{t}{1 + t}.
\]

It is easy to see that (4.3) holds for \(\alpha = 1\). Also, since

\[
M(x, t) = \frac{t}{1 + t},
\]

we have

\[
\lim_{n \to \infty} T^{\infty}_{M, j=n} M \left( x, \frac{\alpha t}{|2|^k} \right) = \lim_{n \to \infty} \left( \lim_{m \to \infty} T^{m}_{M, j=n} M \left( x, \frac{t}{|2|^k} \right) \right) = \lim_{n \to \infty} \lim_{m \to \infty} \left( \frac{t}{t + |2|^k} \right) = 1 \quad (x \in X, t > 0).
\]

Let \(f : X \to Y\) be an odd and \(\Psi\)-approximately AQCQ mapping. Thus, all the conditions of Theorem 4.2 hold, and so there exists a unique additive mapping \(A : X \to Y\) such that

\[
\mu_{f(x)-8f(x/2)-A(x/2)}(t) \geq \frac{t}{t + |2^k|}.
\]

5. Generalized Hyers-Ulam Stability of the Functional Equation (1.4) in Non-Archimedean Random Normed Spaces: An Even Case

Now, we prove the generalized Hyers-Ulam stability of the functional equation \(Df(x, y) = 0\) in non-Archimedean Banach spaces, an even case.

Theorem 5.1. Let \(\mathcal{K}\) be a non-Archimedean field, let \(X\) be a vector space over \(\mathcal{K}\), and let \((Y, \mu, T)\) be a non-Archimedean random Banach space over \(\mathcal{K}\). Let \(f : X \to Y\) be an even mapping, \(f(0) = 0\), and \(\Psi\)-approximately AQCQ mapping. If for some \(\alpha \in \mathbb{R}\), \(\alpha > 0\), and some integer \(k, k > 4\) with \(|2^k| < \alpha\),

\[
\Psi \left( 2^{-k}x, 2^{-k}y, t \right) \geq \Psi(x, y, \alpha t) \quad (x \in X, t > 0),
\]

\[
\lim_{n \to \infty} T^{\infty}_{M, j=n} M \left( 2x, \frac{\alpha t}{|2|^k} \right) = 1 \quad (x \in X, t > 0),
\]

(5.1)
then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-4f(x/2)-Q(x/2)}(t) \geq T_{i=1}^{\infty} M \left( x, \frac{\alpha^{i+1} t}{|16|^{k_i}} \right)$$

(5.2)

for all $x \in X$ and $t > 0$, where

$$M(x, t) := T^{k-1} \left[ \psi \left( \frac{x}{2}, \frac{x}{2}, \frac{t}{|4|} \right), \psi \left( \frac{x}{2}, \frac{t}{|4|} \right), \psi \left( \frac{2k-1}{2}, \frac{2k-1}{2}, \frac{t}{|4|} \right), \psi \left( \frac{2k-1}{2}, \frac{2k-1}{2}, \frac{t}{|4|} \right) \right]$$

(5.3)

Proof. Letting $x = y$ in (4.2), we get

$$\mu_{f(3y)-6f(2y)+15f(y)}(t) \geq \psi(y, y, t)$$

(5.4)

for all $y \in X$ and $t > 0$. Replacing $x$ by $2y$ in (4.2), we get

$$\mu_{f(4y)-4f(2y)+4f(y)}(t) \geq \psi(2y, y, t)$$

(5.5)

for all $y \in X$ and $t > 0$. By (5.4) and (5.5), we have

$$\mu_{f(4y)-20f(2y)+64f(y)}(t) \geq T \left( \mu_{f(3y)-6f(2y)+15f(y)}(t), \mu_{f(4y)-4f(2y)+4f(y)}(t) \right)$$

$$= T \left( \psi \left( \frac{t}{|4|} \right), \psi \left( 2y, y, t \right) \right)$$

(5.6)

for all $y \in X$ and $t > 0$. Letting $y := x/2$ and $g(x) := f(2x) - 4f(x)$ for all $x \in X$ in (5.6), we get

$$\mu_{g(x)-16g(x/2)}(t) \geq T \left( \psi \left( \frac{x}{2}, \frac{x}{2}, \frac{t}{|4|} \right), \psi \left( x, \frac{x}{2}, t \right) \right)$$

(5.7)

for all $x \in X$ and $t > 0$.

The rest of the proof is similar to the proof of Theorem 4.2.

\[\square\]

**Corollary 5.2.** Let $\mathcal{K}$ be a non-Archimedean field, let $X$ be a vector space over $\mathcal{K}$, and let $(Y, \mu, T)$ be a non-Archimedean random Banach space over $\mathcal{K}$ under a $t$-norm $T \in \mathcal{K}$. Let $f : X \to Y$ be an even, $f(0) = 0$, and $\Psi$-approximately AQCC mapping. If, for some $\alpha \in \mathbb{R}$, $\alpha > 0$, and some integer $k, k > 4, with |2^k| < \alpha$,

$$\psi \left( \frac{2^{-k} x, 2^{-k} y}{2^{-k}} t \right) \geq \psi (x, y, at) \quad (x \in X, t > 0),$$

(5.8)
then there exists a unique quartic mapping $Q : X \to Y$ such that

$$
\mu f(x) - 4f(x/2) - Q(x/2) (t) \geq T_{i=n}^{\infty} M\left( x, \frac{a^{i+1}t}{|16|^{ki}} \right)
$$

(5.9)

for all $x \in X$ and $t > 0$.

Proof. Since

$$
\lim_{n \to \infty} M\left( x, \frac{a^t}{|16|^{kj}} \right) = 1 \quad (x \in X, \ t > 0)
$$

(5.10)

and $T$ is of Hadžić type, from Proposition 2.1, it follows that

$$
\lim_{n \to \infty} T_{i=n}^{\infty} M\left( x, \frac{a^t}{|16|^{kj}} \right) = 1 \quad (x \in X, \ t > 0).
$$

(5.11)

Now, we can apply Theorem 5.1 to obtain the result.

Example 5.3. Let $(X, \mu, T_M)$ be non-Archimedean random normed space in which

$$
\mu_x(t) = \frac{t}{t + \|x\|} \quad (x \in X, \ t > 0).
$$

(5.12)

And let $(Y, \mu, T_M)$ be a complete non-Archimedean random normed space (see Example 3.2). Define

$$
\Psi(x, y, t) = \frac{t}{1 + t}.
$$

(5.13)

It is easy to see that (4.3) holds for $\alpha = 1$. Also, since

$$
M(x, t) = \frac{t}{1 + t},
$$

(5.14)

we have

$$
\lim_{n \to \infty} T_{M,j=n}^{\infty} M\left( x, \frac{a^t}{|16|^{kj}} \right) = \lim_{n \to \infty} \left( \lim_{m \to \infty} T_{M,j=n}^{\infty} M\left( x, \frac{t}{|16|^{k_j}} \right) \right)
= \lim_{n \to \infty} \lim_{m \to \infty} \left( \frac{t}{t + |16|^{k_j}} \right)
= 1 \quad (x \in X, \ t > 0).
$$

(5.15)
Let \( f : X \to Y \) be an even, \( f(0) = 0 \), and \( \Psi \)-approximately AQCQ mapping. Thus all the conditions of Theorem 5.1 hold, and so there exists a unique quartic mapping \( Q : X \to Y \) such that

\[
\mu_{f(x)-4f(x/2)-Q(x/2)}(t) \geq \frac{t}{t+|16^k|}. \tag{5.16}
\]

**Theorem 5.4.** Let \( \mathcal{K} \) be a non-Archimedean field, let \( X \) be a vector space over \( \mathcal{K} \) and let \( (Y, \mu, T) \) be a non-Archimedean random Banach space over \( \mathcal{K} \). Let \( f : X \to Y \) be an even mapping, \( f(0) = 0 \) and \( \Psi \)-approximately AQCQ mapping. If for some \( \alpha \in \mathbb{R} \), \( \alpha > 0 \), and some integer \( k \), \( k > 2 \) with \( |2^k| < \alpha \),

\[
\Psi \left( 2^{-k}x, 2^{-k}y, t \right) \geq \Psi (x, y, \alpha t) \quad (x \in X, \ t > 0),
\]

\[
\lim_{n \to \infty} T_{j=n}^\infty \left( 2x, \frac{\alpha^j t}{4^{|j|}} \right) = 1 \quad (x \in X, \ t > 0), \tag{5.17}
\]

then there exists a unique quadratic mapping \( Q : X \to Y \) such that

\[
\mu_{f(x)-16f(x/2)-Q(x/2)}(t) \geq T_{i=1}^\infty \left( x, \frac{\alpha^{i+1} t}{4^{|k|}} \right) \tag{5.18}
\]

for all \( x \in X \) and \( t > 0 \), where

\[
M(x, t) := \left[ \Psi \left( \frac{x}{2}, \frac{x}{2}, \frac{t}{|4|} \right), \Psi \left( \frac{x}{2}, \frac{x}{2}, t \right), \ldots, \Psi \left( \frac{x}{2}, \frac{x}{2}, t \right), \Psi \left( 2^{-k}x, \frac{2^{-k}x}{2}, \frac{t}{|4|} \right), \Psi \left( 2^{-k}x, \frac{2^{-k}x}{2}, t \right) \right].
\]

\[
(x \in X, \ t > 0). \tag{5.19}
\]

**Proof.** Letting \( y := x/2 \) and \( g(x) := f(2x) - 16f(x) \) for all \( x \in X \) in (5.6), we get

\[
\mu_{g(x)-4g(x/2)}(t) \geq T \left( \Psi \left( \frac{x}{2}, \frac{x}{2}, \frac{t}{|4|} \right), \Psi \left( \frac{x}{2}, \frac{x}{2}, t \right) \right) \tag{5.20}
\]

for all \( x \in X \) and \( t > 0 \).

