Research Article

The Split Common Fixed Point Problem for \( \varrho \)-Strictly Pseudononspraying Mappings

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1. Introduction

Throughout this paper, we always assume that \( H \) is a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). Let \( I \) denote the identity operator on \( H \). Let \( H_1 \) and \( H_2 \) be two real Hilbert spaces and let \( A : H_1 \to H_2 \) be a bounded linear operator. Given closed convex subsets \( C \) and \( Q \) of \( H_1 \) and \( H_2 \), respectively.

The split feasibility problem (SFP) (Censor and Elfving 1994 [1]), modeling phase retrieval problems, is to find a point \( x^* \) with the property

\[
x^* \in C, \quad Ax^* \in D.
\]  

(1)

Recently, it has been found that the SFP can also be used to model the intensity-modulated radiation therapy [2–8]. A special case of the SFP (1) is the convexly constrained linear problem:

\[
Ax = b, \quad x \in C.
\]

(2)

This problem, due to its applications in many applied disciplines, has extensively been investigated in the literature ever since Landweber [9] introduced his iterative method in 1951.

Note that the split feasibility problem (1) can be formulated as fixed point equation by using the fact

\[
x^* = P_C (I - \varrho A^*(I - P_Q) A) x^*,
\]

(3)

where \( P_C \) and \( P_Q \) are the projections onto \( C \) and \( Q \), respectively, \( \varrho > 0 \) is any positive constant, and \( A^* \) denotes the adjoint of \( A \); that is, \( x^* \) solves the SFP (1) if and only if \( x^* \) solves the fixed point equation (3) (see [10] for more details). This implies that we can use fixed point algorithms to solve SFP.

In 2002, Byrne [2] proposed his CQ algorithm to solve (1). The sequence \( \{x_n\} \) is generated by the following iteration scheme:

\[
x_{n+1} = P_C (I - \varrho A^*(I - P_Q) A) x_n, \quad n \in N,
\]

(4)

where \( \varrho \in (0, 2/\lambda) \), with \( \lambda \) being the spectral radius of the operator \( A^* A \).

The CQ algorithm (4) is a special case of the K-M algorithm. Due to the fixed point formulation (2) of the SFP, Moudafi [11] applied the K-M algorithm to the operator \( P_C (I - \varrho A^*(I - P_Q) A) \) to obtain a sequence given by

\[
x_{n+1} = (1 - \alpha_n) x_n - \alpha_n P_C (I - \varrho A^*(I - P_Q) A) x_n,
\]

(5)

where \( \varrho \in (0, 2/\lambda) \), with \( \lambda \) being the spectral radius of the operator \( A^* A \), and the sequence \( \{\alpha_n\} \) satisfies the condition \( \sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = +\infty \); he proved weak convergence result of the algorithm (5) in Hilbert spaces.

In 2009, Censor and Segal [12] considered the following algorithm to be solved (1).
Algorithm 1. Initialization: let \( x^* \in H_1 = R^n \) be arbitrary.

Iterative step: for \( n \in N \) let

\[
x_{n+1} = U \left( x_n + \gamma A^* (V - I) A x_n \right), \quad n \in N,
\]

where \( \gamma \in (0, 2/\lambda) \) with \( \lambda \) being the spectral radius of the operator \( A^* A \) and \( U, V \) be a single pair of directed operators.

In 2010, Moudafi [13] extended the Algorithm 1 and introduced the following algorithm with weak convergence for the split common fixed point problem.

Algorithm 2. Initialization: let \( x^* \in H_1 = R^n \) be arbitrary.

Iterative step: for \( n \in N \) let

\[
x_{n+1} = \left( 1 - \alpha_n \right) u_n + \alpha_n U (u_n), \quad n \in N,
\]

where \( \beta \in (0, 1), \alpha_n \in (0, 1) \), and \( \gamma \in (0, 1/\lambda) \) with \( \lambda \) being the spectral radius of the operator \( A^* A \) and \( U, V \) be a pair of quasi-nonexpansive operators.

In 2012, Zhao and He [14] continue to consider the split common fixed point problem with quasi-nonexpansive operators and to use the following algorithm to obtain the strong convergence of the viscosity method for solving the split common fixed point problem.

Algorithm 3. Initialization: let \( x^* \in H_1 = R^n \) be arbitrary.

Iterative step: for \( n \in N \) let

\[
T = U \left( I + \gamma A^* (V - I) A \right) A
\]

\[
x_{n+1} = \alpha_n f (x_n) + (1 - \alpha_n) \times \left( (1 - \omega_n) x_n + \omega_n T x_n \right), \quad n \in N,
\]

where \( f : H \to H \) is a contractive mapping with constant \( \beta \in (0, 1), \alpha_n \in (0, 1) \), and \( \gamma \in (0, 1/\lambda) \) with \( \lambda \) being the spectral radius of the operator \( A^* A \) and \( U, V \) be a pair of quasi-nonexpansive operators.

Motivated and inspired by Censor and Segal [12], Moudafi [11], and Zhao and He [14], we introduce the following relaxed algorithm.

Algorithm 4. Initialization: let \( x^* \in H_1 = R^n \) be arbitrary.

