Research Article

Generalizations of N-Subalgebras in BCK/BCI-Algebras Based on Point N-Structures

Young Bae Jun,1 Kyoung Ja Lee,2 and Min Su Kang3

1 Department of Mathematics Education (and RINS), Gyeongsang National University, Jinju 660-701, Republic of Korea
2 Department of Mathematics Education, Hannam University, Daejeon 306-791, Republic of Korea
3 Department of Mathematics, Hanyang University, Seoul 133-791, Republic of Korea

Correspondence should be addressed to Min Su Kang, sinchangmyun@hanmail.net

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The aim of this article is to obtain more general forms than the papers of Jun et al. (2010); Jun et al. (in press). The notions of N-subalgebras of types (∈, qk), (∈, ∈ ∨ qk), and (q, ∈ ∨ qk) are introduced, and the concepts of qk-support and ∈ ∨ qk-support are also introduced. Several related properties are investigated. Characterizations of N-subalgebra of type (∈, ∈ ∨ qk) are discussed, and conditions for an N-subalgebra of type (∈, ∈ ∨ qk) to be an N-subalgebra of type (∈, ∈) are considered.

1. Introduction

A (crisp) set A in a universe X can be defined in the form of its characteristic function μA : X → {0, 1} yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalizations of the crisp set have been conducted on the unit interval [0, 1], and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fits the crisp point {1} into the interval [0, 1]. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [1] introduced a new function which is called negative-valued function and constructed N-structures. They applied N-structures to BCK/BCI-algebras and discussed N-subalgebras and N-ideals in BCK/BCI-algebras. Jun et al. [2] considered closed ideals in BCH-algebras based on N-structures. To obtain more general form of an N-subalgebra in BCK/BCI-algebras,
Jun et al. [3] defined the notions of \(\mathcal{A}\)-subalgebras of types \((e, e), (e, q), (e, \in \lor q), (q, e), (q, q)\), and \((q, e \in \lor q)\) and investigated related properties. They also gave conditions for an \(\mathcal{A}\)-structure to be an \(\mathcal{A}\)-subalgebra of type \((q, e \in \lor q)\). Jun et al. provided a characterization of an \(\mathcal{A}\)-subalgebra of type \((e, e \in \lor q)\) (see [3, 4]).

In this paper, we try to have more general form of the papers [3, 4]. We introduce the notions of \(\mathcal{A}\)-subalgebras of types \((e, q_k), (e, e \in \lor q_k), (q, e \in \lor q_k)\). We also introduce the concepts of \(q_k\)-support and \(e \in \lor q_k\)-support and investigate several properties. We discuss characterizations of \(\mathcal{A}\)-subalgebra of type \((e, e \in \lor q)\). We consider conditions for an \(\mathcal{A}\)-subalgebra of type \((e, e \in \lor q)\) to be an \(\mathcal{A}\)-subalgebra of type \((e, e)\). The important achievement of the study of \(\mathcal{A}\)-subalgebras of types \((e, q_k), (e, e \in \lor q_k), (q, e \in \lor q_k)\) is that the notions of \(\mathcal{A}\)-subalgebras of types \((e, q), (e, e \in \lor q), (q, e \in \lor q)\) are a special case of \(\mathcal{A}\)-subalgebras of types \((e, q_k), (e, e \in \lor q_k), (q, e \in \lor q_k)\), and thus so many results in the papers [3, 4] are corollaries of our results obtained in this paper.

2. Preliminaries

Let \(K(\tau)\) be the class of all algebras with type \(\tau = (2, 0)\). By a BCI-algebra, we mean a system \(X := (X, *, 0) \in K(\tau)\) in which the following axioms hold:

(i) \(((x * y) * (x * z)) * (z * y) = 0,\)

(ii) \((x * (x * y)) * y = 0,\)

(iii) \(x * x = 0,\)

(iv) \(x * y = y * x = 0 \Rightarrow x = y,\)

for all \(x, y, z \in X\). If a BCI-algebra \(X\) satisfies \(0 * x = 0\) for all \(x \in X\), then we say that \(X\) is a BCK-algebra. We can define a partial ordering \(\leq\) by

\[
(\forall x, y \in X) \quad (x \leq y \iff x * y = 0). \tag{2.1}
\]

In a BCK/BCI-algebra \(X\), the following hold:

(a1) \((\forall x \in X)(x * 0 = x),\)

(a2) \((\forall x, y, z \in X)((x * y) * z = (x * z) * y),\)

for all \(x, y, z \in X\).

A nonempty subset \(S\) of a BCK/BCI-algebras \(X\) is called a subalgebra of \(X\) if \(x * y \in S\) for all \(x, y \in S\). For our convenience, the empty set \(\emptyset\) is regarded as a subalgebra of \(X\).

We refer the reader to the books [5, 6] for further information regarding BCK/BCI-algebras.

For any family \(\{a_i \mid i \in \Lambda\}\) of real numbers, we define

\[
\bigvee \{a_i \mid i \in \Lambda\} := \begin{cases} 
\max \{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite}, \\
\sup \{a_i \mid i \in \Lambda\} & \text{otherwise}, 
\end{cases} \tag{2.2}
\]

\[
\bigwedge \{a_i \mid i \in \Lambda\} := \begin{cases} 
\min \{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite}, \\
\inf \{a_i \mid i \in \Lambda\} & \text{otherwise}.
\end{cases}
\]
Denote by \( \mathcal{F}(X, [-1, 0]) \) the collection of functions from a set \( X \) to \([-1, 0]\). We say that an element of \( \mathcal{F}(X, [-1, 0]) \) is a negative-valued function from \( X \) to \([-1, 0]\) (briefly, \( \mathcal{A} \)-function on \( X \)). By an \( \mathcal{A} \)-structure, we mean an ordered pair \((X, f)\) of \( X \) and an \( \mathcal{A} \)-function \( f \) on \( X \). In what follows, let \( X \) denote a BCK/BCI-algebras and \( f \) an \( \mathcal{A} \)-function on \( X \) unless otherwise specified.