The rest of the proof is similar to the proof of Theorem 5.1. \( \Box \)

**Corollary 5.5.** Let \( \mathcal{K} \) be a non-Archimedean field, let \( X \) be a vector space over \( \mathcal{K} \), and let \( (Y, \mu, T) \) be a non-Archimedean random Banach space over \( \mathcal{K} \) under a \( t \)-norm \( T \in \mathcal{K} \). Let \( f : X \to Y \) be an even, \( f(0) = 0 \), and \( \Psi \)-approximately AQCQ mapping. If, for some \( \alpha \in \mathbb{R} \), \( \alpha > 0 \), and some integer \( k \), \( k > 2 \), with \( |2^k| < \alpha \),

\[
\Psi \left( 2^{-k}x, 2^{-k}y, t \right) \geq \Psi (x, y, \alpha t) \quad (x \in X, \ t > 0), \tag{5.22}
\]
then there exists a unique quadratic mapping \( Q : X \to Y \) such that

\[
\mu f(x) - 16f(x/2) - Q(x/2)(t) \geq T_{i=1}^\infty M \left( x, \frac{\alpha^{i+1}t}{4|k^i|} \right)
\]  

(5.22)

for all \( x \in X \) and \( t > 0 \).

\textbf{Proof.} Since

\[
\lim_{n \to \infty} M \left( x, \frac{\alpha^nt}{4|k^j|} \right) = 1 \quad (x \in X, \ t > 0)
\]  

(5.23)

and \( T \) is of Hadžić type, from Proposition 2.1, it follows that

\[
\lim_{n \to \infty} T_{j=n}^\infty M \left( x, \frac{\alpha^nt}{4|k^j|} \right) = 1 \quad (x \in X, \ t > 0).
\]  

(5.24)

Now, we can apply Theorem 5.4 to obtain the result. \( \square \)

\textbf{Example 5.6.} Let \((X, \mu, T_M)\) be a non-Archimedean random normed space in which

\[
\mu_x(t) = \frac{t}{t + \|x\|} \quad (x \in X, \ t > 0).
\]  

(5.25)

And let \((Y, \mu, T_M)\) be a complete non-Archimedean random normed space (see Example 3.2). Define

\[
\Psi(x, y, t) = \frac{t}{1 + t}.
\]  

(5.26)

It is easy to see that (4.3) holds for \( \alpha = 1 \). Also, since

\[
M(x, t) = \frac{t}{1 + t},
\]  

(5.27)

we have

\[
\lim_{n \to \infty} M \left( x, \frac{\alpha^nt}{4|k^j|} \right) = \lim_{n \to \infty} \left( \lim_{m \to \infty} M \left( x, \frac{t}{4|k^m|} \right) \right)
\]  

\[
= \lim_{n \to \infty} \lim_{m \to \infty} \left( \frac{t}{t + |4k^n|} \right)
\]  

(5.28)

\[
= 1 \quad (x \in X, \ t > 0).
\]
Let $f : X \to Y$ be an even, $f(0) = 0$, and $\Psi$-approximately AQCQ mapping. Thus, all the conditions of Theorem 5.4 hold, and so there exists a unique quadratic mapping $Q : X \to Y$ such that

$$
\mu_{f(x) - 16f(x/2) - Q(x/2)}(t) \geq \frac{t}{t + |4^k|}.
$$

(5.29)

### 6. Latticetic Random Normed Space

Let $\mathcal{L} = (L, \geq_L)$ be a complete lattice, that is, a partially ordered set in which every nonempty subset admits supremum and infimum, and $0_L = \inf L, 1_L = \sup L$. The space of latticetic random distribution functions, denoted by $\Delta^*_L$, is defined as the set of all mappings $F : \mathbb{R} \cup \{-\infty, +\infty\} \to L$ such that $F$ is left continuous and nondecreasing on $\mathbb{R}$, $F(0) = 0_L, F(+\infty) = 1_L$.

$D^*_L \subseteq \Delta^*_L$ is defined as $D^*_L = \{F \in \Delta^*_L : l^-F(+\infty) = 1_L\}$, where $l^-f(x)$ denotes the left limit of the function $f$ at the point $x$. The space $\Delta^*_L$ is partially ordered by the usual pointwise ordering of functions, that is, $F \geq G$ if and only if $F(t) \geq_L G(t)$ for all $t$ in $\mathbb{R}$. The maximal element for $\Delta^*_L$ in this order is the distribution function given by

$$
\varepsilon_0(t) = \begin{cases} 
0_L, & \text{if } t \leq 0, \\
1_L, & \text{if } t > 0.
\end{cases}
$$

(6.1)

In Section 2, we defined $t$-norms on $[0,1]$, and now we extend $t$-norms on a complete lattice.

**Definition 6.1** (see [42]). A triangular norm ($t$-norm) on $L$ is a mapping $\triangledown : (L)^2 \to L$ satisfying the following conditions:

(a) (for all $x \in L$) $(\triangledown(x, 1_L) = x)$ (boundary condition);

(b) (for all $(x, y) \in (L)^2$) $(\triangledown(x, y) = \triangledown(y, x))$ (commutativity);

(c) (for all $(x, y, z) \in (L)^3$) $(\triangledown(x, \triangledown(y, z)) = \triangledown(\triangledown(x, y), z))$ (associativity);

(d) (for all $(x, x', y, y') \in (L)^4$) $(x \leq_L x' \text{ and } y \leq_L y' \Rightarrow \triangledown(x, y) \leq_L \triangledown(x', y'))$ (monotonicity).

Let $\{x_n\}$ be a sequence in $L$ converges to $x \in L$ (equipped order topology). The $t$-norm $\triangledown$ is said to be a continuous $t$-norm if

$$
\lim_{n \to \infty} \triangledown(x_n, y) = \triangledown(x, y)
$$

(6.2)

for all $y \in L$.

A $t$-norm $\triangledown$ can be extended (by associativity) in a unique way to an $n$-array operation taking for $(x_1, \ldots, x_n) \in L^n$ the value $\triangledown(x_1, \ldots, x_n)$ defined by

$$
\triangledown^0_{i=1} x_i = 1, \quad \triangledown^n_{i=1} x_i = \triangledown(\triangledown^{n-1}_{i=1} x_i, x_n) = \triangledown(x_1, \ldots, x_n).
$$

(6.3)
\( \mathcal{T} \) can also be extended to a countable operation taking for any sequence \((x_n)_{n \in \mathbb{N}}\) in \( L \) the value

\[
\mathcal{T}_i^{\infty} x_i = \lim_{n \to \infty} \mathcal{T}_i^n x_i. \tag{6.4}
\]

The limit on the right side of (6.4) exists since the sequence \((\mathcal{T}_i^n x_i)_{n \in \mathbb{N}}\) is nonincreasing and bounded from below.

Note that we put \( \mathcal{T} = T \) whenever \( L = [0,1] \). If \( T \) is a \( t \)-norm, then \( x^{(n)}_T \) is defined for every \( x \in [0,1] \) and \( n \in \mathbb{N} \) by 1 if \( n = 0 \) and \( T(x^{(n-1)}_T, x) \) if \( n \geq 1 \). A \( t \)-norm \( T \) is said to be of Hadzić type, (we denote by \( T \in \mathcal{A} \)) if the family \((x^{(n)}_T)_{n \in \mathbb{N}}\) is equicontinuous at \( x = 1 \) (cf. [27]).