Iterative step: for \( n \in N \) let

\[
T = U \left( I + \gamma A^* (V - I) A \right) A
\]

\[
x_{n+1} = \alpha_n f (x_n) + (1 - \alpha_n) \times \left( (1 - \omega_n) x_n + \omega_n T x_n \right), \quad n \in N,
\]

where \( f : H \to H \) is a contractive mapping with constant \( \beta \in (0, 1), \alpha_n \in (0, 1) \), and \( \gamma \in (0, 1/\lambda) \) with \( \lambda \) being the spectral radius of the operator \( A^* A \) and \( U, V \) be a pair of quasi-nonexpansive operators.

This paper establishes the strong convergence of the sequence given by (9) to the unique solution of solving the split common fixed point problem and the following variational inequality problem \( \text{VIP}(\mu B - \sigma f, T) \):

\[
\text{find } x^* \in \Gamma \text{ such that } \langle (\mu B - \sigma f) x^*, v - x^* \rangle \geq 0, \quad v \in \Gamma.
\]

2. Preliminaries

In this section, we introduce the concepts of contraction mappings, nonexpansive mappings, quasi-nonexpansive mappings, and \( \varphi \)-strictly pseudononsparing mappings and some Lemmas.

Assume that \( C \) is a nonempty closed and convex subset of Hilbert space \( H \). Recall that the (nearest point or metric) projection from \( H \) onto \( K \), that denoted \( P_K \), assigns, to each \( x \in H \), the unique point \( P_K x \in C \) with the property

\[
\| x - P_K x \| = \inf \| x - y \| : y \in C.
\]

Definition 5. A mapping \( T : H \to H \) is said to be

(1) contraction, if \( \| Tx - Ty \| \leq \beta \| x - y \|, \forall x, y \in H \) and \( \beta \in (0, 1) \);

(2) nonexpansive, if \( \| Tx - Ty \| \leq \| x - y \|, \forall x, y \in H \);

(3) quasi-nonexpansive, \( \| Tx - q \| \leq \| x - q \|, \forall (x, q) \in H \times F(T) \).

Remark 6. From the Definition 5, It is easy to see that

(i) iterative methods for quasi-nonexpansive mappings have been extensively investigated; see [13–17];

(ii) a nonexpansive mapping is a quasi-nonexpansive mapping.

Following the terminology of Browder and Petryshyn [18], we obtain the following definitions.

Definition 7. A mapping \( T : D(T) \subseteq H \to H \) is \( \varphi \)-strictly pseudononsparing if there exists \( \varphi \in (0, 1) \) such that

\[
\| Tx - Ty \|^2 \leq \| x - y \|^2 + \varphi \| x - Tx - (y - Ty) \|^2 + 2 \langle x - Tx, y - Ty \rangle,
\]

for all \( x, y \in D(T) \).

Iterative methods for strictly pseudononsparing mapping have been extensively investigated; see [19–23].

Lemma 8 (see [24]). Let \( H \) be a Hilbert spaces, and \( f : H \to H \) is a contractive mapping with constant \( \beta \in (0, 1), B : H \to H \) is \( k \)-Lipschitzian and \( \eta \)-strongly monotone operator with \( k > 0, \eta > 0 \). Then for \( 0 < \sigma < \mu \beta / \beta \),

\[
\langle x - y, (\mu B - \sigma f) x - (\mu B - \sigma f) y \rangle \geq (\mu \eta - \sigma \beta) \| x - y \|^2, \quad x, y \in H.
\]

That is, \( \mu B - \sigma f \) is strongly monotone with coefficient \( \mu \eta - \sigma \beta \).
Lemma 9. Let $H$ be a real Hilbert space. Then the following well-known results hold: for all $x, y \in H$ and $t \in [0, 1]$

(i) $\|x + y\|^2 \leq \|x\|^2 + 2(y, x + y)$;
(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$;
(iii) $\langle x, y \rangle = -(1/2)\|x - y\|^2 + (1/2)\|x\|^2 + (1/2)\|y\|^2$.

Lemma 10. Let $T$ be a $q$-strictly pseudononspreading mapping with $q \in (0, 1)$, and set $T_{\alpha} = (1 - \alpha)I + \alpha T$, $\alpha \in (0, 1)$. The following properties are reached for each $(x, p) \in H \times F_{\alpha}(T)$:

1. $\langle x - T_{\alpha}x, x - p \rangle \geq ((1 - q)/2)\|x - T_{\alpha}x\|^2$ and $\langle x - T_{\alpha}x, p - T_{\alpha}x \rangle \leq ((1 + q)/2)\|x - T_{\alpha}x\|^2$;
2. $\|T_{\alpha}x - p\|^2 \leq \|x - p\|^2 - \alpha(1 - q - \alpha)\|x - T_{\alpha}x\|^2$;
3. $\langle x - T_{\alpha}x, x - p \rangle \geq (\alpha(1 - q)/2)\|x - T_{\alpha}x\|^2$.

