Research Article

On Generalized Bazilevic Functions Related with Conic Regions

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We define and study some generalized classes of Bazilevic functions associated with convex domains. These convex domains are formed by conic regions which are included in the right half plane. Such results as inclusion relationships and integral-preserving properties are proved. Some interesting special cases of the main results are also pointed out.

1. Introduction

Let $A$ denote the class of analytic functions $f(z)$ defined in the unit disc $E = \{z : |z| < 1\}$ and satisfying the conditions $f(0) = 0, f'(0) = 1$. Let $S$ denote the subclass of $A$ consisting of univalent functions in $E$, and let $S^*$ and $C$ be the subclasses of $S$ which contains, respectively, star-like and convex in Bazilevic [1] introduced the class $B(\alpha, \beta, h, g)$ as follows.

Let $f \in A$. Then, $f \in B(\alpha, \beta, h, g)$, $\alpha, \beta$ real and $\alpha > 0$ if

$$f(z) = \left((\alpha + i\beta) \int_0^z h(z) g^\alpha(t) t^{i\beta-1} dt\right)^{1/(\alpha+i\beta)}, \quad (1.1)$$

for some $g \in S^*$ and $\text{Re } h(z) > 0, z \in E$.

The powers appearing in (1.1) are meant as principle values. The functions $f$ in the class $B(\alpha, \beta, h, g)$ are shown to be analytic and univalent, see [1]. $B(\alpha, \beta, h, g)$ is the largest known subclass of univalent functions defined by an explicit formula and contains many of the heavily researched subclasses of $S$. We note the following:

(i) $B(1,0,1,g) = C$,

(ii) $B(1,0,zg'/g,g) = S^*$,
(iii) $B(1, 0, h, g) = K$, where $K$ is the class of close-to-convex functions introduced by Kaplan [2],

(iv) $B(\cos \gamma, \sin \gamma, \cos(3g'/(g + i \sin \gamma)), g)$ is the class of $\gamma$-spiral like functions which are univalent for $|\gamma| < \pi/2$.

For analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, by $f \ast g$ we denote the Hadamard product (convolution) of $f$ and $g$, defined by

$$ (f \ast g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. $$

For $k \in [0, \infty)$, the conic domain $\Omega_k$ is defined in [3] as follows:

$$ \Omega_k = \left\{ u + iv : u > k\sqrt{(u - 1)^2 + v^2} \right\}. $$

For fixed $k$, $\Omega_k$ represents the conic region bounded successively by the imaginary axis ($k = 0$), the right branch of hyperbola ($0 < k < 1$), a parabola ($k = 1$) and an ellipse ($k > 1$).

The following univalent functions, defined by $p_k(z)$ with $p_k(0) = 1$ and $p'_k(0) > 0$, map the unit disc $E$ onto $\Omega_k$:

$$ p_k(z) = \begin{cases} 
\frac{1 + z}{1 - z}, & (k = 0), \\
1 + \frac{2}{\pi^2} \left( \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2, & (k = 1), \\
1 + \frac{2}{1 - k^2} \sin^2 \left[ A(k) \tan z \sqrt{z} \right], & (0 < k < 1), \\
1 + \frac{2}{k^2 - 1} \sin^2 \left( \frac{\pi}{2K(t)} F \left( \frac{\sqrt{z}}{t}, \frac{t}{t'} \right) \right), & (k > 1),
\end{cases} $$

where $A(k) = (2/\pi) \arccos k$, $F(w, t)$ is the Jacobi elliptic integral of the first kind:

$$ F(w, t) = \int_0^w \frac{dx}{\sqrt{1 - x^2 \sqrt{1 - t^2 x^2}}}, $$

and $t \in (0, 1)$ is chosen such that $k = \cosh(\pi K'(t)/2K(t))$, where $K(t)$ is the complete elliptic integral of the first kind, $K(t) = F(1, t), K'(t) = K(\sqrt{1-t^2})$.

It is known that $p_k(z)$ are continuous as regards to $k$ and have real coefficients for $k \in [0, \infty)$.

Let $P(p_k)$ be the subclass of the class $P$ of Carathéodory functions $p(z)$, analytic in $E$ with $p(0) = 1$ and such that $p(z)$ is subordinate to $p_k(z)$, written as $p(z) \prec p_k(z)$ in $E$.