**Definition 6.2** (see [42]). A continuous \( t \)-norm \( \mathcal{T} \) on \( L = [0,1]^2 \) is said to be **continuous \( t \)-representable** if there exist a continuous \( t \)-norm \( * \) and a continuous \( t \)-conorm \( \diamond \) on \( [0,1] \) such that, for all \( x = (x_1, x_2) \), \( y = (y_1, y_2) \in L \),

\[
\mathcal{T}(x, y) = (x_1 * y_1, x_2 \diamond y_2). \tag{6.5}
\]

For example,

\[
\mathcal{T}(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\}), \quad M(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\}) \tag{6.6}
\]

for all \( a = (a_1, a_2), b = (b_1, b_2) \in [0,1]^2 \) are continuous \( t \)-representable. Define the mapping \( \mathcal{T}_\lambda \) from \( L^2 \) to \( L \) by

\[
\mathcal{T}_\lambda(x, y) = \begin{cases} x, & \text{if } y \geq_1 x, \\ y, & \text{if } x \geq_1 y. \end{cases} \tag{6.7}
\]

Recall (see [27, 28]) that if \( \{x_n\} \) is a given sequence in \( L \), \((\mathcal{T}_\lambda)_{n=1}^\infty x_i \) is defined recurrently by \((\mathcal{T}_\lambda)_{i=1}^\infty x_i = x_1 \) and \((\mathcal{T}_\lambda)_{i=1}^\infty x_i = \mathcal{T}_\lambda((\mathcal{T}_\lambda)_{i=1}^{n-1} x_i, x_n) \) for all \( n \geq 2 \).

A negation on \( \mathcal{L} \) is any decreasing mapping \( \mathcal{N} : L \to L \) satisfying \( \mathcal{N}(0_L) = 1_L \) and \( \mathcal{N}(1_L) = 0_L \). If \( \mathcal{N}(\mathcal{N}(x)) = x \), for all \( x \in L \), then \( \mathcal{N} \) is called an **involutive negation**. In the following, \( \mathcal{L} \) is endowed with a (fixed) negation \( \mathcal{N} \).

**Definition 6.3.** A latticeic random normed space (in short LRN-space) is a triple \((X, \mu, \mathcal{T}_\lambda)\), where \( X \) is a vector space and \( \mu \) is a mapping from \( X \) into \( D^+_1 \) such that the following conditions hold:

- **(LRN1)** \( \mu_x(t) = \varepsilon_0(t) \) for all \( t > 0 \) if and only if \( x = 0 \),
- **(LRN2)** \( \mu_{ax}(t) = \mu_x(t/|a|) \) for all \( x \in X, \alpha \neq 0 \) and \( t \geq 0 \),
- **(LRN3)** \( \mu_{x+y}(t+s) \geq_L \mathcal{T}_\lambda(\mu_x(t), \mu_y(s)) \) for all \( x, y \in X \) and \( t, s \geq 0 \).

We note that from (LPN2) it follows that \( \mu_{-x}(t) = \mu_x(t) \) for all \( x \in X \) and \( t \geq 0 \).
Theorem 7.1. Let functional equation

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation.

An Odd Case via Fixed-Point Method

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in random Banach spaces: an odd case.

Theorem 7.1. Let $X$ be a linear space, let $(Y, \mu, \mathcal{T})$ be a complete LRN-space, and $\Phi$ let be a mapping from $X^2$ to $D^+_L (\Phi(x, y)$ is denoted by $\Phi_{x,y})$ such that, for some $0 < \alpha < 1/8$,

$$\Phi_{2x,2y}(t) \leq_L \Phi_{x,y}(at) \quad (x, y \in X, t > 0).$$
Let $f : X \rightarrow Y$ be an odd mapping satisfying

$$
\mu_{df(x,y)}(t) \geq L \Phi_{x,y}(t)
$$

(7.2)

for all $x, y \in X$ and $t > 0$. Then

$$
C(x) := \lim_{n \rightarrow \infty} 8^n \left( f \left( \frac{x}{2^n - 1} \right) - 2f \left( \frac{x}{2^n} \right) \right)
$$

(7.3)

exists for each $x \in X$ and defines a cubic mapping $C : X \rightarrow Y$ such that

$$
\mu_{f(2x)-2f(x)-C(x)}(t) \geq L T\Phi_{x,x} \left( \Phi_{x,x} \left( \frac{1 - 8\alpha}{5\alpha} t \right), \Phi_{2x,x} \left( \frac{1 - 8\alpha}{5\alpha} t \right) \right)
$$

(7.4)

for all $x \in X$ and $t > 0$.

Proof. Letting $x = y$ in (7.2), we get

$$
\mu_{f(3y)-4f(2y)+5f(y)}(t) \geq L \Phi_{y,y}(t)
$$

(7.5)

for all $y \in X$ and $t > 0$. Replacing $x$ by $2y$ in (7.2), we get

$$
\mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t) \geq L \Phi_{2y,y}(t)
$$

(7.6)

for all $y \in X$ and $t > 0$. By (7.5) and (7.6),

$$
\mu_{f(4y)-10f(2y)+16f(y)}(5t) \geq L T\Phi_{y,y} \left( \Phi_{y,y} \left( \frac{1 - 8\alpha}{5\alpha} t \right), \Phi_{2y,y} \left( \frac{1 - 8\alpha}{5\alpha} t \right) \right)
$$

(7.7)

for all $y \in X$ and $t > 0$. Letting $y := x/2$ and $g(x) := f(2x) - 2f(x)$ for all $x \in X$, we get

$$
\mu_{g(x)-8g(x/2)}(5t) \geq L T\Phi_{x/2,x/2} \left( \Phi_{x/2,x/2} \left( t \right), \Phi_{x,x} \left( t \right) \right)
$$

(7.8)

for all $x \in X$ and $t > 0$.

Consider the set

$$
S := \{ h : X \rightarrow Y, \ h(0) = 0 \}
$$

(7.9)

and introduce the generalized metric on $S$:

$$
d(h, k) = \inf \{ u \in \mathbb{R}^+ : \mu_{h(x)-k(x)}(ut) \geq L T\Phi_{x,x} \left( \Phi_{x,x} \left( t \right), \Phi_{2x,x} \left( t \right) \right), \ \forall x \in X, \ \forall t > 0 \}
$$

(7.10)
where, as usual, inf $\emptyset = +\infty$. It is easy to show that $(S, d)$ is complete (see the proof of Lemma 2.1 of [24]).

Now, we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := 8h\left(\frac{x}{2}\right)$$

(7.11)

for all $x \in X$, and we prove that $J$ is a strictly contractive mapping with the Lipschitz constant $8\alpha$.

Let $h, k \in S$ be given such that $d(h, k) < \varepsilon$. Then

$$\mu_{h(x) - k(x)}(\varepsilon t) \geq L_\varepsilon (\Phi_{x,x}(t), \Phi_{2x,x}(t))$$

(7.12)

for all $x \in X$ and $t > 0$. Hence

$$\mu_{Jh(x) - Jk(x)}(8\alpha t) = \mu_{8h(x/2) - 8k(x/2)}(8\alpha t)$$

$$= \mu_{h(x/2) - k(x/2)}(\alpha \varepsilon t)$$

$$\geq L_{\alpha \varepsilon} (\Phi_{x/2,x/2}(\alpha t), \Phi_{2x/2,x/2}(\alpha t))$$

$$\geq L_{\alpha \varepsilon} (\Phi_{x,x}(t), \Phi_{2x,x}(t))$$

(7.13)

for all $x \in X$ and $t > 0$. So, $d(h, k) < \varepsilon$ implies that

$$d(Jh, Jk) \leq \frac{\alpha}{8} \varepsilon.$$

(7.14)

This means that

$$d(Jh, Jk) \leq \frac{\alpha}{8} d(h, k)$$

(7.15)

for all $h, k \in S$. It follows from (7.8) that

$$\mu_{g(x) - g(x/2)}(5\alpha t) \geq L_{\alpha} (\Phi_{x,x}(t), \Phi_{2x,x}(t))$$

(7.16)

for all $x \in X$ and $t > 0$. So, $d(g, Jg) \leq 5\alpha \leq 5/8$.

By Theorem 1.1, there exists a mapping $C : X \to Y$ satisfying the following:

1. $C$ is a fixed point of $J$, that is,

$$C\left(\frac{x}{2}\right) = \frac{1}{8} C(x)$$

(7.17)

for all $x \in X$. Since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping $C$ is a unique fixed point of $J$ in the set

$$M = \{ h \in S : d(h, g) < \infty \}.$$
This implies that $C$ is a unique mapping satisfying (7.17) such that there exists a $u \in (0, \infty)$ satisfying

$$
\mu_{g(x)-C(x)}(ut) \geq 1 \mathcal{T}_n \left( (\Phi_{x,x}(t), \Phi_{2x,x}(t)) \right)
$$

(7.19)

for all $x \in X$ and $t > 0$.

(2) $d(f^n g, C) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$
\lim_{n \rightarrow \infty} 8^n g \left( \frac{x}{2^n} \right) = C(x)
$$

(7.20)

for all $x \in X$.