Proof. Note that property (1) is easily deduced from the Lemma 8(iii) and the fact that $T$ is $q$-strictly pseudononspreading mapping, we obtain

\[
\langle x - T_{\alpha}x, x - p \rangle = -\frac{1}{2}\|T_{\alpha}x - p\|^2 + \frac{1}{2}\|x - T_{\alpha}x\|^2
\]

\[
= -\frac{1}{2}\|T_{\alpha}x - p\|^2 + \frac{1}{2}\|x - T_{\alpha}x\|^2  
\]

\[
\geq -\frac{1}{2}\|T_{\alpha}x - p\|^2 + \frac{1}{2}\|T_{\alpha}x - p\|^2  
\]

\[
\geq \frac{1 - \theta}{2}\|x - T_{\alpha}x\|^2.  
\]

Property (2) is obtained from property (1) and by

\[
\|T_{\alpha}x - p\|^2 = \|x - p\|^2 - 2\alpha\langle x - p, x - T_{\alpha}x \rangle + \alpha^2\|T_{\alpha}x - x\|^2.  
\]

Property (3) is given by $I - T_{\alpha} = \alpha(I - T)$ and property (1).

Lemma 11 (see [25]). Let $\{T_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ which satisfies $T_{n_k} < T_{n_k+1}$ for all $j \geq 0$. Also consider the sequence of integers $\{\delta(n)\}_{n \geq n_0}$ defined by

\[
\delta(n) = \max \{k \leq n | T_k < T_{k+1}\}.  
\]

Then $\{\delta(n)\}_{n \geq n_0}$ is a nondecreasing sequence verifying $\lim_{n \to \infty} \delta(n) = \infty$, for all $n \geq n_0$; it holds that $T_{\delta(n)} < T_{\delta(n)+1}$ and one has

\[
T_n < T_{\delta(n)+1}.  
\]

Lemma 12. Let $K$ be a closed convex subset of a real Hilbert space $H$, given $x \in H$ and $y \in K$. Then $y = P_Kx$ if and only if there holds the inequality

\[
\langle x - y, y - z \rangle \geq 0, \quad \forall z \in K.  
\]

3. Main Results

In what follows, we will focus our attention on the following general two operator split common fixed point problem in real Hilbert space $H$:

\[
\text{find } x^* \in C \text{ such that } Ax^* \in D,  
\]

where $A : H_1 \rightarrow H_2$ is a bounded linear operator, $U : H_1 \rightarrow H_1$ and $V : H_2 \rightarrow H_2$ are two $q_i$-strictly pseudononspreading mappings $i = 1, 2$ with nonempty fixed point sets $F_i(U) = C$ and $F_i(V) = Q$, and denote the solution set of the two-operator SCFP by

\[
\Gamma = \{y \in C; Ay \in Q\}.  
\]

On the other hand, $x^* \in \Gamma$ is also unique solution of solving the variational inequality problem VIP($\mu B - \sigma F, T$):

\[
x^* \in \Gamma, \quad \langle (\mu B - \sigma F)x^*, y - x^* \rangle \geq 0, \quad y \in \Gamma,  
\]

where $B : H \rightarrow H$ is $h$-strongly monotone and $k$-Lipschitzian on $H$ with $k > 0, h > 0$. Let $0 < \mu < 2k/\sqrt{2}, 0 < \sigma < h(\eta - \mu k^2/2))/\beta = \tau/\beta$.

Before stating our main convergence result, we establish the boundedness of the iterates given by (9).

Lemma 13. The sequence $\{x_n\}$ is generated by (9), and let $U : H_1 \rightarrow H_1$ and $V : H_2 \rightarrow H_2$ be two $q_i$-strictly pseudononspreading mappings on $H_i, i = 1, 2$, and $f : H \rightarrow H$ is a contractive mapping with constant $\beta \in (0, 1)$, $\{\omega_i\} \subset (0, 1)$ and $0 < \omega_i < \omega_i < 1/2, i = 1, 2$. Then $\{x_n\}$ is bounded.

Proof. Set $T_{\omega_i} = (1 - \omega_i)I + \omega_i T$. Then $x_{n+1} = \omega_i Af(x_n) + (I - \omega_i B)T_{\omega_i}x_n$. 

\[\square\]
Taking \( y \in \Gamma \), that is, \( y \in F_{ix}(U) \) and \( Ay \in F_{ix}(V) \). We obtain
\[
\|x_{n+1} - y\| = \|\alpha_n \sigma (f (x_n) - f (y)) + \alpha_n (\sigma f (y) - \mu By) + (1 - \mu \alpha_n) B (T_{\alpha_n} x_n - y)\|
\leq \alpha_n \sigma \|f (x_n) - f (y)\| + \alpha_n \|\sigma f (y) - \mu By\|
+ (1 - \alpha_n \tau) \|T_{\alpha_n} x_n - y\|
\leq \alpha_n \sigma \|x_n - y\| + \alpha_n \|\sigma f (y) - \mu By\|
+ (1 - \alpha_n \tau) \|T_{\alpha_n} x_n - y\|.
\] (22)

From the definition of \( T_{\alpha_n} \), we have
\[
\|T_{\alpha_n} x_n - y\|^2 = \|x_n - y + \alpha_n (T_{\alpha_n} x_n - x_n)\|^2
= \|x_n - y\|^2 - 2\alpha_n (x_n - y, x_n - T_{\alpha_n} x_n)
+ \alpha_n^2 \|T_{\alpha_n} x_n - x_n\|^2.
\] (23)