We define the following.
2. Preliminary Results

Lemma 2.1 (see [3]). Let $0 \leq k < \infty$, and let $\beta_0$, \(\delta\) be any complex numbers with $\beta_0 \neq 0$ and Re$(\beta_0 k / (k + 1) + \delta) > 0$. If $h(z)$ is analytic in $E, h(0) = 1$ and satisfies

$$\left\{ h(z) + \frac{zh'(z)}{\beta_0 h(z) + \delta} \right\} < p_k(z)$$

and $q_k(z)$ is analytic solution of

$$\left\{ q_k(z) + \frac{zq_k'(z)}{\beta_0 q_k(z) + \delta} \right\} = p_k(z),$$

then $q_k(z)$ is univalent,

$$h(z) < q_k(z) < p_k(z),$$

and $q_k(z)$ is the best dominant of (2.1).

Lemma 2.2 (see [9]). Let $q(z)$ be convex in $E$ and $j : E \to \mathbb{C}$ with Re $j(z) > 0$, $z \in E$. If $p(z)$ is analytic in $E$ with $p(0) = 1$ and satisfies \{p(z) + j(z) \cdot zp'(z)\} < q(z), then $p(z) < q(z)$.

Lemma 2.3 (see [9]). Let $u = u_1 + iu_2, v = v_1 + iv_2$, and let $\psi(u, v)$ be a complex-valued function satisfying the conditions

(i) $\psi(u, v)$ is continuous in a domain $D \subset \mathbb{C}^2$, 

(ii) \((0, 1) \in D\) and \(\Re \varphi(1, 0) > 0\),

(iii) \(\Re \varphi(i u_z, v_1) \leq 0\), whenever \((i u_z, v_1) \in D\) and \(v_1 \leq -(1/2)(1 + u_z^2)\).

If \(h(z) = 1 + c_1 z + c_2 z^2 + \cdots\) is a function analytic in \(E\) such that \((h(z), z h'(z)) \in D\) and \(\Re \varphi(h(z), z h'(z)) > 0\) for \(z \in E\), then \(\Re h(z) > 0\) in \(E\).

### 3. Main Results

**Theorem 3.1.** Let \((1/(1 - \gamma))|z f'(z)/f(z) - \gamma| \in \mathcal{P}_m(p_k)\), for \(z \in E\) and \(\gamma \in [0, 1]\). Define

\[
g(z) = \left[(c + 1)z^{-c} \int_0^z t^{c-1}f^a(t)dt\right]^{1/a}, \quad a > 0, \quad c \in \mathbb{C}, \quad \Re c \geq 0.
\]  

(3.1)

Then, \((1/(1 - \gamma))|z g'(z)/g(z) - \gamma| \in \mathcal{P}_m(p_k)\) in \(E\). In particular \(g \in k - \cup R_m\) in \(E\).

**Proof.** Let

\[
\frac{z g'(z)}{g(z)} = (1 - \gamma)p(z) + \gamma,
\]  

(3.2)

where \(p(z)\) is analytic in \(E\) with \(p(0) = 1\), and let

\[
p(z) = \left(\frac{m}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right)p_2(z).
\]  

(3.3)

From (3.1) and (3.2), we have

\[
g^a(z) = [a(1 - \gamma)p(z) + c + a\gamma] = f^a(z).
\]  

(3.4)

Logarithmic differentiation of (3.4) and some computation yield

\[
p(z) + \frac{zp'(z)}{a(1 - \gamma)p(z) + (c + a\gamma)} = \frac{1}{1 - \gamma}\left\{\frac{zf'(z)}{f(z)} - \gamma\right\}.
\]  

(3.5)

That is

\[
p(z) + \frac{zp'(z)}{a(1 - \gamma)p(z) + (c + a\gamma)} \in \mathcal{P}_m(p_k) \quad \text{in} \ E.
\]  

(3.6)

Let \(\phi_{a,b}(z) = z + \sum_{n=2}^\infty z^n/((n-1)a + b)\). Then,

\[
\left(\frac{p(z) \cdot \phi_{a,b}(z)}{z}\right) = p(z) + \frac{a(zp(z))}{p(z) + b}.
\]  

(3.7)
Using convolution technique (3.7) with \( a = 1/\alpha(1 - \gamma) \), \( b = (c + a\gamma)/\alpha(1 - \gamma) \), we obtain, from (3.3) and (3.6),

\[
\left\{ p_i(z) + \frac{zp_i'(z)}{\alpha(1 - \gamma)p_i(z) + (c + a\gamma)} \right\} \times p_k(z) \quad \text{in } E, \ i = 1, 2. \tag{3.8}
\]