(3) $d(h, C) \leq (1/(1 - 8\alpha)) d(h, Jh)$ with $h \in M$, which implies the inequality

$$
d(g, C) \leq \frac{5\alpha}{1 - 8\alpha},
$$

(7.21)

from which it follows that

$$
\mu_{g(x)-C(x)} \left( \frac{5\alpha}{1 - 8\alpha} t \right) \geq 1 \mathcal{T}_n \left( (\Phi_{x,x}(t), \Phi_{2x,x}(t)) \right).
$$

(7.22)

This implies that the inequality (7.4) holds. From $Dg(x, y) = Df(2x, 2y) - 2Df(x, y)$, by (7.2), we deduce that

$$
\mu_{Df(2x, 2y)}(t) \geq 1 \mathcal{T}_n (\Phi_{2x,2y}(t)), \quad \mu_{-2Df(x, y)}(t) = \mu_{Df(2x, 2y)} \left( \frac{t}{2} \right) \geq 1 \mathcal{T}_n (\Phi_{x,y}(t)) \left( \frac{t}{2} \right)
$$

(7.23)

and so, by (LRN3) and (7.1), we obtain

$$
\mu_{Dg(x,y)}(3t) \geq 1 \mathcal{T}_n \left( \mu_{Df(2x,2y)}(t), \mu_{-2Df(x, y)}(2t) \right)
\geq 1 \mathcal{T}_n \left( (\Phi_{2x,2y}(t), \Phi_{x,y}(t)) \right) \geq 1 \mathcal{T}_n (\Phi_{2x,2y}(t)).
$$

(7.24)

It follows that

$$
\mu_{8^n Dg(x/2^n, y/2^n)}(3t) = \mu_{g(x/2^n, y/2^n)} \left( \frac{3}{8^n t} \right)
\geq \Phi_{x/2^{n-1}, y/2^{n-1}} \left( \frac{t}{8^n} \right) \geq 1 \mathcal{T}_n \left( (\Phi_{x,y}(t), \Phi_{x,y}(t)) \right) \geq 1 \mathcal{T}_n (\Phi_{2x,2y}) \left( \frac{t}{8^n} \right)^{n-1}
$$

(7.25)

for all $x, y \in X$, $t > 0$ and $n \in \mathbb{N}$.
Since \( \lim_{n \to \infty} \Phi_{x,y}(3/8(t/(8\alpha)^{n-1})) = 1 \) for all \( x, y \in X \) and \( t > 0 \), by Theorem 2.4, we deduce that

\[
\mu_{DC(x,y)}(3t) = 1 \tag{7.26}
\]

for all \( x, y \in X \) and \( t > 0 \). Thus the mapping \( C : X \to Y \) satisfies (1.4).

Now, we have

\[
C(2x) - 8C(x) = \lim_{n \to \infty} \left[ 8^n g\left( \frac{x}{2^{n-1}} \right) - 8^{n+1} g\left( \frac{x}{2^n} \right) \right] = 0 \tag{7.27}
\]

for all \( x \in X \). Since the mapping \( x \to C(2x) - 2C(x) \) is cubic (see Lemma 2.2 of [14]), from the equality \( C(2x) = 8C(x) \), we deduce that the mapping \( C : X \to Y \) is cubic.

**Corollary 7.2.** Let \( \theta \geq 0 \) and let \( p \) be a real number with \( p > 3 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \). Let \( f : X \to Y \) be an odd mapping satisfying

\[
\mu_{Df(x,y)}(2x - 2f(x) - C(x)) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{7.28}
\]

for all \( x, y \in X \) and \( t > 0 \). Note that \( (X, \mu, T_M) \) is a complete LRN-space, in which \( L = [0,1] \), then

\[
C(x) := \lim_{n \to \infty} 8^n \left( f\left( \frac{x}{2^{n-1}} \right) - 2f\left( \frac{x}{2^n} \right) \right) \tag{7.29}
\]

exists for each \( x \in X \) and defines a cubic mapping \( C : X \to Y \) such that

\[
\mu_{f(2x) - 2f(x) - C(x)}(t) \geq \frac{(2^p - 8)t}{(2^p - 8)t + 5(1 + 2^p)\theta\|x\|^p} \tag{7.30}
\]

for all \( x \in X \) and \( t > 0 \).

**Proof.** The proof follows from Theorem 7.1 by taking

\[
\Phi_{x,y}(t) := \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{7.31}
\]
for all \( x, y \in X \) and \( t > 0 \). Then we can choose \( \alpha = 2^{-p} \), and we get

\[
\mu_{f(2x)-2f(x)-C(x)}(t) \geq \min\left(\frac{(1-2^{3-p})t}{(1-2^{3-p})t + 5 \cdot 2^{-p}\theta(2\|x\|^p)}, \frac{(1-2^{3-p})t}{(1-2^{3-p})t + 5 \cdot 2^{-p}\theta(\|2x\|^p + \|x\|^p)}\right)
\]

\[
\geq \frac{(1-2^{3-p})t}{(1-2^{3-p})t + 5 \cdot 2^{-p}\theta(\|2x\|^p + \|x\|^p)}
\]

\[
= \frac{(2^p - 8)t}{(2^p - 8)t + 5 \cdot (2^p + 1)\theta\|x\|^p},
\]

which is the desired result. \( \square \)

**Theorem 7.3.** Let \( X \) be a linear space, let \( (Y, \mu, \mathcal{C}) \) be a complete LRN-space, and let \( \Phi \) be a mapping from \( X^2 \) to \( D^+_L \) \( (\Phi(x, y) \) is denoted by \( \Phi_{x,y}) \) such that, for some \( 0 < \alpha < 8 \),

\[
\Phi_{x/2,y/2}(t) \leq_L \Phi_{x,y}(at) \quad (x, y \in X, \ t > 0).
\]

Let \( f : X \to Y \) be an odd mapping satisfying (7.2), then

\[
C(x) := \lim_{n \to \infty} \frac{1}{8^n} \left(f\left(2^{n+1}x\right) - 2f(2^nx)\right)
\]

exists for each \( x \in X \) and defines a cubic mapping \( C : X \to Y \) such that

\[
\mu_{f(2x)-2f(x)-C(x)}(t) \geq_L \mathcal{T}_L\left(\Phi_{x,x}\left(\frac{8-\alpha}{5}t\right), \Phi_{2x,x}\left(\frac{8-\alpha}{5}t\right)\right)
\]

for all \( x \in X \) and \( t > 0 \).

**Proof.** Let \( (S, d) \) be the generalized metric space defined in the proof of Theorem 7.1.

Consider the linear mapping \( J : S \to S \) such that

\[
Jh(x) := \frac{1}{8}h(2x)
\]

for all \( x \in X \), and we prove that \( J \) is a strictly contractive mapping with the Lipschitz constant \( \alpha/8 \).

Let \( h, k \in S \) be given such that \( d(h, k) < \varepsilon \), then

\[
\mu_{h(x)-k(x)}(\varepsilon t) \geq_L \mathcal{T}_L\left(\Phi_{x,x}(t), \Phi_{2x,x}(t)\right)
\]
for all $x \in X$ and $t > 0$. Hence

\[
\mu_{h(x) - k(x)}\left(\frac{\alpha}{8} t\right) = \mu_{h(2x) - k(2x)}\left(\frac{\alpha}{8} t\right) = \mu_{h(2x) - k(2x)}(\alpha t) \\
\geq L \mathcal{T}_\alpha(\Phi_{2x,2x}(at), \Phi_{4x,2x}(at)) \\
\geq \mathcal{T}_\alpha(\Phi_{x,x}(t), \Phi_{2x,x}(t))
\]

(7.38)

for all $x \in X$ and $t > 0$. So, $d(h, k) < \varepsilon$ implies that

\[
d(Jh, Jk) \leq \frac{\alpha}{8} \varepsilon.
\]

(7.39)

This means that

\[
d(Jh, Jk) \leq \frac{\alpha}{8} d(h, k)
\]

(7.40)

for all $g, h \in S$. Letting $g(x) := f(2x) - 2f(x)$ for all $x \in X$, from (7.8), we get that

\[
\mu_{g(x) - (1/8)g(2x)}\left(\frac{5}{8} t\right) \geq L \mathcal{T}_\alpha(\Phi_{x,x}(t), \Phi_{2x,x}(t))
\]

(7.41)

for all $x \in X$ and $t > 0$. So, $d(g, Jg) \leq 5/8$.

By Theorem 1.1, there exists a mapping $C : X \to Y$ satisfying the following:

1. $C$ is a fixed point of $J$, that is,

\[
C(2x) = 8C(x)
\]

(7.42)

for all $x \in X$. Since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping $C$ is a unique fixed point of $J$ in the set

\[
M = \{ h \in S : d(h, g) < \infty \}.
\]

(7.43)

This implies that $C$ is a unique mapping satisfying (7.42) such that there exists a $u \in (0, \infty)$ satisfying

\[
\mu_{g(x) - C(x)}(ut) \geq L \mathcal{T}_\alpha(\Phi_{x,x}(t), \Phi_{2x,x}(t))
\]

(7.44)

for all $x \in X$ and $t > 0$.

2. $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equalit

\[
\lim_{n \to \infty} \frac{1}{8^n} g(2^n x) = C(x)
\]

(7.45)

for all $x \in X$. 
(3) $d(h, C) \leq (1/(1 - \alpha/8))d(h, Jh)$ for every $h \in M$, which implies the inequality

$$d(g, C) \leq \frac{5}{8 - \alpha},$$

(7.46)

from which it follows that

$$\mu_{g(x) - C(x)}(\frac{5}{8 - \alpha} t) \geq \mathcal{T}_C(\Phi_{x,x}(t), \Phi_{2x,x}(t))$$

(7.47)

for all $x \in X$ and $t > 0$. This implies that the inequality (7.35) holds.