On the other hand, we obtain
\[
\|T_{\alpha_n} x_n - y\|^2 = \|U (I + yA^* (V - I) A) x_n - y\|^2
\leq \|(I + yA^* (V - I) A)x_n - y\|^2
= \|x_n - y\|^2 + y^2 \|A^* (V - I) Ax_n\|^2
+ 2y (x_n - y, A^* (V - I) Ax_n)
= \|x_n - y\|^2 + y^2 \langle (V - I) Ax_n, A^* (V - I) Ax_n \rangle
+ 2y (x_n - y, A^* (V - I) Ax_n).
\] (24)

According to the definition of \( \lambda \), we have
\[
y^2 \langle (V - I) Ax_n, A^* (V - I) Ax_n \rangle
\leq \lambda y^2 \langle (V - I) Ax_n, (V - I) Ax_n \rangle
= \lambda y^2 \| (V - I) Ax_n \|^2.
\] (25)

Now, by using property (i) of Lemma 9, we obtain
\[
2y (x_n - y, A^* (V - I) Ax_n)
= 2y \langle A (x_n - y), (V - I) Ax_n \rangle
= 2y \langle A (x_n - y), (V - I) Ax_n \rangle
- (V - I) Ax_n, (V - I) Ax_n
= 2 y \langle V Ax_n - Ay, (V - I) Ax_n \rangle
\leq 2y \left( \frac{1 - \eta_2}{2} \| (V - I) Ax_n \|^2 \right)
\leq \| (V - I) Ax_n \|^2
\leq -\gamma (1 - \eta_2) \| (V - I) Ax_n \|^2.
\] (26)

Combining (24)–(26), we obtain
\[
\|T_{\alpha_n} x_n - y\|^2 = \|x_n - y\|^2 + \lambda y^2 \| (V - I) Ax_n \|^2
- y (1 - \eta_2) \| (V - I) Ax_n \|^2
= \|x_n - y\|^2 - y ((1 - \eta_2) - \lambda y)
\times \| (V - I) Ax_n \|^2
\leq \|x_n - y\|^2.
\] (27)

From property (i) of Lemma 9 and (23), we get
\[
\|T_{\alpha_n} x_n - y\|^2 \leq \|x_n - y\|^2 - (1 - \eta_2) \omega_n \|x_n - T_{\alpha_n} x_n\|^2
+ \omega_n^2 \|T_{\alpha_n} x_n - x_n\|^2
= \|x_n - y\|^2 - \omega_n (1 - \eta_2 - \omega_n)
\times \|x_n - T_{\alpha_n} x_n\|^2
\leq \|x_n - y\|^2.
\] (28)

Combining (22), (23), and (28), we have
\[
\|x_n+1 - y\| \leq \alpha_n \sigma \|x_n - y\| + \alpha_n \|f (y) - \mu By\|
+ (1 - \alpha_n \tau) \|T_{\alpha_n} x_n - y\|
= [1 - \alpha_n (\tau - \sigma \beta)] \|x_n - y\|
+ \alpha_n \|f (y) - \mu By\|
\leq \max \left\{ \|x_n - y\|, \frac{1}{\tau - \sigma \beta} \|f (y) - \mu By\| \right\}.
\] (29)

It follows from (29) and induction that
\[
\|x_{n+1} - y\| \leq \max \left\{ \|x_0 - y\|, \frac{1}{\tau - \sigma \beta} \|f (y) - \mu By\| \right\},
\] (30)

and hence \( \{x_n\} \) is bounded. \( \Box \)

Now we are in position to claim the main convergence result.
**Theorem 14.** Given a bounded linear operator \( A : H_1 \to H_2 \), let \( U : H_1 \to H_1 \) and \( V : H_2 \to H_2 \) be two \( \varphi \)-strictly pseudomonotone mappings, and \( i = 1, 2 \) with fixed point \( F_i(U) = C \) and \( F_i(V) = Q \). Assume that \( U - I \) and \( V - I \) are demiclosed at origin. Let \( B : H \to H \) be \( \eta \)-strongly monotone and \( k \)-Lipschitzian on \( H \) with \( k > 0, \eta > 0 \), and \( f : H \to H \) is a contractive mapping with constant \( \beta \in (0, 1) \).

Assume that \( \{x_n\} \) is the sequence given by Algorithm 4 with \( y \in (0,1/\lambda) \), \( 0 < \varphi < \omega_2 < 1/2 \), and \( i = 1, 2 \) such that \( 0 < \lim \inf_{n \to \infty} \omega_n \leq \lim \sup_{n \to \infty} \omega_n < 1/2 \) and \( \alpha_n \in (0, 1) \) such that \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \). If \( \Gamma \neq \emptyset \), then the sequence \( \{x_n\} \) strongly converges to a split common fixed point \( y \in \Gamma \), verifying \( y = P_\Gamma(I - \mu B + \sigma f)(y) \) which equivalently solves the following variational inequality problem:

\[
y \in \Gamma, \quad \langle (\mu B - \sigma f) y, x^* - y \rangle \geq 0, \quad x^* \in \Gamma.
\]