**Definition 2.1** (see [1]). By a subalgebra of \( X \) based on \( \mathcal{A} \)-function \( f \) (briefly, \( \mathcal{A} \)-subalgebra of \( X \)), we mean an \( \mathcal{A} \)-structure \((X, f)\) in which \( f \) satisfies the following assertion:

\[
(\forall x, y \in X) \quad (f(x * y) \leq \bigvee \{f(x), f(y)\}).
\]  

(2.3)

For any \( \mathcal{A} \)-structure \((X, f)\) and \( t \in [-1, 0]\), the set

\[
C(f; t) := \{x \in X \mid f(x) \leq t\}
\]

(2.4)

is called a closed \( t \)-support of \((X, f)\), and the set

\[
O(f; t) := \{x \in X \mid f(x) < t\}
\]

(2.5)

is called an open \( t \)-support of \((X, f)\).

Using the similar method to the transfer principle in fuzzy theory (see [7, 8]), Jun et al. [2] considered transfer principle in \( \mathcal{A} \)-structures as follows.

**Theorem 2.2** (see [2]; \( \mathcal{A} \)-transfer principle). An \( \mathcal{A} \)-structure \((X, f)\) satisfies the property \( \overline{\mathcal{D}} \) if and only if for all \( \alpha \in [-1, 0] \),

\[
C(f; \alpha) \neq \emptyset \implies C(f; \alpha) \text{ satisfies the property } \mathcal{D}.
\]

(2.6)

**Lemma 2.3** (see [1]). An \( \mathcal{A} \)-structure \((X, f)\) is an \( \mathcal{A} \)-subalgebra of \( X \) if and only if every open \( t \)-support of \((X, f)\) is a subalgebra of \( X \) for all \( t \in [-1, 0] \).

### 3. General Form of \( \mathcal{A} \)-Subalgebras with Type \((\in, \in \vee q)\)

In what follows, let \( t \) and \( k \) denote arbitrary elements of \([-1, 0]\) and \((-1,0]\), respectively, unless otherwise specified.

Let \((X, f)\) be an \( \mathcal{A} \)-structure in which \( f \) is given by

\[
f(y) = \begin{cases} 
0 & \text{if } y \neq x, \\
t & \text{if } y = x.
\end{cases}
\]

(3.1)

In this case, \( f \) is denoted by \( x_t \), and we call \((X, x_t)\) a point \( \mathcal{A} \)-structure. For any \( \mathcal{A} \)-structure \((X, g)\), we say that a point \( \mathcal{A} \)-structure \((X, x_t)\) is an \( \mathcal{A}_\in \)-subset (resp., \( \mathcal{A}_q \)-subset) of \((X, g)\) if \( g(x) \leq t \) (resp., \( g(x) + t + 1 < 0 \)). If a point \( \mathcal{A} \)-structure \((X, x_t)\) is an \( \mathcal{A}_\in \)-subset of \((X, g)\) or an \( \mathcal{A}_q \)-subset of \((X, g)\), we say \((X, x_t)\) is an \( \mathcal{A}_{\in \vee q} \)-subset of \((X, g)\). We say that a point \( \mathcal{A} \)-structure
(X, x_i) is an \( \mathcal{A}_{q_k} \)-subset of (X, g) if \( g(x) + t - k + 1 < 0 \). Clearly, every \( \mathcal{A}_{q_k} \)-subset with \( k = 0 \) is an \( \mathcal{A}_q \)-subset. Note that if \( k, r \in (-1, 0] \) with \( k < r \), then every \( \mathcal{A}_{q_1} \)-subset is an \( \mathcal{A}_{q_r} \)-subset.

**Definition 3.1.** An \( \mathcal{A} \)-structure \( (X, f) \) is called an \( \mathcal{A} \)-subalgebra of type \( \mathcal{A} \) if

(i) \((\varepsilon, \varepsilon) \) (resp., \((\varepsilon, q) \) and \((\varepsilon, \varepsilon \lor q) \) if whenever two point \( \mathcal{A} \)-structures \( (X, x_i) \) and \( (X, y_i) \) are \( \mathcal{A}_\varepsilon \)-subsets of \( (X, f) \), then the point \( \mathcal{A} \)-structure \( (X, (x \land y \lor \{i, l\}_2)) \) is an \( \mathcal{A}_\varepsilon \)-subset (resp., \( \mathcal{A}_q \)-subset and \( \mathcal{A}_{\lor_q} \)-subset) of \( (X, f) \).

(ii) \((q, \varepsilon) \) (resp., \((q, q) \) and \((q, \varepsilon \lor q) \) if whenever two point \( \mathcal{A} \)-structures \( (X, x_i) \) and \( (X, y_i) \) are \( \mathcal{A}_q \)-subsets of \( (X, f) \), then the point \( \mathcal{A} \)-structure \( (X, (x \land y \lor \{i, l\}_2)) \) is an \( \mathcal{A}_q \)-subset (resp., \( \mathcal{A}_q \)-subset and \( \mathcal{A}_{\lor_q} \)-subset) of \( (X, f) \).

**Definition 3.2.** An \( \mathcal{A} \)-structure \( (X, f) \) is called an \( \mathcal{A} \)-subalgebra of type \( \mathcal{A} \) if whenever two point \( \mathcal{A} \)-structures \( (X, x_i) \) and \( (X, y_i) \) are \( \mathcal{A}_\varepsilon \)-subsets (resp., \( \mathcal{A}_q \)-subsets) of \( (X, f) \), then the point \( \mathcal{A} \)-structure \( (X, (x \land y \lor \{i, l\}_2)) \) is an \( \mathcal{A}_\varepsilon \)-subset (resp., \( \mathcal{A}_q \)-subset and \( \mathcal{A}_{\lor_q} \)-subset) of \( (X, f) \).

**Example 3.3.** Consider a BCI-algebra \( X = \{0, a, b, c\} \) with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Let \((X, f)\) be an \( \mathcal{A} \)-structure in which \( f \) is defined by

\[
f = \begin{pmatrix}
0 & a & b & c \\
-0.6 & -0.7 & -0.3 & -0.3
\end{pmatrix}
\]  

(3.2)

It is routine to verify that \((X, f)\) is an \( \mathcal{A} \)-subalgebra of type \((\varepsilon, \varepsilon \lor q_0.2)\).