From

$$\mu_{Dg(x,y)}(3t) \geq \mathcal{T}_C(\Phi_{2x,2y}(t), \Phi_{x,y}(t)) \geq \mathcal{T}_C(\Phi_{2x,2y}(t), \Phi_{x,y}(t))$$

(7.48)

by (7.33), we deduce that

$$\mu_{8^{-n}Dg(2^n x, 2^n y)}(3t) = \mu_{Dg(2^n x, 2^n y)}(3 \cdot 2^n t) \geq \mathcal{T}_C(\Phi_{2x,2y}(8^n t), \Phi_{x,y}(8^n t)) \geq \mathcal{T}_C(\Phi_{x,y}(8^n t), \Phi_{x,y}(\frac{t}{\alpha}))$$

(7.49)

for all $x, y \in X$, $t > 0$, and $n \in \mathbb{N}$. As $n \to \infty$, we deduce that

$$\mu_{DC(x,y)}(3t) = 1$$

(7.50)

for all $x, y \in X$ and $t > 0$. Thus the mapping $C : X \to Y$ satisfies (1.4).

Now, we have

$$C(2x) - 8C(x) = \lim_{n \to \infty} \left[ \frac{1}{8^n g(2^{n+1} x)} - \frac{1}{8^{n-1} g(2^n x)} \right] = \lim_{n \to \infty} \left[ \frac{1}{8^n g(2^{n+1} x)} - \frac{1}{8^{n-1} g(2^n x)} \right] = 0$$

(7.51)

for all $x \in X$. Since the mapping $x \to C(2x) - 2C(x)$ is cubic (see Lemma 2.2 of [14]), from the equality $C(2x) = 8C(x)$, we deduce that the mapping $C : X \to Y$ is cubic. 

\[ \square \]

**Corollary 7.4.** Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 3$. Let $X$ be a normed vector space with norm $\| \cdot \|$. Let $f : X \to Y$ be an odd mapping satisfying (7.28), then

$$C(x) := \lim_{n \to \infty} \frac{1}{8^n} \left( f(2^{n+1} x) - 2f(2^n x) \right)$$

(7.52)
exists for each \( x \in X \) and defines a cubic mapping \( C : X \to Y \) such that

\[
\mu_{f(2x)-2f(x)-C(x)}(t) \geq \frac{(8-2^p)t}{(8-2^p)t + 5(1+2^p)\theta\|x\|^p}
\]  

(7.53)

for all \( x \in X \) and \( t > 0 \). Note that \((X, \mu, T_M)\) is a complete LRN-space, in which \( L = [0, 1] \).

**Proof.** The proof follows from Theorem 7.3 by taking

\[
\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}
\]  

(7.54)

for all \( x, y \in X \) and \( t > 0 \). Then we can choose \( \alpha = 2^p \), and we get the desired result. \( \square \)

**Theorem 7.5.** Let \( X \) be a linear space, let \((Y, \mu, \mathcal{T}_\beta)\) be a complete LRN-space, and let \( \Phi \) be a mapping from \( X^2 \) to \( D_1^\alpha (\Phi(x, y) \text{ is denoted by } \Phi_{x,y}) \) such that, for some \( 0 < \alpha < 1/2 \),

\[
\Phi_{2x,2y}(t) \leq L \Phi_{x,y}(at) \quad (x, y \in X, \ t > 0).
\]  

(7.55)

Let \( f : X \to Y \) be an odd mapping satisfying (7.2), then

\[
A(x) := \lim_{n \to \infty} 2^n \left( f \left( \frac{x}{2^{n-1}} \right) - 8 f \left( \frac{x}{2^n} \right) \right)
\]  

(7.56)

exists for each \( x \in X \) and defines an additive mapping \( A : X \to Y \) such that

\[
\mu_{f(2x)-8f(x)-A(x)}(t) \geq \lambda \left( \Phi_{x,x} \left( \frac{1-2\alpha}{5\alpha} t \right), \Phi_{2x,2x} \left( \frac{1-2\alpha}{5\alpha} t \right) \right)
\]  

(7.57)

for all \( x \in X \) and \( t > 0 \).

**Proof.** Let \((S, d)\) be the generalized metric space defined in the proof of Theorem 7.1. Letting \( y := x/2 \) and \( g(x) := f(2x) - 8f(x) \) for all \( x \in X \) in (7.7), we get

\[
\mu_{g(x)-2g(x/2)}(5t) \geq \lambda \left( \Phi_{x/2,x/2}(t), \Phi_{x,x}(t) \right)
\]  

(7.58)

for all \( x \in X \) and \( t > 0 \).

Now, we consider the linear mapping \( J : S \to S \) such that

\[
Jh(x) := 2h \left( \frac{x}{2} \right)
\]  

(7.59)

for all \( x \in X \). It is easy to see that \( J \) is a strictly contractive self-mapping on \( S \) with the Lipschitz constant \( 2\alpha \).
It follows from (7.58) and (7.55) that
\[ \mu_{g(x)-2g(x/2)}(5at) \geq T_M(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \] \tag{7.60}
for all \( x \in X \) and \( t > 0 \). So, \( d(g, Jg) \leq 5\alpha < \infty \).

By Theorem 1.1, there exists a mapping \( A : X \to Y \) satisfying the following:

1. A is a fixed point of \( J \), that is,
\[ A\left(\frac{x}{2}\right) = \frac{1}{2} A(x) \] \tag{7.61}
for all \( x \in X \). Since \( g : X \to Y \) is odd, \( A : X \to Y \) is an odd mapping. The mapping \( A \) is a unique fixed point of \( J \) in the set
\[ M = \{ h \in \mathcal{S} : d(h, g) < \infty \}. \] \tag{7.62}
This implies that \( A \) is a unique mapping satisfying (7.61) such that there exists a \( u \in (0, \infty) \) satisfying
\[ \mu_{g(x)-A(x)}(ut) \geq L \mathcal{T}_x(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \] \tag{7.63}
for all \( x \in X \) and \( t > 0 \).

2. \( d(J^n g, A) \to 0 \) as \( n \to \infty \). This implies the equality
\[ \lim_{n \to \infty} 2^n g\left(\frac{x}{2^n}\right) = A(x) \] \tag{7.64}
for all \( x \in X \).

3. \( d(h, A) \leq (1/(1-2\alpha))d(h, Jh) \) for each \( h \in M \), which implies the inequality
\[ d(g, A) \leq \frac{5\alpha}{1-2\alpha}. \] \tag{7.65}
This implies that the inequality (7.57) holds. Since \( \mu_{Dg(x,y)}(3t) \geq L \Phi_{2x,2y}(t) \), it follows that
\[ \mu_{2^n Dg(x/2^n, y/2^n)}(3t) = \mu_{Dg(x/2^n, y/2^n)}\left(\frac{3t}{2^n}\right) \geq \Phi_{x/2^{n-1}, y/2^{n-1}} \left(\frac{t}{2^n}\right) \geq L \Phi_{x,y}\left(\frac{1}{2} \left(\frac{t}{(2\alpha)^{n-1}}\right)\right) \] \tag{7.66}
for all \( x, y \in X, t > 0, \) and \( n \in \mathbb{N}. \) As \( n \to \infty, \) we deduce that
\[
\mu_{DA(x,y)}(3t) = 1.2
\] (7.67)
for all \( x, y \in X \) and \( t > 0. \) Thus, the mapping \( A : X \to Y \) satisfies (1.4).

Now, we have
\[
A(2x) - 2A(x) = \lim_{n \to \infty} \left[ 2^n g \left( \frac{x}{2^{n-1}} \right) - 2^{n+1} g \left( \frac{x}{2^n} \right) \right]
\]
\[
= 2 \lim_{n \to \infty} \left[ 2^{n-1} g \left( \frac{x}{2^{n-1}} \right) - 2^n g \left( \frac{x}{2^n} \right) \right] = 0
\] (7.68)
for all \( x \in X. \) Since the mapping \( x \to A(2x) - 8A(x) \) is additive (see Lemma 2.2 of [14]), from the equality \( A(2x) = 2A(x), \) we deduce that the mapping \( A : X \to Y \) is additive. \( \square \)

**Corollary 7.6.** Let \( \theta \geq 0 \) and let \( p \) be a real number with \( p > 1. \) Let \( X \) be a normed vector space with norm \( \| \cdot \|. \) Let \( f : X \to Y \) be an odd mapping satisfying (7.28), then
\[
A(x) := \lim_{n \to \infty} 2^n \left( f \left( \frac{x}{2^{n-1}} \right) - 8f \left( \frac{x}{2^n} \right) \right)
\] (7.69)
exists for each \( x \in X \) and defines an additive mapping \( A : X \to Y \) such that
\[
\mu_{f(2x)-8f(x)-A(x)}(t) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 5(1 + 2^p)\theta \|x\|^p}
\] (7.70)
for all \( x \in X \) and \( t > 0, \) where \( (X, \mu, T_M) \) is a complete LRN-space in which \( L = [0,1]. \)