**Proof.** Let \( y \) be the solution of (31). From (9) we obtain that

\[
x_{n+1} - x_n + \alpha_n (\mu B x_n - \sigma f(x_n)) = (I - \mu \alpha_n B) (T_{\omega_n} x_n - x_n), \tag{32}
\]

hence

\[
\langle x_{n+1} - x_n + \alpha_n (\mu B x_n - \sigma f(x_n)), x_n - y \rangle
\]

\[
= \langle (I - \mu \alpha_n B) (T_{\omega_n} x_n - x_n), x_n - y \rangle
\]

\[
= (1 - \alpha_n) \langle T_{\omega_n} x_n - x_n - x_n, x_n - y \rangle
\]

\[
+ \alpha_n \langle (I - \mu B) (T_{\omega_n} x_n - x_n), x_n - y \rangle
\]

\[
\leq (1 - \alpha_n) \langle T_{\omega_n} x_n - x_n, x_n - y \rangle
\]

\[
+ \alpha_n \langle (I - \mu B) (T_{\omega_n} x_n - x_n), x_n - y \rangle
\]

\[
\leq (1 - \alpha_n) \langle T_{\omega_n} x_n - x_n, x_n - y \rangle
\]

\[
+ \alpha_n (1 - \tau) \| T_{\omega_n} x_n - x_n \| \| x_n - y \|.
\]

By (28), we obtain that

\[
\langle x_n - T_{\omega_n} x_n, x_n - y \rangle = \frac{\omega_n^2}{2} \| x_n - T_{\omega_n} x_n \|^2 + \frac{\omega_n^2}{2} \| x_n - y \|^2 - \frac{\omega_n^2}{2} \| T_{\omega_n} x_n - y \|^2
\]

\[
\geq \frac{\omega_n^2}{2} \| x_n - T_{\omega_n} x_n \|^2 + \frac{\omega_n^2}{2} \| x_n - y \|^2
\]

\[
- \frac{\omega_n^2}{2} \| x_n - y \|^2
\]

\[
+ \frac{\omega_n^2}{2} (1 - \varphi_2 - \omega_n) \| x_n - T_{\omega_n} x_n \|^2
\]

\[
= \frac{\omega_n}{2} (1 - \varphi_2) \| x_n - T_{\omega_n} x_n \|^2.
\]

It follows from (33) that

\[
\langle x_{n+1} - x_n + \alpha_n (\mu B x_n - \sigma f(x_n)), x_n - y \rangle
\]

\[
\leq \frac{\omega_n}{2} (1 - \varphi_2) \| x_n - T_{\omega_n} x_n \|^2
\]

\[
+ \omega_n \alpha_n (1 - \tau) \| T_{\omega_n} x_n - x_n \| \| x_n - y \|.
\]

or equivalently

\[
- \langle x_{n+1} - x_n, x_n - y \rangle \leq - \alpha_n \langle (\mu B - \sigma f) x_n, x_n - y \rangle
\]

\[
- \frac{\omega_n}{2} (1 - \varphi_2) (1 - \alpha_n) \| x_n - T_{\omega_n} x_n \|^2
\]

\[
+ \omega_n \alpha_n (1 - \tau) \| T_{\omega_n} x_n - x_n \| \| x_n - y \|.
\]

Furthermore, using the classical equality (iii) in Lemma 10 and setting \( T_{\omega_n} = (1/2) \| x_n - y \|^2 \), we have

\[
\langle x_{n+1} - x_n, x_n - y \rangle = T_{\omega_n} - T_{\omega_n+1} + \frac{1}{2} \| x_{n+1} - x_n \|^2.
\]

So that (36) can be equivalently rewritten as

\[
T_{\omega_n} - T_{\omega_n+1} = \frac{1}{2} \| x_{n+1} - x_n \|^2
\]

\[
\leq - \alpha_n \langle (\mu B - \sigma f) x_n, x_n - y \rangle
\]

\[
- \frac{\omega_n}{2} (1 - \varphi_2) (1 - \alpha_n) \| x_n - T_{\omega_n} x_n \|^2
\]

\[
+ \omega_n \alpha_n (1 - \tau) \| T_{\omega_n} x_n - x_n \| \| x_n - y \|.
\]

Now using (32) again, we have

\[
\| x_{n+1} - x_n \|^2 = \alpha_n \langle (\sigma f(x_n) - \mu B x_n), x_{n+1} - x_n \rangle
\]

\[
+ (1 - \mu \alpha_n B) \| T_{\omega_n} x_n - x_n \|^2.
\]

Since \( B : H \to H \) is \( \eta \)-strongly monotone and \( k \)-Lipschitzian on \( H \), hence it is a classical matter to see that

\[
\| x_{n+1} - x_n \|^2 \leq 2 \alpha_n^2 \| \sigma f(x_n) - \mu B x_n \|^2
\]

\[
+ 2 (1 - \alpha_n \tau)^2 \| T_{\omega_n} x_n - x_n \|^2,
\]

which by \( \| T_{\omega_n} x_n - x_n \| = \omega_n \| x_n - T_{\omega_n} x_n \| \) yields

\[
\frac{1}{2} \| x_{n+1} - x_n \|^2 \leq \alpha_n^2 \| \sigma f(x_n) - \mu B x_n \|^2
\]

\[
+ (1 - \alpha_n \tau)^2 \omega_n^2 \| x_n - T_{\omega_n} x_n \|^2.
\]

Then from (38) and (41), we have

\[
T_{\omega_n} - T_{\omega_n+1} \leq \alpha_n \left[ \alpha_n \| \sigma f(x_n) - \mu B x_n \|^2
\]

\[
- \langle (\mu B - \sigma f) x_n, x_n - y \rangle
\]

\[
+ \omega_n (1 - \tau) \| T_{\omega_n} x_n - x_n \| \| x_n - y \| \right].
\]