Note that if \( k, r \in (-1, 0] \) with \( k < r \), then every \( \mathcal{A} \)-subalgebra of type \((\varepsilon, \varepsilon \lor q_k)\) is an \( \mathcal{A} \)-subalgebra of type \((\varepsilon, \varepsilon \lor q_r)\), but the converse is not true as seen in the following example.

**Example 3.4.** The \( \mathcal{A} \)-subalgebra \((X, f)\) of type \((\varepsilon, \varepsilon \lor q_0.2)\) in Example 3.3 is not of type \((\varepsilon, \varepsilon \lor q_0.4)\) since \((X, a \cdot 0.65)\) and \((X, a \cdot 0.68)\) are \( \mathcal{A}_\varepsilon \)-subsets of \((X, f)\), but

\[
(X, (a \cdot a) \lor \{-0.65, -0.68\})
\]  

is not an \( \mathcal{A}_{q_0.4} \)-subset of \((X, f)\).

**Theorem 3.5.** Every \( \mathcal{A} \)-subalgebra of type \((\varepsilon, \varepsilon)\) is of type \((\varepsilon, \varepsilon \lor q_k)\).

**Proof.** Straightforward.

Taking \( k = 0 \) in Theorem 3.5 induces the following corollary.
Corollary 3.6. Every \( \mathcal{A} \)-subalgebra of type \((\varepsilon, \varepsilon)\) is of type \((\varepsilon, \varepsilon \lor q)\).

The converse of Theorem 3.5 is not true as seen in the following example.

Example 3.7. Consider the \( \mathcal{A} \)-subalgebra \((X, f)\) of type \((\varepsilon, \varepsilon \lor q_{0.2})\) which is given in Example 3.3. Then \((X, f)\) is not an \( \mathcal{A} \)-subalgebra of type \((\varepsilon, \varepsilon)\) since \((X, a_{0.65})\) and \((X, a_{0.68})\) are \( \mathcal{A}_e \)-subsets of \((X, f)\), but \((X, (a \ast a)_{\lor (-0.65, -0.68)})\) is not an \( \mathcal{A}_e \)-subset of \((X, f)\).

Definition 3.8. An \( \mathcal{A} \)-structure \((X, f)\) is called an \( \mathcal{A} \)-subalgebra of type \((\varepsilon, q_k)\) if whenever two point \( \mathcal{A} \)-structure \((X, x_i)\) and \((X, y_i)\) are \( \mathcal{A}_e \)-subsets of \((X, f)\) then the point \( \mathcal{A} \)-structure \((X, (x \ast y)_{\lor (i_1, i_2)})\) is an \( \mathcal{A}_{q_k} \)-subset of \((X, f)\).

Theorem 3.9. Every \( \mathcal{A} \)-subalgebra of type \((\varepsilon, q_k)\) is of type \((\varepsilon, \varepsilon \lor q_k)\).

Proof. Straightforward. \( \square \)

Taking \( k = 0 \) in Theorem 3.9 induces the following corollary.

Corollary 3.10. Every \( \mathcal{A} \)-subalgebra of type \((\varepsilon, q)\) is of type \((\varepsilon, \varepsilon \lor q)\).

The converse of Theorem 3.9 is not true as seen in the following example.

Example 3.11. Consider the \( \mathcal{A} \)-subalgebra \((X, f)\) of type \((\varepsilon, \varepsilon \lor q_{0.2})\) which is given in Example 3.3. Then \((X, a_{0.65})\) and \((X, b_{0.25})\) are \( \mathcal{A} \)-subsets of \((X, f)\), but

\[
\left( X, (a \ast b)_{\lor (-0.65, -0.25)} \right) = (X, c_{0.2})
\]  \quad (3.5)

is not an \( \mathcal{A}_{q_k} \)-subset of \((X, f)\) for \( k = -0.2 \) since \( f(c) - 0.25 - 0.2 + 1 > 0 \).

We consider a characterization of an \( \mathcal{A} \)-subalgebra of type \((\varepsilon, \varepsilon \lor q_k)\).

Theorem 3.12. An \( \mathcal{A} \)-structure \((X, f)\) is an \( \mathcal{A} \)-subalgebra of type \((\varepsilon, \varepsilon \lor q_k)\) if and only if it satisfies

\[
(\forall x, y \in X) \quad \left( f(x \ast y) \leq \sqrt{f(x) f(y) + \frac{k-1}{2}} \right).
\]  \quad (3.6)

Proof. Let \((X, f)\) be an \( \mathcal{A} \)-structure of type \((\varepsilon, \varepsilon \lor q_k)\). Assume that (3.6) is not valid. Then there exists \( a, b \in X \) such that

\[
f(a \ast b) > \sqrt{f(a) f(b) + \frac{k-1}{2}}.
\]  \quad (3.7)

If \( \sqrt{f(a) f(b)} > (k - 1)/2 \), then \( f(a \ast b) > \sqrt{f(a) f(b)} \). Hence

\[
f(a \ast b) > t \geq \sqrt{f(a) f(b)}
\]  \quad (3.8)
for some \( t \in [-1, 0) \). It follows that point \( \mathcal{N} \)-structures \((X, a_i)\) and \((X, b_i)\) are \( \mathcal{N}_e \)-subsets of \((X, f)\), but the point \( \mathcal{N} \)-structure \((X, (a * b)_1)\) is not an \( \mathcal{N}_e \)-subset of \((X, f)\). Moreover,

\[
 f(a * b) + t - k + 1 > 2t - k + 1 = 0,
\]

and so \((X, (a * b)_i)\) is not an \( \mathcal{N}_q \)-subset of \((X, f)\). Consequently, \((X, (a * b)_i)\) is not an \( \mathcal{N}_{\ev q} \)-subset of \((X, f)\). This is a contradiction. If \( \sqrt{|f(a), f(b)|} \leq (k - 1)/2 \), then \( f(a) \leq (k - 1)/2, f(b) \leq (k - 1)/2 \) and \( f(a * b) > (k - 1)/2 \). Thus \((X, a_{(k-1)/2})\) and \((X, b_{(k-1)/2})\) are \( \mathcal{N}_e \)-subsets of \((X, f)\), but \((X, (a * b)_{(k-1)/2})\) is not an \( \mathcal{N}_e \)-subset of \((X, f)\). Also,

\[
 f(a * b) + \frac{k - 1}{2} - k + 1 > \frac{k - 1}{2} + \frac{k - 1}{2} - k + 1 = 0,
\]

that is, \((X, (a * b)_{(k-1)/2})\) is not an \( \mathcal{N}_q \)-subset of \((X, f)\). Hence \((X, (a * b)_{(k-1)/2})\) is not an \( \mathcal{N}_{\ev q} \)-subset of \((X, f)\), a contradiction. Therefore (3.6) is valid.