**Proof.** The proof follows from Theorem 7.5 by taking
\[
\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta (\|x\|^p + \|y\|^p)}
\] (7.71)
for all \( x, y \in X \) and \( t > 0. \) Then we can choose \( \alpha = 2^{-p}, \) and we get the desired result. \( \square \)

**Theorem 7.7.** Let \( X \) be a linear space, let \( (Y, \mu, T_M) \) be a complete LRN-space, and let \( \Phi \) be a mapping from \( X^2 \) to \( D_L^* (\Phi(x,y) \) is denoted by \( \Phi_{x,y}) \) such that, for some \( 0 < \alpha < 2, \)
\[
\Phi_{x,y}(at) \geq \Phi_{x/2,y/2}(t) \quad (x, y \in X, \ t > 0).
\] (7.72)

Let \( f : X \to Y \) be an odd mapping satisfying (7.2), then
\[
A(x) := \lim_{n \to \infty} \frac{1}{2^n} \left( f \left( 2^{n+1}x \right) - 8f \left( 2^n x \right) \right)
\] (7.73)
exists for each \( x \in X \) and defines an additive mapping \( A : X \to Y \) such that

\[
\mu_{f(2x) - 8f(x) - A(x)}(t) \geq L T \wedge \left( \Phi_{x,x}(t), \Phi_{2x,x}(t) \right)
\]

(7.74)

for all \( x \in X \) and \( t > 0 \).

**Proof.** Let \((S, d)\) be the generalized metric space defined in the proof of Theorem 7.1. Consider the linear mapping \( J : S \to S \) such that

\[
Jh(x) := \frac{1}{2} h(2x)
\]

(7.75)

for all \( x \in X \). It is easy to see that \( J \) is a strictly contractive self-mapping on \( S \) with the Lipschitz constant \( \alpha/2 \). Let \( g(x) = f(2x) - 8f(x) \), from (7.58), it follows that

\[
\mu_{g(x) - 1/2g(2x)} \left( \frac{5}{2} t \right) \geq L T \wedge \left( \Phi_{x,x}(t), \Phi_{2x,x}(t) \right)
\]

(7.76)

for all \( x \in X \) and \( t > 0 \). So, \( d(g, Jg) \leq 5/2 \). By Theorem 1.1, there exists a mapping \( A : X \to Y \) satisfying the following:

1. \( A \) is a fixed point of \( J \), that is,

\[
A(2x) = 2A(x)
\]

(7.77)

for all \( x \in X \). Since \( h : X \to Y \) is odd, \( A : X \to Y \) is an odd mapping. The mapping \( A \) is a unique fixed point of \( J \) in the set

\[
M = \{ h \in S : d(h, g) < \infty \}.
\]

(7.78)

This implies that \( A \) is a unique mapping satisfying (7.77) such that there exists a \( u \in (0, \infty) \) satisfying

\[
\mu_{g(x) - A(x)}(ut) \geq L T \wedge \left( \Phi_{x,x}(t), \Phi_{2x,x}(t) \right)
\]

(7.79)

for all \( x \in X \) and \( t > 0 \).

2. \( d(J^n g, A) \to 0 \) as \( n \to \infty \). This implies the equality

\[
\lim_{n \to \infty} \frac{1}{2^n} g(2^n x) = A(x)
\]

(7.80)

for all \( x \in X \).
(3) \( d(h, A) \leq (1/(1 - \alpha/2))d(h, Jh) \), which implies the inequality
\[
d(g, A) \leq \frac{5}{2 - \alpha}.
\] (7.81)

This implies that the inequality (7.74) holds.

Proceeding as in the proof of Theorem 7.5, we obtain that the mapping \( A : X \rightarrow Y \) satisfies (1.4). Now, we have
\[
A(2x) - 2A(x) = \lim_{n \to \infty} \left[ \frac{1}{2^n} g\left(2^{n+1}x\right) - \frac{1}{2^{n-1}} g\left(2^n x\right) \right]
\]
\[
= 2 \lim_{n \to \infty} \left[ \frac{1}{2^{n+1}} g\left(2^{n+1}x\right) - \frac{1}{2^n} g\left(2^n x\right) \right] = 0
\] (7.82)
for all \( x \in X \). Since the mapping \( x \rightarrow A(2x) - 8A(x) \) is additive (see Lemma 2.2 of [14]), from the equality \( A(2x) = 2A(x) \), we deduce that the mapping \( A : X \rightarrow Y \) is additive. \( \square \)

**Corollary 7.8.** Let \( \theta \geq 0 \) and let \( p \) be a real number with \( 0 < p < 1 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \). Let \( f : X \rightarrow Y \) be an odd mapping satisfying (7.28), then
\[
A(x) := \lim_{n \to \infty} \frac{1}{2^n} \left( f\left(2^{n+1}x\right) - 8f(2^n x) \right)
\] (7.83)
exists for each \( x \in X \) and defines an additive mapping \( A : X \rightarrow Y \) such that
\[
\mu_{f(2x)-8f(x)-A(x)}(t) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 5(1 + 2^p)\theta\|x\|^p}
\] (7.84)
for all \( x \in X \) and \( t > 0 \), where \( (X, \mu, T_M) \) is a complete LRN-space in which \( L = [0,1] \).

**Proof.** The proof follows from Theorem 7.7 by taking
\[
\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}
\] (7.85)
for all \( x, y \in X \) and \( t > 0 \). Then we can choose \( \alpha = 2^p \), and we get the desired result. \( \square \)

**8. Generalized Hyers-Ulam Stability of the Functional Equation** (1.4): An Even Case via Fixed-Point Method

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation \( Df(x, y) = 0 \) in random Banach spaces, an even case.
Let $X$ be a linear space, let $(Y, \mu, \mathcal{T})$ be a complete LRN-space, and let $\Phi$ be a mapping from $X^2$ to $D^+_L$ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 1/16$,

$$\Phi_{x,y}(at) \geq_L \Phi_{2x,2y}(t) \quad (x, y \in X, \ t > 0). \quad (8.1)$$

Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (7.2), then

$$Q(x) := \lim_{n \to \infty} 16^n \left( f \left( \frac{x}{2^n} \right) - 4f \left( \frac{x}{2^n} \right) \right) \quad (8.2)$$

exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$\mu_{f^2-4f}Q(t) \geq_L \tau_n \left( \Phi_{x,x} \left( \frac{1-16\alpha}{5\alpha} t \right), \Phi_{2x,2x} \left( \frac{1-16\alpha}{5\alpha} t \right) \right) \quad (8.3)$$

for all $x \in X$ and $t > 0$.

**Proof.** Letting $x = y$ in (7.2), we get

$$\mu_{f(3y)-6f(2y)+15f(y)}(t) \geq_L \Phi_{y,y}(t) \quad (8.4)$$

for all $y \in X$ and $t > 0$. Replacing $x$ by $2y$ in (7.2), we get

$$\mu_{f(4y)-4f(3y)+4f(2y)+4f(y)}(t) \geq_L \Phi_{2y,y}(t) \quad (8.5)$$

for all $y \in X$ and $t > 0$. By (8.4) and (8.5),

$$\mu_{f^2-20f}Q(t) \geq_L \tau_n \left( \mu_{f^2}(2x)+64f(x) \left( \frac{5t}{2} \right), \mu_{f^2}(3x)+6f(2x)+15f(x) \left( \frac{4t}{2} \right), \mu_{f^2}(4x)+4f(3x)+4f(2x)+4f(x) \left( \frac{t}{2} \right) \right) \quad (8.6)$$

for all $x \in X$ and $t > 0$. Letting $g(x) := f(2x) - 4f(x)$ for all $x \in X$, we get

$$\mu_{g^2}Q(t) \geq_L \tau_n \left( \Phi_{x,x/2}(t), \Phi_{x,x/2}(t) \right) \quad (8.7)$$

for all $x \in X$ and $t > 0$. Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 7.1.

Now we consider the linear mapping $J : S \to S$ such that $Jh(x) := 16h(x/2)$ for all $x \in X$. It is easy to see that $J$ is a strictly contractive self-mapping on $S$ with the Lipschitz constant $16\alpha$. It follows from (8.7) that

$$\mu_{g^2}Q(t) \geq_L \tau_n \left( \Phi_{x,x}(t), \Phi_{x,x}(t) \right) \quad (8.8)$$

for all $x \in X$ and $t > 0$. So,

$$d(g, Jg) \leq 5\alpha \leq \frac{5}{16} < \infty. \quad (8.9)$$
By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

1) $Q$ is a fixed point of $J$, that is,

$$Q\left(\frac{x}{2}\right) = \frac{1}{16}Q(x) \quad (8.10)$$

for all $x \in X$. Since $g : X \to Y$ is even with $g(0) = 0$, $Q : X \to Y$ is an even mapping with $Q(0) = 0$. The mapping $Q$ is a unique fixed point of $J$ in the set

$$M = \{ h \in S : d(h, g) < \infty \}. \quad (8.11)$$

This implies that $Q$ is a unique mapping satisfying (8.10) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{g(x)-Q(x)}(ut) \geq \lambda_{\lambda, (\Phi_{x,x}(t), \Phi_{2x,x}(t))} \quad (8.12)$$

for all $x \in X$ and $t > 0$.