The rest of the proof will be divided into two parts.
Case 1. Suppose that there exists \( n_0 \) such that \( \{T_n\}_{n \geq n_0} \) is nonincreasing. In this situation, \( \{T_n\} \) is then convergent because it is also nonnegative (hence it is bounded from below), so that \( \lim_{n \to \infty} (T_{n+1} - T_n) = 0 \); hence, in light of (42) together with \( \lim_{n \to \infty} \alpha_n = 0 \), the boundedness of \( \{x_n\} \) and \( 0 \leq \inf_{n \to \infty} \alpha_n \leq \sup_{n \to \infty} \alpha_n < 1/2 \), we obtain
\[
\lim_{n \to \infty} \|x_n - Tx_n\| = 0. \tag{43}
\]
It also follows from (42) that
\[
T_n - T_{n+1} \geq \alpha_n \left( -\alpha_n \|\sigma f(x_n) - \mu Bx_n\|^2 + \left( (\mu B - \sigma f)x_n, x_n - y \right) - \omega_n (1 - \tau) \|T_n x_n - x_n\| \|x_n - y\| \right), \tag{44}
\]
Then, by \( \sum_{n=0}^{\infty} \alpha_n = \infty \), we obviously deduce that
\[
\liminf_{n \to \infty} \alpha_n \left( -\alpha_n \|\sigma f(x_n) - \mu Bx_n\|^2 + \left( (\mu B - \sigma f)x_n, x_n - y \right) - \omega_n (1 - \tau) \|T_n x_n - x_n\| \|x_n - y\| \right) \leq 0
\]
or equivalently (as \( \alpha_n \|\sigma f(x_n) - \mu Bx_n\|^2 \to 0 \) and \( \lim_{n \to \infty} (T_n + T_{n+1}) = 0 \)). From (45), we get
\[
\liminf_{n \to \infty} \left( (\mu B - \sigma f)x_n, x_n - y \right) \leq 0. \tag{46}
\]
Moreover, by Lemma 8, we have
\[
2 (\mu \eta - \sigma \beta) T_n + (\mu B - \sigma f) y, x_n - y \right) \leq 0. \tag{47}
\]
which by (46) entails
\[
\liminf_{n \to \infty} 2 (\mu \eta - \sigma \beta) T_n + (\mu B - \sigma f) y, x_n - y \right) \leq 0. \tag{48}
\]
Hence, recalling that \( \lim_{n \to \infty} T_n = T \) exists, we equivalently obtain
\[
2 (\mu \eta - \sigma \beta) \lim_{n \to \infty} T_n + \liminf_{n \to \infty} (\mu B - \sigma f) y, x_n - y \right) \leq 0. \tag{49}
\]
Namely,
\[
2 (\mu \eta - \sigma \beta) \lim_{n \to \infty} T_n \leq -\liminf_{n \to \infty} (\mu B - \sigma f) y, x_n - y. \tag{50}
\]
Now we prove that
\[
\liminf_{n \to \infty} (\mu B - \sigma f) y, x_n - y \right) \leq 0. \tag{51}
\]
It follows from (27) and (43) that
\[
y (1 - \epsilon_2) \|V - I\| A x_n^2 \leq \|x_n - y\|^2 - \|Tx_n - y\|^2 = (\|x_n - y\|^2 - \|Tx_n - y\|^2)
\]
\[
\times (\|x_n - y\| + \|Tx_n - y\|)
\]
\[
= (\|x_n - Tx_n\|) (\|x_n - y\| + \|Tx_n - y\|) \to 0, \quad (n \to \infty) ,
\]
and hence
\[
\lim_{n \to \infty} (V - I) A x_n = 0. \tag{53}
\]
Taking \( x^* \in \omega_w (x_n) \), from the demiclosedness of \( V - I \) at 0, we have
\[
V (A x^*) = A x^* . \tag{54}
\]
Now, by setting \( u_n = x_n + y A^* V - I A x_n \), it follows that \( x^* \in \omega_w (u_n) \). On the other hand,
\[
\|U (u_n) - u_n\| = \|Tx_n - x_n - y A^* V - I A x_n\| \leq \|Tx_n - x_n\| + y \|A^*\| \|V - I A x_n\| \to 0, \tag{55}
\]
which, combined with the demiclosedness of \( U - I \) at 0, yields
\[
U x^* = x^* . \tag{56}
\]
Hence, \( x^* \in C \) and \( x^* \in \Gamma \). We can take subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \lim_{n \to \infty} x_{n_k} = x^* \) and
\[
\liminf_{n \to \infty} (\sigma f - \mu B) y, x_{n_k} - y \right) = \lim_{j \to \infty} \left( (\sigma f - \mu B) y, x_{n_j} - y \right), \tag{57}
\]
which leads to
\[
\liminf_{n \to \infty} (\sigma f - \mu B) y, x_{n_k} - y \right) = (\sigma f - \mu B) y, x^* - y \right) \leq 0. \tag{58}
\]
By (50), we have \( \lim_{n \to \infty} \Gamma_n = 0 \), and hence \( \{x_n\} \) converges strongly to \( y \).