Conversely, suppose that (3.6) is valid. Let \( x, y \in X \) and \( t_1, t_2 \in [-1, 0) \) be such that two point \( \mathcal{N} \)-structures \((X, x_i)\) and \((X, y_i)\) are \( \mathcal{N}_e \)-subsets of \((X, f)\). Then

\[
 f(x * y) \leq \sqrt{\left\{ f(x), f(y), \frac{k - 1}{2} \right\}} \leq \sqrt{\left\{ t_1, t_2, \frac{k - 1}{2} \right\}}.
\]

Assume that \( t_1 \geq (k - 1)/2 \) or \( t_2 \geq (k - 1)/2 \). Then \( f(x * y) \leq \sqrt{t_1, t_2} \), and so \((X, (x * y)_{\sqrt{t_1, t_2}})\) is an \( \mathcal{N}_e \)-subset of \((X, f)\). Now suppose that \( t_1 < (k - 1)/2 \) and \( t_2 < (k - 1)/2 \). Then \( f(x * y) \leq (k - 1)/2 \), and thus

\[
 f(x * y) + \sqrt{t_1, t_2} - k + 1 < \frac{k - 1}{2} + \frac{k - 1}{2} - k + 1 = 0,
\]

that is, \((X, (x * y)_{\sqrt{t_1, t_2}})\) is an \( \mathcal{N}_q \)-subset of \((X, f)\). Therefore \((X, (x * y)_{\sqrt{t_1, t_2}})\) is an \( \mathcal{N}_{\ev q} \)-subset of \((X, f)\) and consequently \((X, f)\) is an \( \mathcal{N} \)-subalgebra of type \((\varepsilon, \in \lor q)\).

**Corollary 3.13** (see [3]). An \( \mathcal{N} \)-structure \((X, f)\) is an \( \mathcal{N} \)-subalgebra of type \((\varepsilon, \in \lor q)\) if and only if it satisfies

\[
 (\forall x, y \in X) \quad f(x * y) \leq \sqrt{\left\{ f(x), f(y), -0.5 \right\}}.
\]

**Proof.** It follows from taking \( k = 0 \) in Theorem 3.12.

We provide conditions for an \( \mathcal{N} \)-structure to be an \( \mathcal{N} \)-subalgebra of type \((q, \in \lor q)\).

**Theorem 3.14.** Let \( S \) be a subalgebra of \( X \) and let \((X, f)\) be an \( \mathcal{N} \)-structure such that

(a) \( (\forall x \in X)(x \in S \Rightarrow f(x) \leq (k - 1)/2) \),

(b) \( (\forall x \in X)(x \notin S \Rightarrow f(x) = 0) \).

Then \((X, f)\) is an \( \mathcal{N} \)-subalgebra of type \((q, \in \lor q)\).
Proof. Let \( x, y \in X \) and \( t_1, t_2 \in [-1, 0) \) be such that two point \( \mathcal{A} \)-structures \((X, x_0)\) and \((X, y_0)\) are \( \mathcal{A}_k\)-subsets of \((X, f)\). Then \( f(x) + t_1 + 1 < 0 \) and \( f(y) + t_2 + 1 < 0 \). Thus \( x \ast y \in S \) because if it is impossible, then \( x \notin S \) or \( y \notin S \). Thus \( f(x) = 0 \) or \( f(y) = 0 \), and so \( t_1 < -1 \) or \( t_2 < -1 \). This is a contradiction. Hence \( f(x \ast y) \leq (k-1)/2 \). If \( \cup \{t_1, t_2\} < (k-1)/2 \), then \( f(x \ast y) + \cup \{t_1, t_2\} - k + 1 < (k-1)/2 + (k-1)/2 - k + 1 = 0 \) and so the point \( \mathcal{A}\)-structure \((X, (x \ast y)_{\cup \{t_1, t_2\}})\) is an \( \mathcal{A}_k\)-subset of \((X, f)\). If \( \cup \{t_1, t_2\} \geq (k-1)/2 \), then \( f(x \ast y) \leq (k-1)/2 \leq \cup \{t_1, t_2\} \) and so the point \( \mathcal{A}\)-structure \((X, (x \ast y)_{\cup \{t_1, t_2\}})\) is an \( \mathcal{A}_e\)-subset of \((X, f)\). Therefore the point \( \mathcal{A}\)-structure \((X, (x \ast y)_{\cup \{t_1, t_2\}})\) is an \( \mathcal{A}_{e\cup k}\)-subset of \((X, f)\). This shows that \((X, f)\) is an \( \mathcal{A}\)-subalgebra of type \((q_2, \in \cup q_k)\).  

Taking \( k = 0 \) in Theorem 3.14, we have the following corollary.

**Corollary 3.15** (see [3]). Let \( S \) be a subalgebra of \( X \) and let \((X, f)\) be an \( \mathcal{A}\)-structure such that

1. \((\forall x \in X)(x \in S \Rightarrow f(x) \leq -0.5)\),
2. \((\forall x \in X)(x \notin S \Rightarrow f(x) = 0)\).

Then \((X, f)\) is an \( \mathcal{A}\)-subalgebra of type \((q_2, \in \cup q_k)\).

**Theorem 3.16.** Let \((X, f)\) be an \( \mathcal{A}\)-subalgebra of type \((q_2, \in \cup q_k)\). If \( f \) is not constant on the open 0-support of \((X, f)\), then \( f(x) \leq (k-1)/2 \) for some \( x \in X \). In particular, \( f(0) \leq (k-1)/2 \).