2) $d(J^n g, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n \to \infty} 16^n g\left(\frac{x}{2^n}\right) = Q(x) \quad (8.13)$$

for all $x \in X$.

3) $d(h, Q) \leq (1/(1 - 16\alpha))d(h, Jh)$ for every $h \in M$, which implies the inequality

$$d(g, Q) \leq \frac{5\alpha}{1 - 16\alpha}. \quad (8.14)$$

This implies that the inequality (8.3) holds.

Proceeding as in the proof of Theorem 7.1, we obtain that the mapping $Q : X \to Y$ satisfies (1.4). Now, we have

$$Q(2x) - 16Q(x) = \lim_{n \to \infty} \left[ 16^n g\left(\frac{x}{2^n}\right) - 16^{n+1} g\left(\frac{x}{2^{n+1}}\right) \right]$$

$$= 16 \lim_{n \to \infty} \left[ 16^n g\left(\frac{x}{2^n}\right) - 16^{n+1} g\left(\frac{x}{2^{n+1}}\right) \right] = 0 \quad (8.15)$$

for all $x \in X$. Since the mapping $x \to Q(2x) - 4Q(x)$ is quartic, we get that the mapping $Q : X \to Y$ is quartic.

\[\square\]

**Corollary 8.2.** Let $\theta \geq 0$ and let $p$ be a real number with $p > 4$. Let $X$ be a normed vector space with norm $\| \cdot \|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (7.28), then

$$Q(x) := \lim_{n \to \infty} 16^n \left( f\left(\frac{x}{2^n}\right) - 4f\left(\frac{x}{2^n}\right) \right) \quad (8.16)$$
Theorem 8.3. Let \( x \in X \) and defines a quartic mapping \( Q : X \to Y \) such that

\[
\mu_{f(2x)-4f(x)-Q(x)}(t) \geq \frac{(2^p - 16)t}{(2^p - 16)t + 5(1 + 2^p)\theta\|x\|^p}
\] (8.17)

for all \( x \in X \) and \( t > 0 \), where \( (X, \mu, T_M) \) is a complete LRN-space in which \( L = [0, 1] \).

Proof. The proof follows from Theorem 8.1 by taking

\[
\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}
\] (8.18)

for all \( x, y \in X \) and \( t > 0 \). Then we can choose \( \alpha = 2^{-p} \), and we get the desired result. \( \square \)

Theorem 8.3. Let \( X \) be a linear space, let \( (Y, \mu, \mathcal{T}_\lambda) \) be a complete LRN-space, and let \( \Phi \) be a mapping from \( X^2 \) to \( D_L^+(\Phi(x,y)) \) is denoted by \( \Phi_{x,y} \) such that, for some \( 0 < \alpha < 16 \),

\[
\Phi_{x,y}(at) \geq \Phi_{x/2,y/2}(t) \quad (x, y \in X, \ t > 0).
\] (8.19)

Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (7.2), then

\[
Q(x) := \lim_{n \to \infty} \frac{1}{16^n} \left( f \left( 2^{n+1}x \right) - 4f(2^n x) \right)
\] (8.20)

exists for each \( x \in X \) and defines a quartic mapping \( Q : X \to Y \) such that

\[
\mu_{f(2x)-4f(x)-Q(x)}(t) \geq \mathcal{L} \mathcal{T}_{\lambda} \left( \Phi_{x,x} \left( \frac{16 - \alpha}{5} t \right), \Phi_{2x,x} \left( \frac{16 - \alpha}{5} t \right) \right)
\] (8.21)

for all \( x \in X \) and \( t > 0 \).

Proof. In the generalized metric space \((S,d)\) defined in the proof of Theorem 7.1, we consider the linear mapping \( J : S \to S \) such that

\[
Jh(x) := \frac{1}{16} h(2x)
\] (8.22)

for all \( x \in X \). It is easy to see that \( J \) is a strictly contractive self-mapping on \( S \) with the Lipschitz constant \( \alpha/16 \).

Letting \( g(x) := f(2x) - 4f(x) \) for all \( x \in X \), by (8.7), we get

\[
\mu_{g(x)-(1/16)g(2x)} \left( \frac{5}{16} t \right) \geq \mathcal{L} \mathcal{T}_{\lambda} \left( \Phi_{x,x}(t), \Phi_{2x,x}(t) \right)
\] (8.23)

for all \( x \in X \) and \( t > 0 \). So, \( d(g, Jg) \leq 5/16 \).
By Theorem 1.1, there exists a mapping $Q : X \rightarrow Y$ satisfying the following:

(1) $Q$ is a fixed point of $J$, that is,

$$Q(2x) = 16Q(x) \tag{8.24}$$

for all $x \in X$. Since $g : X \rightarrow Y$ is even with $g(0) = 0$, $Q : X \rightarrow Y$ is an even mapping with $Q(0) = 0$. The mapping $Q$ is a unique fixed point of $J$ in the set

$$M = \{ h \in S : d(h, g) < \infty \}. \tag{8.25}$$

This implies that $Q$ is a unique mapping satisfying (8.24) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{g(x)-Q(x)}(ut) \geq \mathcal{T}_{\lambda}(\Phi_{x,x}(t), \Phi_{2x,x}(t)) \tag{8.26}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n g, Q) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$\lim_{n \rightarrow \infty} \frac{1}{16^n} g(2^n x) = Q(x) \tag{8.27}$$

for all $x \in X$.

(3) $d(g, Q) \leq (16/(16 - \alpha))d(g, Jg)$ for each $h \in M$, which implies the inequality

$$d(g, Q) \leq 5/(16 - \alpha). \tag{8.28}$$

This implies that the inequality (8.21) holds.

Proceeding as in the proof of Theorem 7.3, we obtain that the mapping $Q : X \rightarrow Y$ satisfies (1.4). Now, we have

$$Q(2x) - 16Q(x) = \lim_{n \rightarrow \infty} \left[ \frac{1}{16^n} \theta^\ast \left( \frac{2^{n+1} x}{16^{n-1}} \right) - \frac{1}{16^n} \theta^\ast (2^n x) \right]$$

$$= 16 \lim_{n \rightarrow \infty} \left[ \frac{1}{16^n} \theta^\ast \left( \frac{2^{n+1} x}{16^{n+1}} \right) - \frac{1}{16^n} \theta^\ast (2^n x) \right] = 0 \tag{8.29}$$

for all $x \in X$. Since the mapping $x \rightarrow Q(2x) - 4Q(x)$ is quartic, we get that the mapping $Q : X \rightarrow Y$ is quartic. 

**Corollary 8.4.** Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 4$. Let $X$ be a normed vector space with norm $\| \cdot \|$. Let $f : X \rightarrow Y$ be an even mapping satisfying $f(0) = 0$ and (7.28), then

$$Q(x) := \lim_{n \rightarrow \infty} \frac{1}{16^n} \left( f \left( \frac{2^{n+1} x}{16^n} \right) - 4f(2^n x) \right) \tag{8.30}$$
exists for each \( x \in X \) and defines a quartic mapping \( Q : X \to Y \) such that

\[
\mu_{f(2x)-4f(x)-Q(x)}(t) \geq \frac{(16-2^p)t}{(16-2^p)t + 5(1+2^p)\theta\|x\|^p}
\]  

(8.31)

for all \( x \in X \) and \( t > 0 \), where \((X, \mu, T_M)\) is a complete LRN-space in which \( L = [0,1] \).

**Proof.** The proof follows from Theorem 8.3 by taking

\[
\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}
\]  

(8.32)

for all \( x, y \in X \) and \( t > 0 \). Then we can choose \( \alpha = 2^p \), and we get the desired result. \( \square \)

**Theorem 8.5.** Let \( X \) be a linear space, let \((Y, \mu, T_\lambda)\) be a complete LRN-space, and let \( \Phi \) be a mapping from \( X^2 \) to \( D^p_L(\Phi(x, y) \) is by denoted \( \Phi_{x,y} \) such that, for some \( 0 < \alpha < 1/4 \),

\[
\Phi_{x,y}(at) \geq L \Phi_{2x,2y}(t) \quad (x, y \in X, \ t > 0).
\]  

(8.33)

Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (7.2), then

\[
T(x) := \lim_{n \to \infty} 4^n \left( f\left( \frac{x}{2^{n+1}} \right) - 16f\left( \frac{x}{2^{n+1}} \right) \right)
\]  

(8.34)

exists for each \( x \in X \) and defines a quadratic mapping \( T : X \to Y \) such that

\[
\mu_{f(2x)-16f(x)-T(x)}(t) \geq L T(\Phi_{x,x}\left( \frac{1-4\alpha}{5\alpha}t \right), \Phi_{2x,x}\left( \frac{1-4\alpha}{5\alpha}t \right))
\]  

(8.35)

for all \( x \in X \) and \( t > 0 \).