Case 2. Suppose that there exists a subsequence \( \{T_n\}_{n \geq n_0} \) of \( \{T_n\}_{n \geq 0} \) such that \( \lim_{n \to \infty} \frac{(1 - \epsilon_2)(1 - \alpha_{\delta(n)}) - \omega_n^2 (1 - \alpha_{\delta(n)} \tau)^{2}}{\|x_{\delta(n)} - Tx_{\delta(n)}\|} \leq \alpha_{\delta(n)} \|\sigma f (x_{\delta(n)}) - \mu B x_{\delta(n)}\| \]
\[
+ \omega_n (1 - \tau) \|Tx_{\delta(n)} - x_{\delta(n)}\| \|x_{\delta(n)} - y\|. \tag{59}
\]
By the boundedness of \( \{x_n\} \) and \( \lim_{n \to \infty} x_n = 0 \), we immediately obtain
\[
\lim_{n \to \infty} \|x_{\delta(n)} - Tx_{\delta(n)}\| = 0. \tag{60}
\]

Using (9), we have
\[
\|x_{\delta(n)+1} - x_{\delta(n)}\| \leq \alpha_{\delta(n)} \|\sigma f(x_{\delta(n)}) - \mu Bx_{\delta(n)}\| + |1 - \alpha_{\delta(n)}| \|T_{\omega_{\delta(n)}} x_{\delta(n)} - x_{\delta(n)}\| \leq \alpha_{\delta(n)} \|\sigma f(x_{\delta(n)}) - \mu Bx_{\delta(n)}\| + |1 - \alpha_{\delta(n)}| \omega_{\delta(n)} \|x_{\delta(n)} - x_{\delta(n)}\|,
\]
which together with (60) and \( \lim_{n \to \infty} \alpha_n = 0 \) yields
\[
\lim_{n \to \infty} \|x_{\delta(n)+1} - x_{\delta(n)}\| = 0. \tag{62}
\]

Similar to Case 1, we have
\[
\lim_{n \to \infty} \inf_{n} \langle (\mu B - \sigma f) y, x_{\delta(n)} - y \rangle \geq 0. \tag{63}
\]

Now by (59) we clearly have
\[
\alpha_{\delta(n)} \|\sigma f(x_{\delta(n)}) - \mu Bx_{\delta(n)}\|^2 + \alpha (1 - \tau) \langle x_{\delta(n)} - y, x_{\delta(n)} - y \rangle \geq \langle (\mu B - \sigma f) x_{\delta(n)}, x_{\delta(n)} - y \rangle,
\]
which in the light of (47) yields
\[
2(\mu \eta - \sigma \beta) \mathcal{T}_{\delta(n)} + \langle (\mu B - \sigma f) y, x_{\delta(n)} - y \rangle \leq \alpha_{\delta(n)} \|\sigma f(x_{\delta(n)}) - \mu Bx_{\delta(n)}\|^2 + \alpha (1 - \tau) \langle x_{\delta(n)} - y, x_{\delta(n)} - y \rangle. \tag{64}
\]

Hence (as \( \lim_{n \to \infty} \alpha_{\delta(n)} = 0 \) and \( \lim_{n \to \infty} \|T_{\delta(n)} x_{\delta(n)} - x_{\delta(n)}\| = 0 \) it follows that
\[
2(\mu \eta - \sigma \beta) \limsup_{n \to \infty} \mathcal{T}_{\delta(n)} \leq -\liminf_{n \to \infty} \langle (\mu B - \sigma f) y, x_{\delta(n)} - y \rangle. \tag{66}
\]

From (59) and (63), we obtain
\[
\lim_{n \to \infty} \langle (\mu B - \sigma f) y, x_{\delta(n)} - y \rangle \geq 0, \tag{67}
\]
which by (60) yields \( \limsup_{n \to \infty} \mathcal{T}_{\delta(n)} = 0 \), so that \( \lim_{n \to \infty} \mathcal{T}_{\delta(n)} = 0 \). Combining (62), we have \( \lim_{n \to \infty} \mathcal{T}_{\delta(n)} = 0 \). Then, recalling that \( \mathcal{T}_{n} < \mathcal{T}_{\delta(n)+1} \) (by Lemma 11), we get \( \lim_{n \to \infty} \mathcal{T}_{n} = 0 \), so that \( x_n \to y \) strongly. In addition, the variational inequality (50) and (67) can be written as
\[
\langle (I - \mu B + \sigma f) y - y, x - y \rangle \leq 0, \quad x^* \in \Gamma. \tag{68}
\]

So, by the Lemma 12, it is equivalent to the fixed point equation
\[
P_{\Gamma} (I - \mu B + \sigma f) y = y. \tag{69}
\]

4. Application in Other Nonlinear Operators

In order to define our motivations, we recall some definitions of classed of operators as follows

**Definition 15.** \( T : D(T) \subseteq H \to H \) is said to be

(1) nonspreading in \([26, 27]\), if
\[
\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C, \tag{70}
\]

(2) demicontractive in \([28]\), if there exists a constant \( \alpha < 1 \) such that
\[
\|Tx - q\|^2 \leq \|x - q\|^2 + \alpha \|x - Tx\|^2, \quad \forall (x, q) \in H \times F_{ix}(T). \tag{71}
\]

Remark 16. Iemoto and Takahashi \([29]\) proved that (70) is equivalent to
\[
\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2 \langle x - Tx, y - Ty \rangle, \quad \forall x, y \in C. \tag{72}
\]

Iterative methods for nonspreading mapping have been extensively investigated; see \([30–34]\).