Since \( \text{Re}\{ (\alpha(1 - \gamma)k/(k + 1)) + c + a\gamma \} \geq 0 \), we apply Lemma 2.1 with \( \beta_0 = \alpha(1 - \gamma) \), \( \delta = c + a\gamma \) to obtain \( p_i(z) < q_k(z) < p_k(z) \), where \( q_k(z) \) is the best dominant and is given as

\[
q_k(z) = \left[ \beta_0 \int_0^1 \left( u^{\beta_0+\delta-1} \exp\left( \int_{z}^{u} \frac{p_k(u) - 1}{u} du \right) \right)^{\beta_0} dt \right]^{-1} \frac{1}{\beta_0}, \tag{3.9}
\]

Consequently, \( p \in P_m(p_k) \) in \( E \), and this completes the result.

As a special case, we prove the following.

**Corollary 3.2.** Let \( k = 0 \) and let \((1/(1 - \gamma_1))\{zf'(z)/f(z) - \gamma_1\} \in P_m \) in \( E \). Then, for \( g \) defined by (3.1), \((1/(1 - \gamma))\{zg'(z)/g(z) - \gamma \} \in P_m \) in \( E \) where

\[
\gamma = \frac{2}{\left\{ (2c - 2\alpha\gamma_1 + 1) + \sqrt{(2c - 2\alpha\gamma_1 + 1)^2 + 8\alpha} \right\}}. \tag{3.10}
\]

**Proof.** We can write

\[
\frac{zf'(z)}{f(z)} = (1 - \gamma_1)h(z) + \gamma, \tag{3.11}
\]

where \( h \in P_m \) in \( E \).

Now proceeding as before, we have, with

\[
\frac{zg'(z)}{g(z)} = (1 - \gamma)p(z) + \gamma = \left( 1 - \gamma \right) p_1(z) + \gamma = \left( \frac{m}{4} + \frac{1}{2} \right) \left( (1 - \gamma)p_1(z) + \gamma \right) - \left( \frac{m}{4} - \frac{1}{2} \right) \left( (1 - \gamma)p_2(z) + \gamma \right) \tag{3.12}
\]

\[
(1 - \gamma)p(z) + \gamma + \frac{(1 - \gamma)zp'(z)}{\alpha(1 - \gamma)p(z) + (c + a\gamma)} = \frac{zf'(z)}{f(z)}. \tag{3.13}
\]

Using convolution technique together with (3.11), we obtain

\[
\text{Re}\left\{ (1 - \gamma)p_i(z) + (\gamma - \gamma_1) + \frac{(1 - \gamma)zp_i'(z)}{\alpha(1 - \gamma)p_i(z) + (c + a\gamma)} \right\} > 0, \tag{3.14}
\]

for \( i = 1, 2 \).
We construct the functional \( \psi(u, v) \) by taking \( u = p(z), v = zp'(z) \) as

\[
\psi(u, v) = (1 - \gamma)u + (\gamma - \gamma_1) + \frac{(1 - \gamma)v}{\alpha(1 - \gamma)u + (c + a\gamma)}.
\] (3.15)

The first two conditions of Lemma 2.3 are clearly satisfied. We verify condition (iii) as follows.

\[
\text{Re} \psi(\alpha, v_1) = (\gamma - \gamma_1) + \text{Re} \left\{ \frac{(1 - \gamma)v_1}{i\alpha(1 - \gamma)u_2 + (c + a\gamma)} \right\},
\]

\[
= (\gamma - \gamma_1) + \frac{(c + a\gamma)(1 - \gamma)v_1}{(c + a\gamma)^2 + \alpha^2(1 - \gamma)^2u_2^2},
\]

\[
\leq (\gamma - \gamma_1) + \frac{(c + a\gamma)(1 - \gamma)(1 + u_2^2)}{2[(c + a\gamma)^2 + \alpha^2(1 - \gamma)^2u_2^2]}, \quad \left( v_1 \leq -\frac{1 + u_2^2}{2} \right),
\]

\[
= \frac{A + Bu_2^2}{2C},
\]

where

\[
A = 2(\gamma - \gamma_1)(c + a\gamma)^2 - (1 - \gamma)(c + a\gamma), B = 2\alpha^2(\gamma - \gamma_1)(1 - \gamma)^2 - (1 - \gamma)(c + a\gamma), C = (c + a\gamma)^2 + \alpha^2(1 - \gamma)^2u_2^2 > 0.
\]

\( \text{Re} \psi(\alpha, v_1) \leq 0 \) if and only if \( A \leq 0, B \leq 0 \). From \( A \leq 0 \), we obtain \( \gamma \) as given by (3.10) and \( B \leq 0 \) ensures that \( \gamma \in [0, 1) \).