**Proof.** Let \((S, d)\) be the generalized metric space defined in the proof of Theorem 7.1.

Letting \( g(x) := f(2x) - 16f(x) \) for all \( x \in X \) in (8.6), we get

\[
\mu_{g(x)-4g(x/2)}(5t) \geq L T(\Phi_{x,x/2}(t), \Phi_{x,x/2}(t))
\]  

(8.36)

for all \( x \in X \) and \( t > 0 \). It is easy to see that the linear mapping \( J : S \to S \) such that

\[
Jh(x) := 4h\left( \frac{x}{2} \right)
\]  

(8.37)

for all \( x \in X \), is a strictly contractive self-mapping with the Lipschitz constant \( 4\alpha \).

It follows from (8.36) that

\[
\mu_{g(x)-4g(x/2)}(5\alpha t) \geq L T(\Phi_{x,x}(t), \Phi_{2x,x}(t))
\]  

(8.38)

for all \( x \in X \) and \( t > 0 \). So, \( d(g, Jg) \leq 5\alpha < \infty \).
By Theorem 1.1, there exists a mapping \( T : X \to Y \) satisfying the following:

(1) \( T \) is a fixed point of \( J \), that is,

\[
T\left( \frac{x}{2} \right) = \frac{1}{4} T(x)
\]

for all \( x \in X \). Since \( g : X \to Y \) is even with \( g(0) = 0 \), \( T : X \to Y \) is an even mapping with \( T(0) = 0 \). The mapping \( T \) is a unique fixed point of \( J \) in the set \( M = \{ h \in S : d(h, g) < \infty \} \). This implies that \( T \) is a unique mapping satisfying (8.39) such that there exists a \( u \in (0, \infty) \) satisfying

\[
\mu_{g(x)-T(x)}(ut) \geq \lambda_\lambda(\Phi_{x,x}(t), \Phi_{2x,x}(t))
\]

for all \( x \in X \) and \( t > 0 \).

(2) \( d(f^n g, T) \to 0 \) as \( n \to \infty \). This implies the equality

\[
\lim_{n \to \infty} 4^n g\left( \frac{x}{2^n} \right) = T(x)
\]

for all \( x \in X \).

(3) \( d(h, T) \leq \frac{1}{1 - 4\alpha} d(h, Jh) \) for each \( h \in M \), which implies the inequality

\[
d(g, T) \leq \frac{5\alpha}{1 - 4\alpha}.
\]

This implies that the inequality (8.35) holds.

Proceeding as in the proof of Theorem 7.1, we obtain that the mapping \( T : X \to Y \) satisfies (1.4). Now, we have

\[
T(2x) - 4T(x) = \lim_{n \to \infty} \left[ 4^n g\left( \frac{x}{2^{n-1}} \right) - 4^{n+1} g\left( \frac{x}{2^n} \right) \right]
\]

\[
= 4 \lim_{n \to \infty} \left[ 4^{n-1} g\left( \frac{x}{2^{n-1}} \right) - 4^n g\left( \frac{x}{2^n} \right) \right] = 0
\]

for all \( x \in X \). Since the mapping \( x \to T(2x) - 16T(x) \) is quadratic, we get that the mapping \( T : X \to Y \) is quadratic. \( \square \)

**Corollary 8.6.** Let \( \theta \geq 0 \) and let \( p \) be a real number with \( p > 2 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \). Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (7.28), then

\[
T(x) := \lim_{n \to \infty} 4^n \left( f\left( \frac{x}{2^{n-1}} \right) - 16f\left( \frac{x}{2^n} \right) \right)
\]

(8.44)
exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that
\begin{equation}
\mu_{f(2x)-16f(x)-T(x)}(t) \geq \frac{(2^n - 4)t}{(2^n - 4)t + 5(1 + 2^n)\theta \|x\|^p}
\end{equation}
(8.45)
for all $x \in X$ and $t > 0$.

**Proof.** The proof follows from Theorem 8.5 by taking
\begin{equation}
\Phi_{x,y}(t) := \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}
\end{equation}
(8.46)
for all $x, y \in X$. Then we can choose $\alpha = 2^{-r}$, and we get the desired result. $\square$

**Theorem 8.7.** Let $X$ be a linear space, let $(Y, \mu, T, M)$ be a complete RN-space, and let $\Phi$ be a mapping from $X^2$ to $D^+$ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that, for some $0 < \alpha < 4$,
\begin{equation}
\Phi_{x,y}(at) \geq \Phi_{x/2,y/2}(t) \quad (x, y \in X, \ t > 0).
\end{equation}
(8.47)

Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (7.2), then
\begin{equation}
T(x) := \lim_{n \to \infty} \frac{1}{4^n} \left( f \left( 2^{n+1}x \right) - 16f(2^n x) \right)
\end{equation}
(8.48)
eexists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that
\begin{equation}
\mu_{f(2x)-16f(x)-T(x)}(t) \geq T_M \left( \Phi_{x,x} \left( \frac{4 - \alpha}{5} t \right), \Phi_{2x,x} \left( \frac{4 - \alpha}{5} t \right) \right)
\end{equation}
(8.49)
for all $x \in X$ and $t > 0$.

**Proof.** Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 7.1. It is easy to see that the linear mapping $J : S \to S$ such that
\begin{equation}
Jh(x) := \frac{1}{4} h(2x)
\end{equation}
(8.50)
for all $x \in X$ is a strictly contractive self-mapping with the Lipschitz constant $\alpha/4$.

Letting $g(x) := f(2x) - 16f(x)$ for all $x \in X$, from (8.36), we get
\begin{equation}
\mu_{g(x) - 1/4g(2x)} \left( \frac{5}{4} t \right) \geq T_M \left( \Phi_{x,x}(t), \Phi_{2x,x}(t) \right)
\end{equation}
(8.51)
for all $x \in X$ and $t > 0$. So, $d(g, Jg) \leq 5/4$. 
By Theorem 1.1, there exists a mapping $T : X \to Y$ satisfying the following:

(1) $T$ is a fixed point of $f$, that is,

$$T(2x) = 4T(x)$$ (8.52)

for all $x \in X$. Since $g : X \to Y$ is even with $g(0) = 0$, $T : X \to Y$ is an even mapping with $T(0) = 0$. The mapping $T$ is a unique fixed point of $f$ in the set

$$M = \{ h \in S : d(h, g) < \infty \}.$$ (8.53)

This implies that $T$ is a unique mapping satisfying (8.52) such that there exists a $u \in (0, \infty)$ satisfying

$$\mu_{g(x) - T(x)}(ut) \geq T_M(\Phi_{x, x}(t), \Phi_{2x, x}(t))$$ (8.54)

for all $x \in X$ and $t > 0$.

(2) $d(J^n g, T) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n \to \infty} \frac{1}{4^n} g(2^n x) = T(x)$$ (8.55)

for all $x \in X$.

(3) $d(h, T) \leq 1/(1 - \alpha/4))d(h, Jh)$ for each $h \in M$, which implies the inequality

$$d(g, T) \leq 5/(4 - \alpha).$$ (8.56)

This implies that the inequality (8.49) holds.

Proceeding as in the proof of Theorem 2.3, we obtain that the mapping $Q : X \to Y$ satisfies (1.4). Now, we have

$$T(2x) - 4T(x) = \lim_{n \to \infty} \frac{1}{4^n} g\left(2^{n+1} x\right) - \frac{1}{4^{n-1}} g\left(2^{n} x\right)$$

$$= 4 \lim_{n \to \infty} \frac{1}{4^n} g\left(2^{n+1} x\right) - \frac{1}{4^n} g\left(2^{n} x\right) = 0$$ (8.57)

for all $x \in X$. Since the mapping $x \to T(2x) - 16T(x)$ is quadratic, we get that the mapping $T : X \to Y$ is quadratic. \qed

**Corollary 8.8.** Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 2$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (7.28). Then

$$T(x) := \lim_{n \to \infty} \frac{1}{4^n} \left(f\left(2^{n+1} x\right) - 16f(2^n x)\right)$$ (8.58)
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exists for each \( x \in X \) and defines a quadratic mapping \( T : X \rightarrow Y \) such that

\[
\mu_{f(2x) - 16f(x) - T(x)}(t) \geq \frac{(4 - 2^p)t}{(4 - 2^p)t + 5(1 + 2^p)\theta\|x\|^p}
\] (8.59)

for all \( x \in X \) and \( t > 0 \), where \( (X, \mu, T_M) \) is a complete LRN-space in which \( L = [0, 1] \).

Proof. The proof follows from Theorem 8.5 by taking

\[
\Phi_{x,y}(t) := \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}
\] (8.60)

for all \( x, y \in X \) and \( t > 0 \). Then we can choose \( \alpha = 2^p \), and we get the desired result. \( \square \)

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