Proof. Assume that \( f(x) > (k-1)/2 \) for all \( x \in X \). Since \( f \) is not constant on the open 0-support of \((X, f)\), there exists \( x \in O(f; 0) \) such that \( t_x = f(x) \neq f(0) = t_0 \). Then either \( t_0 < t_x \) or \( t_0 > t_x \). For the case \( t_0 < t_x \), choose \( r < (k-1)/2 \) such that \( t_0 + t_x - r - k + 1 < 0 < t_x + r - k + 1 \). Then the point \( \mathcal{A}\)-structure \((X, 0, r)\) is an \( \mathcal{A}_k\)-subset of \((X, f)\). Since \((X, x, 0)\) is an \( \mathcal{A}_e\)-subset of \((X, f)\), it follows from (a1) that the point \( \mathcal{A}\)-structure \((X, (x \ast 0)_{\cup \{r-1\}}) = (X, x, r)\) is an \( \mathcal{A}_{e\cup k}\)-subset of \((X, f)\). But, \( f(x) > (k-1)/2 > r \) implies that the point \( \mathcal{A}\)-structure \((X, x, r)\) is not an \( \mathcal{A}_e\)-subset of \((X, f)\). Also, \( f(x) + r - k + 1 = t_x + r - k + 1 > 0 \) implies that the point \( \mathcal{A}\)-structure \((X, x, r)\) is not an \( \mathcal{A}_k\)-subset of \((X, f)\). This is a contradiction. Assume that \( t_0 > t_x \) and take \( r < (k-1)/2 \) such that \( t_x + r - k + 1 < 0 < t_0 + r - k + 1 \). Then \((X, x, r)\) is an \( \mathcal{A}_e\)-subset of \((X, f)\). Since

\[
\begin{align*}
f(x \ast x) &= f(0) = t_0 > -r + k - 1 > -\frac{k-1}{2} + k - 1 = \frac{k-1}{2} > r,
\end{align*}
\]

\((X, (x \ast x)_{\cup \{r-1\}})\) is not an \( \mathcal{A}_e\)-subset of \((X, f)\). Since

\[
\begin{align*}
f(x \ast x) + \cup \{r, r\} - k + 1 &= f(0) + r - k + 1 = t_0 + r - k + 1 > 0,
\end{align*}
\]

\((X, (x \ast x)_{\cup \{r-1\}})\) is not an \( \mathcal{A}_e\)-subset of \((X, f)\). Hence \((X, (x \ast x)_{\cup \{r, r\}})\) is not an \( \mathcal{A}_{e\cup k}\)-subset of \((X, f)\), which is a contradiction. Therefore \( f(x) \leq (k-1)/2 \) for some \( x \in X \). We now prove that \( f(0) \leq (k-1)/2 \). Assume that \( f(0) = t_0 > (k-1)/2 \). Note that there exists \( x \in X \) such that \( f(x) = t_x \leq (k-1)/2 \) and so \( t_x < t_0 \). Choose \( t_0 < t_0 \) such that \( t_x + t_1 - k + 1 < 0 < t_0 + t_1 - k + 1 \). Then \( f(x) + t_0 - k + 1 = t_x + t_1 - k + 1 < 0 \), and thus the point \( \mathcal{A}\)-structure \((X, x_0)\) is an \( \mathcal{A}_k\)-subset of \((X, f)\). Now we have

\[
\begin{align*}
f(x \ast x) + \cup \{t_1, t_1\} - k + 1 &= f(0) + t_1 - k + 1 = t_0 + t_1 - k + 1 > 0
\end{align*}
\]
and \( f(x \ast x) = f(0) = t_0 > t_1 = \sqrt{\{t_1, t_1\}} \). Hence \((X, (x \ast x)_{\sqrt{\{t_1, t_1\}}})\) is not an \( \mathcal{A}_{\mathbb{N} \lor q_k} \)-subset of \((X, f)\). This is a contradiction, and therefore \( f(0) \leq (k-1)/2 \).

**Corollary 3.17** (see [3]). Let \((X, f)\) be an \( \mathcal{A} \)-subalgebra of type \((q, \in \lor q_k)\). If \( f \) is not constant on the open 0-support of \((X, f)\), then \( f(x) \leq -0.5 \) for some \( x \in X \). In particular, \( f(0) \leq -0.5 \).

**Theorem 3.18.** An \( \mathcal{A} \)-structure \((X, f)\) is an \( \mathcal{A} \)-subalgebra of type \((\varepsilon, \in \lor q_k)\) if and only if for every \( t \in [(k-1)/2, 0] \) the nonempty closed \( t \)-support of \((X, f)\) is a subalgebra of \( X \).

**Proof.** Assume that \((X, f)\) is an \( \mathcal{A} \)-subalgebra of type \((\varepsilon, \in \lor q_k)\) and let \( t \in [(k-1)/2, 0] \) be such that \( C(f; t) \neq \emptyset \). Let \( x, y \in C(f; t) \). Then \( f(x) \leq t \) and \( f(y) \leq t \). It follows from Theorem 3.12 that

\[
f(x \ast y) \leq \bigvee \left\{ f(x), f(y), \frac{k-1}{2} \right\} \leq \bigvee \left\{ t, \frac{k-1}{2} \right\} = t
\]

so that \( x \ast y \in C(f; t) \). Therefore \( C(f; t) \) is a subalgebra of \( X \).

Conversely, let \((X, f)\) be an \( \mathcal{A} \)-structure such that the nonempty closed \( t \)-support of \((X, f)\) is a subalgebra of \( X \) for all \( t \in [(k-1)/2, 0] \). If there exist \( a, b \in X \) such that \( f(a \ast b) > \bigvee \left\{ f(a), f(b), (k-1)/2 \right\} \), then we can take \( s \in [-1, 0] \) such that

\[
f(a \ast b) > s \geq \bigvee \left\{ f(a), f(b), \frac{k-1}{2} \right\}.
\]

Thus \( a, b \in C(f; s) \) and \( s \geq (k-1)/2 \). Since \( C(f, s) \) is a subalgebra of \( X \), it follows that \( a \ast b \in C(f; s) \) so that \( f(a \ast b) \leq s \). This is a contradiction, and therefore \( f(x \ast y) \leq \bigvee \left\{ f(x), f(y), (k-1)/2 \right\} \) for all \( x, y \in X \). Using Theorem 3.12, we conclude that \((X, f)\) is an \( \mathcal{A} \)-subalgebra of type \((\varepsilon, \in \lor q_k)\).