Remark 17. From the Definition 5 (3), Definition 7, and Definition 15, we have the following facts.

(i) Observe that every nonspreading mapping is 0-strictly pseudononspreading.

(ii) If \( T \) is nonspreading mapping and the set of fixed point is nonempty, then \( T \) is quasi-nonexpansive mapping.

(iii) Every nonspreading mapping with a nonempty fixed point set \( F_{ix}(T) \) is demicontractive (see \([28]\)).

**Corollary 18.** Given a bounded linear operator \( A : H_1 \to H_2 \), let \( U : H_1 \to H_1 \) and \( V : H_2 \to H_2 \) be two nonspreading mappings with fixed point \( F_{ix}(U) = C \) and \( F_{ix}(V) = Q \). Assume that \( U - I \) and \( V - I \) are demicontractive at origin. Let \( B : H \to H \) be \( \eta \)-strongly monotone and \( k \)-Lipschitzian on \( H \) with \( k > 0 \), \( \eta > 0 \), and let \( f : H \to H \) be a contractive mapping with constant \( \beta \in (0, 1) \). Let \( \{x_n\} \) be the sequence given by (9) with \( y \in (0, 1/\lambda) \), \( 0 < q_0 < \omega_\eta < 1/2 \), and \( i = 1, 2 \) such that \( 0 < \lim_{n \to \infty} \omega_n \leq \limsup_{n \to \infty} \omega_n < 1/2 \) and \( \alpha_n \in (0, 1) \) such that \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{i=1}^{\infty} \alpha_n = \infty \). If \( \Gamma \neq \emptyset \), then the sequence \( \{x_n\} \) strongly converges to a split common fixed point \( x^* \in \Gamma \), verifying \( x^* = P_{\Gamma}(I - \mu B + \sigma f(x^*)) \) which equivalently solves the following variational inequality problem:
\[
x^* \in \Gamma, \quad \langle (\mu B - \sigma f)x^*, y - x^* \rangle \geq 0, \quad y \in \Gamma. \tag{73}
\]

**Proof.** Form the proof of the Theorem 14, we can easily certify this theorem by nonspreading mapping (i.e., nonspreading is 0-strictly pseudononspreading).

From the Remark 17(ii) and the Corollary 18, we have the following corollary.
Corollary 19. Given a bounded linear operator \( A : H_1 \to H_2 \), let \( U : H_1 \to H_1 \) and \( V : H_2 \to H_2 \) be two quasi-nonexpansive mappings with fixed point \( F_\omega(U) = C \) and \( F_\nu(V) = Q \). Assume that \( U - I \) and \( V - I \) are demicontractions and \( B : H \to H \) be a contractive mapping with constant \( \beta \in (0, 1) \). Let \( (x_n) \) be the sequence given by (8) with \( \gamma \in (0, 1/2) \), \( \omega_n \in (0, 1/2) \) such that \( 0 < \liminf_{n \to \infty} \omega_n \leq \limsup_{n \to \infty} \omega_n < 1/2 \) and \( \sigma_n \in (0, 1) \) such that \( \lim_{n \to \infty} \sigma_n = 0 \) and \( \sum_{n=1}^{\infty} \sigma_n = \infty \). If \( \Gamma \neq \emptyset \), then the sequence \((x_n)\) strongly converges to a split common fixed point \( x^* \in \Gamma \), verifying \( x^* = P_{\Gamma}(I - \mu B + \sigma f)(x^*) \) which equivalently solves the following variational inequality problem:

\[
x^* \in \Gamma, \quad \langle (\mu B - \sigma f)x^*, y - x^* \rangle \geq 0, \quad y \in \Gamma. \tag{74}
\]

If \( \sigma = \mu = 1 \) and \( B = 1 \) in (9), thus \( k = \eta = 1 \) and \( \beta \in (0, 1/2) \), and then we obtain (8) and (corollary) following the corollary. On the other hand, this corollary was proven by Zhao and He [14].

Corollary 20. Given a bounded linear operator \( A : H_1 \to H_2 \), let \( U : H_1 \to H_1 \) and \( V : H_2 \to H_2 \) be two quasi-nonexpansive mappings with fixed point \( F_\omega(U) = C \) and \( F_\nu(V) = Q \). Assume that \( U - I \) and \( V - I \) are demicontractions and \( B : H \to H \) be a contractive mapping with constant \( \beta \in (0, 1/2) \). Let \( (x_n) \) be the sequence given by (8) with \( \gamma \in (0, 1/2) \), \( \omega_n \in (0, 1/2) \) such that \( 0 < \liminf_{n \to \infty} \omega_n \leq \limsup_{n \to \infty} \omega_n < 1/2 \), and \( \alpha_n \in (0, 1) \) such that \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \). If \( \Gamma \neq \emptyset \), then the sequence \((x_n)\) strongly converges to a split common fixed point \( x^* \in \Gamma \), verifying \( x^* = P_{\Gamma}(I - \mu B + \sigma f)(x^*) \) which equivalently solves the following variational inequality problem:

\[
x^* \in \Gamma, \quad \langle (I - f)x^*, y - x^* \rangle \geq 0, \quad y \in \Gamma. \tag{75}
\]

References


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