Review Article

Quasi-Contractive Mappings in Modular Metric Spaces

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We prove the existence of fixed point and uniqueness of quasi-contractive mappings in modular metric spaces which was introduced by Cirić

1. Introduction and Preliminaries

In this paper, we prove the existence and uniqueness of fixed points of quasi-contractive mappings in modular metric spaces which develop the theory of metric spaces generated by modulars. Throughout the paper $\mathcal{X}$ is a nonempty set and $\lambda > 0$. The notion of a metric modular was introduced by Chistyakov [1] as follows.

Definition 1.1. A function $\omega : (0, \infty) \times \mathcal{X} \times \mathcal{X} \to [0, \infty]$ is said to be a metric modular on $\mathcal{X}$ (or, simply, a modular if no ambiguity arises) if it satisfies three axioms:

(i) for any $x, y \in \mathcal{X}$, $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$;

(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$, and $x, y \in \mathcal{X}$;

(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y \in \mathcal{X}$.

Definition 1.2. Let $(\mathcal{X}, \omega)$ be a metric modular space.
(1) A sequence \( \{x_n\} \) in \( \mathcal{X}_\omega \) is said to be \( \omega \)-convergent to a point \( x \in \mathcal{X} \) if, for all \( \lambda > 0 \),

\[
\omega_\lambda(x_n, x) \longrightarrow 0 \tag{1.1}
\]

as \( n \to \infty \).

(2) A subset \( \mathcal{C} \) of \( \mathcal{X}_\omega \) is said to be \( \omega \)-closed if the \( \omega \)-limit of a \( \omega \)-convergent sequence of \( \mathcal{C} \) always belongs to \( \mathcal{C} \).

(3) A subset \( \mathcal{C} \) of \( \mathcal{X}_\omega \) is said to be \( \omega \)-complete if every \( \omega \)-Cauchy sequence in \( \mathcal{C} \) is \( \omega \)-convergent and its \( \omega \)-limit is in \( \mathcal{C} \).

**Definition 1.3.** The metric modular \( \omega \) is said to have the Fatou property if

\[
\omega_\lambda(x, y) \leq \liminf_{n \to \infty} \omega(x_n, y) \tag{1.2}
\]

for all \( y \in \mathcal{X}_\omega \) and \( \lambda \in (0, \infty) \), where \( \{x_n\} \) \( \omega \)-converges to \( x \).

### 2. Main Results

**Definition 2.1.** Let \( (\mathcal{X}, \omega) \) be a metric modular space, and let \( \mathcal{C} \) be a nonempty subset of \( \mathcal{X}_\omega \). The self-mapping \( T : \mathcal{C} \to \mathcal{C} \) is said to be quasi-contraction if there exists \( 0 < k < 1 \) such that

\[
\omega_\lambda(T(x), T(y)) \leq k \max\{\omega_\lambda(x, y), \omega_\lambda(x, T(x)), \omega_\lambda(y, T(y)), \omega_\lambda(x, T(y)), \omega_\lambda(T(x), y)\} \tag{2.1}
\]

for any \( x, y \in \mathcal{X} \) and \( \lambda \in (0, \infty) \).

Let \( T : \mathcal{C} \to \mathcal{C} \) be a mapping, and let \( \mathcal{C} \) be a nonempty subset of \( \mathcal{X}_\omega \). For any \( x \in \mathcal{C} \), define the orbit

\[
\mathcal{O}(x) = \{x, T(x), T^2(x), \ldots\} \tag{2.2}
\]

and its \( \omega \)-diameter by

\[
\delta_\omega(x) = \text{diam}(\mathcal{O}(x)) = \sup\{\omega_\lambda(T^n(x), T^m(x)) : n, m \in \mathbb{N}\}. \tag{2.3}
\]

**Lemma 2.2.** Let \( (\mathcal{X}, \omega) \) be a metric modular space, and let \( \mathcal{C} \) be a nonempty subset of \( \mathcal{X}_\omega \). Let \( T : \mathcal{C} \to \mathcal{C} \) be a quasi-contraction mapping, and let \( x \in \mathcal{C} \) be such that \( \delta_\omega(x) < \infty \). Then, for any \( n \geq 1 \), one has

\[
\delta_\omega(T(x)) \leq k^n \delta_\omega(x), \tag{2.4}
\]

where \( k \) is the constant associated with the mapping of \( T \). Moreover, one has

\[
\omega_\lambda(T^n(x), T^{n+m}(x)) \leq k^n \delta_\omega(x) \tag{2.5}
\]
for any $n, m \geq 1$ and $\lambda \in (0, \infty)$.

Proof. For each $n, m \geq 1$, we have

$$
\omega_\lambda(T^n(x), T^m(y)) \leq k \max \left\{ \omega_1\left(T^{n-1}(x), T^{m-1}(y)\right), \omega_1\left(T^n(x), T^{n-1}(x)\right), \omega_1\left(T^{m-1}(y), T^{m}(y)\right), \omega_1\left(T^n(x), T^{m}(y)\right), \omega_1\left(T^n(x), T^{m-1}(y)\right) \right\}
$$

(2.6)

for any $x, y \in \mathcal{C}$ and $\lambda \in (0, \infty)$. This obviously implies that

$$
\delta_\omega(T^n(x)) \leq k\delta_\omega(T^{n-1}(x))
$$

(2.7)

for any $n \geq 1$. Hence, for any $n \geq 1$, we have

$$
\delta_\omega(T^n(x)) \leq k^n\delta_\omega(x).
$$

(2.8)

Moreover, for any $n, m \geq 1$, we have

$$
\omega_\lambda(T^n(x), T^{n+m}(x)) \leq \delta_\omega(T^n(x)) \leq k^n\delta_\omega(x).
$$

(2.9)

This completes the proof. $\square$

The next lemma is helpful to prove the main result in this paper.