Now proceeding as before, it follows from (3.12) that \( p \in P_m \), and this proves our result.

By assigning certain permissible values to different parameters, we obtain several new and some known result.

**Corollary 3.3.** Let \( f \in k - \cup R_2 = k - \cup ST \). Then, it is known that \( f \in S^*(\gamma_1), \gamma_1 = k/(k + 1) \) and, from Corollary 3.2, it follows that \( g \in S^*(\gamma) \) where \( \gamma \) is given by (3.10). Also a starlike function is \( k \)-uniformly convex for \( |z| < r_k \),

\[
r_k = \frac{1}{2(k + 1) + \sqrt{4k^2 + 6k + 3}}, \text{ see [8].}
\] (3.17)

Therefore, for \( f \in k - \cup R_2 \), it follows that \( (1/(1 - \gamma))(zg'(z))'/g'(z) - \gamma) \prec p_k \) for \( |z| < r_k \), where \( \gamma \) is given by (3.10).

As special cases we note the following.

(i) For \( k = 0 \), we have \( r_0 = 1/(2 + \sqrt{3}) \) and \( f \in S^*(0) \) implies that \( g \in C(\gamma_k) \), with

\[
\gamma_k = \frac{2}{2(c + 1) + \sqrt{2(c + 1)^2 + 8\alpha}}.
\] (3.18)
(ii) When \( k = 1 \), we have \( \gamma_1 = 1/2, \gamma = 2/((2c - \alpha + 1) + \sqrt{(2c - \alpha + 1)^2 + 8\alpha}) \) and \( r_1 = 1/(4 + \sqrt{13}) \).

Theorem 3.4. Let \( F \in k - \cup B_m(\alpha, \beta, p, f), f \in k - \cup R_2, p \in P_m(p_k). \) Define, for \( \text{Re}[a/k + 1] + (c + i\beta) > 0 \),

\[
G(z) = \left[(c + 1)z^{-c} \int_0^z t^{-1} F^{a+i\beta}(t) dt \right]^{1/(a+i\beta)}.
\]

Then, \( G \in k - \cup B_m(\alpha, \beta, h, g) \) in \( E \), where \( g(z) \) is given by (3.1), and \( h(z) \) is analytic in \( E \) with \( h(0) = 1 \).

Proof. Set

\[
\frac{zG'(z)G^{a+i\beta-1}(z)}{z^\beta g^a(z)} = h(z) = \left(\frac{m}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) h_2(z).
\]

We note that \( h(z) \) is analytic in \( E \) with \( h(0) = 1 \). From (3.20), we have

\[
z^\beta g^a(z) \left\{ zh'(z) + h(z) \left[ a \frac{zg'(z)}{g(z)} + c + i\beta \right] \right\} = zF'(z) F^{a+i\beta-1}(z),
\]

using (3.1), we note that

\[
\left(\frac{f(z)}{g(z)}\right)^a = a \frac{zg'(z)}{g(z)} + c + i\beta.
\]

From (3.21) and (3.22), it follows that

\[
\left\{ h(z) + \frac{zh'(z)}{ah_0(z) + (c + i\beta)} \right\} \in P_m(p_k),
\]

where \( h_0(z) = zg'(z)/g(z) \) is analytic in \( E \) since \( g \in k - \cup R_2 \) by Theorem 3.1.

It can be easily seen that \( g \in S^*(k/(k + 1)) \) and \( \text{Re}\{azg'(z)/g(z) + c + i\beta\} > 0 \).

Now, using (3.8), we can easily derive

\[
\left\{ h_i(z) + j(z)(zh'_i(z)) \right\} < p_k(z) \quad \text{in} \quad E, \quad i = 1, 2,
\]

where \( 1/j(z) = \{azg'(z)/g(z) + c + i\beta\} \) and \( \text{Re} j(z) > 0 \).