Taking \( k = 0 \) in Theorem 3.18, we have the following corollary.

**Corollary 3.19** (see [4]). An \( \mathcal{A} \)-structure \((X, f)\) is an \( \mathcal{A} \)-subalgebra of type \((\varepsilon, \in \lor q)\) if and only if for every \( t \in [-0.5, 0] \) the nonempty closed \( t \)-support of \((X, f)\) is a subalgebra of \( X \).

**Theorem 3.20.** Let \( S \) be a subalgebra of \( X \). For any \( t \in [(k-1)/2, 0) \), there exists an \( \mathcal{A} \)-subalgebra \((X, f)\) of type \((\varepsilon, \in \lor q_k)\) for which \( S \) is represented by the closed \( t \)-support of \((X, f)\).

**Proof.** Let \((X, f)\) be an \( \mathcal{A} \)-structure in which \( f \) is given by

\[
f(x) = \begin{cases} t & \text{if } x \in S, \\ 0 & \text{if } x \notin S, \end{cases}
\]

for all \( x \in X \) where \( t \in [(k-1)/2, 0) \). Assume that \( f(a \ast b) > \bigvee \left\{ f(a), f(b), (k-1)/2 \right\} \) for some \( a, b \in X \). Since the cardinality of the image of \( f \) is 2, we have \( f(a \ast b) = 0 \) and \( \bigvee \left\{ f(a), f(b), (k-1)/2 \right\} = t \). Since \( t \geq (k-1)/2 \), it follows that \( f(a) = t = f(b) \) so that \( a, b \in S \). Since \( S \) is a subalgebra of \( X \), we obtain \( a \ast b \in S \) and so \( f(a \ast b) = t < 0 \). This is a contradiction. Therefore \( f(x \ast y) \leq \bigvee \left\{ f(x), f(y), (k-1)/2 \right\} \) for all \( x, y \in X \). Using Theorem 3.12, we conclude that
(X, f) is an $\mathcal{A}$-subalgebra of type $(\varepsilon, \in \forall \psi_k)$. Obviously, $S$ is represented by the closed $t$-support of $(X, f)$.}

**Corollary 3.21** (see [4]). Let $S$ be a subalgebra of $X$. For any $t \in [-0.5, 0)$, there exists an $\mathcal{A}$-subalgebra $(X, f)$ of type $(\varepsilon, \in \forall \psi_q)$ for which $S$ is represented by the closed $t$-support of $(X, f)$.

**Proof.** It follows from taking $k = 0$ in Theorem 3.20.

Note that every $\mathcal{A}$-subalgebra of type $(\varepsilon, \in)$ is an $\mathcal{A}$-subalgebra of type $(\varepsilon, \in \forall \psi_k)$, but the converse is not true in general (see Example 3.7). Now, we give a condition for an $\mathcal{A}$-subalgebra of type $(\varepsilon, \in \forall \psi_k)$ to be an $\mathcal{A}$-subalgebra of type $(\varepsilon, \in)$.

**Theorem 3.22.** Let $(X, f)$ be an $\mathcal{A}$-subalgebra of type $(\varepsilon, \in \forall \psi_k)$ such that $f(x) > (k - 1)/2$ for all $x \in X$. Then $(X, f)$ is an $\mathcal{A}$-subalgebra of type $(\varepsilon, \in)$.

**Proof.** Let $x, y \in X$ and $t \in [-1, 0)$ be such that $(X, x_t)$ and $(X, y_t)$ are $\mathcal{A}_e$-subsets of $(X, f)$. Then $f(x) \leq t_1$ and $f(y) \leq t_2$. It follows from Theorem 3.12 and the hypothesis that

$$f(x * y) \leq \bigvee \left\{ f(x), f(y), \frac{k - 1}{2} \right\} = \bigvee \{ f(x), f(y) \} \leq \bigvee \{ t_1, t_2 \}$$

(3.20)

so that $(X, (x * y)_V \{ t_1, t_2 \})$ is an $\mathcal{A}_e$-subset of $(X, f)$. Therefore $(X, f)$ is an $\mathcal{A}$-subalgebra of type $(\varepsilon, \in)$.

**Corollary 3.23** (see [4]). Let $(X, f)$ be an $\mathcal{A}$-structure of type $(\varepsilon, \in \forall \psi_q)$ such that $f(x) > -0.5$ for all $x \in X$. Then $(X, f)$ is an $\mathcal{A}$-subalgebra of type $(\varepsilon, \in)$.

**Proof.** It follows from taking $k = 0$ in Theorem 3.22.

**Theorem 3.24.** Let $\{ (X, f_i) \mid i \in A \}$ be a family of $\mathcal{A}$-subalgebras of type $(\varepsilon, \in \forall \psi_k)$. Then $(X, \bigcup_{i \in A} f_i)$ is an $\mathcal{A}$-subalgebra of type $(\varepsilon, \in \forall \psi_k)$, where $\bigcup_{i \in A} f_i$ is an $\mathcal{A}$-function on $X$ given by $(\bigcup_{i \in A} f_i)(x) = \bigvee_{i \in A} f_i(x)$ for all $x \in X$.