**Lemma 2.3.** Let $(X, \omega)$ be a modular metric space, and let $\mathcal{C}$ be a $\omega$-complete nonempty subset of $X_\omega$. Let $T: \mathcal{C} \to \mathcal{C}$ be quasi-contractive mapping, and let $x \in \mathcal{C}$ be such that $\delta_\omega(x) < \infty$. Then $\{T^n(x)\}$ $\omega$-converges to a point $\nu \in \mathcal{C}$. Moreover, one has

$$
\omega_\lambda(T^n(x) - \nu) \leq k^n\delta_\omega(x)
$$

(2.10)

for all $n \geq 1$ and $\lambda \in (0, \infty)$.

Proof. From Lemma 2.2, we know that $\{T^n(x)\}$ is a $\omega$-Cauchy sequence in $\mathcal{C}$. Since $\mathcal{C}$ is $\omega$-complete, then there exists $\nu \in \mathcal{C}$ such that $\{T^n(x)\}$ $\omega$-converges to $\nu$. Since

$$
\omega_\lambda(T^n(x), T^{n+m}(x)) \leq k^n\delta_\omega(x)
$$

(2.11)

for any $n, m \geq 1$ and $\omega$ satisfies the Fatou property, and letting $m \to \infty$, we have

$$
\omega_\lambda(T^n(x), \nu) \leq \liminf_{m \to \infty} \omega_\lambda(T^n(x), T^{n+m}(x)) \leq k^n\delta_\omega(x).
$$

(2.12)

This completes the proof. $\square$
Next, we prove that \( v \) is, in fact, a fixed point of \( T \) and it is unique provided some extra assumptions.

**Theorem 2.4.** Let \( T, \mathcal{E}, \) and \( x \) be as in Lemma 2.3. Suppose that \( \omega_1(v, T(v)) < \infty \) and \( \omega_1(x, T(x)) < \infty \) for all \( \lambda \in (0, \infty) \). Then the \( \omega \)-limit of \( \{T^n(x)\} \) is a fixed point of \( T \), that is, \( T(v) = v \). Moreover, if \( v^* \) is any fixed point of \( T \) in \( \mathcal{E} \) such that \( \omega_1(v, v^*) < \infty \) for all \( \lambda \in (0, \infty) \), then one has \( v = v^* \).

**Proof.** We have

\[
\omega_1(T(x), T(v)) \leq k \max\{\omega_1(x, v), \omega_1(x, T(x)), \omega_1(v, T(v)), \omega_1(x, T(v)), \omega_1(T(x), v)\}. \tag{2.13}
\]

From Lemma 2.3, it follows that

\[
\omega_1(T(x), T(v)) \leq k \max\{\delta_\omega(x), \omega_1(v, T(v)), \omega_1(x, T(v))\}. \tag{2.14}
\]

Suppose that, for each \( n \geq 1 \),

\[
\omega_1(T^n(x), T(v)) \leq \max\{k^n \delta_\omega(x), k \omega_1(v, T(v)), k^n \omega_1(x, T(v))\}. \tag{2.15}
\]

Then we have

\[
\omega_1(T^{n+1}(x), T(v)) \\
\leq k \max\{\omega_1(T^n(x), v), \omega_1(T^n(x), T^{n+1}(x)), \omega_1(v, T(v)), \omega_1(T^n(x), T(v)), \omega_1(T^{n+1}(x), v)\}. \tag{2.16}
\]

Hence we have

\[
\omega_1(T^{n+1}(x), T(v)) \leq k \max\{k^n \delta_\omega(x), k \omega_1(v, T(v)), \omega_1(T^n(x), T(v))\}. \tag{2.17}
\]

Using our previous assumption, we get

\[
\omega_1(T^{n+1}(x), T(v)) \leq \max\{k^{n+1} \delta_\omega(x), k \omega_1(v, T(v)), k^{n+1} \omega_1(x, T(v))\}. \tag{2.18}
\]

Thus, by induction, we have

\[
\omega_1(T^n(x), T(v)) \leq \max\{k^n \delta_\omega(x), k \omega_1(v, T(v)), k^n \omega_1(x, T(v))\} \tag{2.19}
\]

for any \( n \geq 1 \) and \( \lambda \in (0, \infty) \). Therefore, we have

\[
\limsup_{n \to \infty} \omega_1(T^n(x), T(x)) \leq \omega(v, T(v)) \tag{2.20}
\]
for all $\lambda \in (0, \infty)$. Using the Fatou property for the metric modular $\omega$, we get

$$\omega_1(v, T(v)) \lim_{n \to \infty} \omega_1(T^n(x), T(v)) \leq k\omega(v, T(v))$$

(2.21)

for all $\lambda \in (0, \infty)$. Since $k < 1$, we get $\omega_1(v, T(v)) = 0$ for all $\lambda \in (0, \infty)$, and so $T(v) = v$.

Let $v^\ast$ be another fixed point of $T$ such that $\omega_1(v, v^\ast) < \infty$ for all $\lambda \in (0, \infty)$. Then we have

$$\omega_1(v, v^\ast) = \omega_1(T(v), T(v^\ast)) \leq k\omega_1(v, v^\ast),$$

(2.22)

which implies that

$$\omega_1(v, v^\ast) = 0$$

(2.23)

for all $\lambda \in (0, \infty)$. Hence $v = v^\ast$. This complete the proof.

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\section*{References}

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