Applying Lemma 2.2, it follows from (3.24) \( h_i(z) < p_k(z) \) in \( E \) and therefore \( h \in P_m(p_k) \) in \( E \). This completes the proof. \( \square \)
Theorem 3.5. Let $f(z)$ be given by (1.1) with $h(z) = 1, \{ (a^2 + \beta^2)^{1/2} e^{i\gamma} (zg'/g) \} \in P_m(p_k)$

$(a^2 + \beta^2)^{1/2} e^{i\gamma} = a + i\beta, |\gamma| < \pi/2$. Then, for $z \in E$

(i) $e^{i\gamma} (zf'(z)/f(z)) = \cos \gamma (p(z)) + i \sin \gamma, \ p \in P_m(p_k),$

(ii) For $a' + i\beta' = t(a + i\beta), \ t \geq 1,$

$$k - \cup B_m (\alpha, \beta, 1, g) \subset kt - \cup B_m (\alpha', \beta', 1, g). \quad (3.25)$$

Proof. (i) From (1.1), we have

$$1 + \frac{zf''(z)}{f'(z)} + (a - 1 + i\beta) \frac{zf'(z)}{f(z)} = \left( a \frac{zg'(z)}{g(z)} + i\beta \right) = H_2(z), \quad H_2 \in P_m(p_k) \text{ in } E. \quad (3.26)$$

Define a function $p(z)$ analytic in $E$ by

$$e^{i\gamma} \frac{zf'(z)}{f(z)} = \cos \gamma (p(z)) + i \sin \gamma, \quad \gamma = \tan^{-1} \frac{\beta}{\alpha}. \quad (3.27)$$

We can easily check that $p(0) = 1$.

Now, from (3.26) and (3.27), we have

$$\left[ \frac{zp'(z)}{p(z) + i \tan \gamma} + ap(z) + i\beta \right] \in P_m(p_k) \text{ in } E. \quad (3.28)$$

That is

$$\left[ \frac{azp'(z)}{ap(z) + i\beta} + ap(z) + i\beta \right] \in P_m(p_k), \quad (3.29)$$

and, with $h(z) = ap(z) + i\beta = (m/4 + 1/2)h_1(z) - (m/4 - 1/2)h_2(z)$, we apply convolution technique used before to have

$$\left\{ h_1(z) + \frac{zh_1'(z)}{h_1(z)} \right\} < p_k(z) \text{ in } E. \quad (3.30)$$

Applying Lemma, it follows that

$$h_1(z) < q_k(z) < p_k(z), \quad z \in E, \quad (3.31)$$

where $q_k(z)$ is the best dominant and is given by

$$q_k(z) = \left[ \int_0^{z^1} \exp \left( \int_0^{u} \frac{p_k(u) - 1}{u} du \right) \right]^{-1}. \quad (3.32)$$

From (3.31), we have $h(z) = (ap(z) + i\beta) \in P_m(p_k)$ in $E$, and this proves part (i).
(ii) From part (i), we have
\[
\left(\alpha^2 + \beta^2\right)^{1/2} e^{i\gamma} \frac{zf''(z)}{f(z)} = H_1(z), \quad H_1 \in P_m(p_k) \quad \text{in } E.
\] (3.33)

Now,
\[
1 + \frac{zf''(z)}{f'(z)} + (\alpha' - 1 + i\beta') \frac{zf'(z)}{f(z)}
= \left\{1 + \frac{zf''(z)}{f'(z)} + (\alpha - 1 + i\beta) \frac{zf'(z)}{f(z)}\right\}
+ (t - 1)\left(\alpha^2 + \beta^2\right)^{1/2} e^{i\gamma} \frac{zf'(z)}{f(z)},
\] (3.34)
\[
= H_2(z) + (t - 1)H_1(z), \quad H_i \in P_m(p_k), \quad i = 1, 2,
= t \left[\left(1 - \frac{1}{t}\right)H_1(z) + \frac{1}{t}H_2(z)\right],
= tH, \quad t \geq 1,
\]

\(H \in P_m(p_k),\) since \(P_m(p_k)\) is convex set, see [8].

Therefore, \(f \in kt - \cup B_m(\alpha', \beta', 1, g)\) for \(z \in E.\) This completes the proof. \(\Box\)

As a special case, with \(m = 2, k = 0,\) we obtain a result proved in [10].

By assigning certain permissible values to the parameters \(\alpha, \beta\) and \(m,\) we have several other new results.

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