**Proof.** Let $x, y \in X$ and $t_1, t_2 \in [-1, 0)$ be such that $(X, x_t)$ and $(X, y_t)$ are $\mathcal{A}_e$-subsets of $(X, \bigcup_{i \in A} f_i)$. Assume that $(X, (x * y)_V \{ t_1, t_2 \})$ is not an $\mathcal{A}_e \forall \psi_q$-subset of $(X, \bigcup_{i \in A} f_i)$. Then $(X, (x * y)_V \{ t_1, t_2 \})$ is neither an $\mathcal{A}_e$-subset nor an $\mathcal{A}_e \forall \psi_k$-subset of $(X, \bigcup_{i \in A} f_i)$. Hence $(\bigcup_{i \in A} f_i)(x * y) > \bigvee \{ t_1, t_2 \}$ and

$$\left( \bigcup_{i \in A} f_i \right)(x * y) + \bigvee \{ t_1, t_2 \} - k + 1 \geq 0,$$

(3.21)

which imply that

$$\left( \bigcup_{i \in A} f_i \right)(x * y) > \frac{k - 1}{2}.$$

(3.22)

Let $A_1 := \{ i \in A \mid (X, (x * y)_V \{ t_1, t_2 \}) \text{ is an } \mathcal{A}_e\text{-subset of } (X, f_i) \}$ and $A_2 := \{ i \in A \mid (X, (x * y)_V \{ t_1, t_2 \}) \text{ is an } \mathcal{A}_e \forall \psi_k\text{-subset of } (X, f_i) \} \cap \{ j \in A \mid (X, (x * y)_V \{ t_1, t_2 \}) \text{ is not an } \mathcal{A}_e\text{-subset} \}$.
of \((X, f_i)\). Then \(\Lambda = A_1 \cup A_2\) and \(A_1 \cap A_2 = \emptyset\). If \(A_2 = \emptyset\), then \((X, (x \ast y)_{\cup \{t_1, t_2\}})\) is an \(\mathcal{N}_e\)-subset of \((X, f_i)\) for all \(i \in \Lambda\), that is, \(f_i(x \ast y) \leq \sqrt{\{t_1, t_2\}}\) for all \(i \in \Lambda\). Thus \((\bigcup_{i \in \Lambda} f_i)(x \ast y) \leq \sqrt{\{t_1, t_2\}}\). This is a contradiction. Hence \(A_2 \neq \emptyset\), and so for every \(i \in A_2\), we have \(f_i(x \ast y) > \sqrt{\{t_1, t_2\}}\) and \(f_i(x \ast y) + \sqrt{\{t_1, t_2\}} - k + 1 < 0\). It follows that \(\sqrt{\{t_1, t_2\}} < (k - 1)/2\). Since \((X, x_i)\) is an \(\mathcal{N}_e\)-subset of \((X, \bigcup_{i \in \Lambda} f_i)\), we have

\[
f_i(x) \leq \left( \bigcup_{i \in \Lambda} f_i \right)(x) \leq t_1 \leq \sqrt{\{t_1, t_2\}} < \frac{k - 1}{2}
\]

for all \(i \in \Lambda\). Similarly, \(f_i(y) < (k - 1)/2\) for all \(i \in \Lambda\). Next suppose that \(t := f_i(x \ast y) > (k - 1)/2\). Taking \((k - 1)/2 < r < t\), we know that \((X, x_r)\) and \((X, y_r)\) are \(\mathcal{N}_e\)-subsets of \((X, f_i)\), but \((X, (x \ast y)_{\sqrt{\{r, r\}}}) = (X, (x \ast y), r)\) is not an \(\mathcal{N}_{ev, q}\)-subset of \((X, f_i)\). This contradicts that \((X, f_i)\) is an \(\mathcal{N}\)-subalgebra of type \((\varepsilon, \in \mathbb{V}, q_k)\). Hence \(f_i(x \ast y) \leq (k - 1)/2\) for all \(i \in \Lambda\), and so \((\bigcup_{i \in \Lambda} f_i)(x \ast y) \leq (k - 1)/2\) which contradicts (3.22). Therefore \((X, (x \ast y)_{\sqrt{\{r, r\}}})\) is an \(\mathcal{N}_{ev, q, r}\)-subset of \((X, \bigcup_{i \in \Lambda} f_i)\) and consequently \((X, \bigcup_{i \in \Lambda} f_i)\) is an \(\mathcal{N}\)-subalgebra of type \((\varepsilon, \in \mathbb{V}, q_k)\). □

For any \(\mathcal{N}\)-structure \((X, f)\) and \(t \in [-1, 0)\), the \(q\)-support and the \(\in \mathbb{V}, q\)-support of \((X, f)\) related to \(t\) are defined to be the sets (see [4])

\[
\mathcal{N}_q(f; t) := \{ x \in X \mid (X, x_i) \text{ is an } \mathcal{N}_q\text{-subset of } (X, f) \},
\]

\[
\mathcal{N}_{ev, q}(f; t) := \{ x \in X \mid (X, x_i) \text{ is an } \mathcal{N}_{ev, q}\text{-subset of } (X, f) \},
\]

respectively. Note that the \(\in \mathbb{V}, q\)-support is the union of the closed support and the \(q\)-support, that is,

\[
\mathcal{N}_{ev, q}(f; t) = C(f; t) \cup \mathcal{N}_q(f; t), \quad t \in [-1, 0).
\]

The \(q_k\)-support and the \(\in \mathbb{V}, q_k\)-support of \((X, f)\) related to \(t\) are defined to be the sets

\[
\mathcal{N}_{q_k}(f; t) := \{ x \in X \mid (X, x_i) \text{ is an } \mathcal{N}_{q_k}\text{-subset of } (X, f) \},
\]

\[
\mathcal{N}_{ev, q_k}(f; t) := \{ x \in X \mid (X, x_i) \text{ is an } \mathcal{N}_{ev, q_k}\text{-subset of } (X, f) \},
\]

respectively. Clearly, \(\mathcal{N}_{ev, q_k}(f; t) = C(f; t) \cup \mathcal{N}_{q_k}(f; t)\) for all \(t \in [-1, 0)\).

**Theorem 3.25.** An \(\mathcal{N}\)-structure \((X, f)\) is an \(\mathcal{N}\)-subalgebra of type \((\varepsilon, \in \mathbb{V}, q_k)\) if and only if the \(\in \mathbb{V}, q_k\)-support of of \((X, f)\) related to \(t\) is a subalgebra of \(X\) for all \(t \in [-1, 0)\).

**Proof.** Suppose that \((X, f)\) is an \(\mathcal{N}\)-subalgebra of type \((\varepsilon, \in \mathbb{V}, q_k)\). Let \(x, y \in \mathcal{N}_{ev, q_k}(f; t)\) for \(t \in [-1, 0)\). Then \((X, x_i)\) and \((X, y_i)\) are \(\mathcal{N}_{ev, q_k}\)-subsets of \((X, f)\). Hence \(f(x) \leq t\) or \(f(x) + t - k + 1 < 0\), and \(f(y) \leq t\) or \(f(y) + t - k + 1 < 0\). Then we consider the following four cases:

- (c1) \(f(x) \leq t\) and \(f(y) \leq t\),
- (c2) \(f(x) \leq t\) and \(f(y) + t - k + 1 < 0\),
- (c3) \(f(x) + t - k + 1 < 0\) and \(f(y) \leq t\),
- (c4) \(f(x) + t - k + 1 < 0\) and \(f(y) + t - k + 1 < 0\).
Combining (3.6) and (c1), we have $f(x * y) \leq \sqrt{t, (k-1)/2}$. If $t \geq (k-1)/2$, then $f(x * y) \leq t$ and so $(X, (x * y)_i)$ is an $\mathcal{N}_e$-subset of $(X, f)$. Hence $x * y \in C(f; t) \subseteq \mathcal{N}_{eq_k}(f; t)$. If $t < (k-1)/2$, then $f(x * y) \leq (k-1)/2$ and so $f(x * y) + t - k + 1 < ((k-1)/2) + ((k-1)/2) - k + 1 = 0$, that is, $(X, (x * y)_i)$ is an $\mathcal{N}_{q_k}$-subset of $(X, f)$. Therefore $x * y \in \mathcal{N}_{q_k}(f; t) \subseteq \mathcal{N}_{eq_k}(f; t)$. For the case (c2), assume that $t < (k-1)/2$. Then

$$f(x * y) \leq \sqrt{\left\{ f(x), f(y), \frac{k-1}{2} \right\}}$$

$$\leq \sqrt{\left\{ f(y), \frac{k-1}{2} \right\}} = \sqrt{\left\{ f(y), \frac{k-1}{2} \right\}}$$

$$= \begin{cases} f(y) & \text{if } f(y) > \frac{k-1}{2}, \\ t & \text{if } f(y) \leq \frac{k-1}{2}, \end{cases} \text{ for } k-1 - t,$$

and so $f(x * y) + t - k + 1 < 0$. Thus $(X, (x * y)_i)$ is an $\mathcal{N}_{q_k}$-subset of $(X, f)$. If $t \geq (k-1)/2$, then

$$f(x * y) \leq \sqrt{\left\{ f(x), f(y), \frac{k-1}{2} \right\}}$$

$$\leq \sqrt{\left\{ t, f(y), \frac{k-1}{2} \right\}} = \sqrt{\left\{ t, f(y) \right\}}$$

$$= \begin{cases} f(y) & \text{if } f(y) > t, \\ t & \text{if } f(y) \leq t, \end{cases} \text{ for } k-1 - t,$$

and thus $x * y \in \mathcal{N}_{q_k}(f; t)$ or $x * y \in C(f; t)$. Consequently, $x * y \in \mathcal{N}_{eq_k}(f; t)$. For the case (c3), it is similar to the case (c2). Finally, for the case (c4), if $t \geq (k-1)/2$, then $k-1 - t \leq (k-1)/2 \leq t$. Hence

$$f(x * y) \leq \sqrt{\left\{ f(x), f(y), \frac{k-1}{2} \right\}} \leq \sqrt{\left\{ k-1 - t, \frac{k-1}{2} \right\}} = \frac{k-1}{2} \leq t,$$

which implies that $x * y \in C(f; t)$. If $t < (k-1)/2$, then $t < (k-1)/2 < k-1 - t$. Therefore

$$f(x * y) \leq \sqrt{\left\{ f(x), f(y), \frac{k-1}{2} \right\}} \leq \sqrt{\left\{ k-1 - t, \frac{k-1}{2} \right\}} = k-1 - t,$$

that is, $f(x * y) + t - k + 1 < 0$, which means that $(X, (x * y)_i)$ is an $\mathcal{N}_{q_k}$-subset of $(X, f)$. Consequently, the $\in \mathcal{N}_{q_k}$-support of $(X, f)$ related to $t$ is a subalgebra of $X$ for all $t \in [-1, 0)$.
Conversely, let \((X, f)\) be an \(\mathcal{A}\)-structure for which the \(\in \vee q_k\)-support of \((X, f)\) related to \(t\) is a subalgebra of \(X\) for all \(t \in [-1, 0)\). Assume that there exist \(a, b \in X\) such that \(f(a \ast b) > \sqrt{f(a), f(b), (k - 1)/2}\). Then

\[
f(a \ast b) > s \geq \sqrt{\left\{ f(a), f(b), \frac{k - 1}{2} \right\}}
\]

(3.33)

for some \(s \in [(k - 1)/2, 0)\). It follows that \(a, b \in C(f; s) \subseteq \mathcal{A}_{\vee q_k}(f; s)\) but \(a \ast b \not\in C(f; s)\). Also, \(f(a \ast b) + s - k + 1 > 2s - k + 1 \geq 0\), that is, \(a \ast b \not\in \mathcal{A}_q(f; s)\). Thus \(a \ast b \not\in \mathcal{A}_{\vee q_k}(f; s)\) which is a contradiction. Therefore

\[
f(x \ast y) \leq \sqrt{\left\{ f(x), f(y), \frac{k - 1}{2} \right\}}
\]

(3.34)

for all \(x, y \in X\). Using Theorem 3.12, we conclude that \((X, f)\) is an \(\mathcal{A}\)-subalgebra of type \((\in, \in \vee q)\).

If we take \(k = 0\) in Theorem 3.25, we have the following corollary.

**Corollary 3.26** (see [4]). An \(\mathcal{A}\)-structure \((X, f)\) is an \(\mathcal{A}\)-subalgebra of type \((\in, \in \vee q)\) if and only if the \(\in \vee q\)-support of \((X, f)\) related to \(t\) is a subalgebra of \(X\) for all \(t \in [-1, 0)\).

**References**


