Research Article

Completing a $2 \times 2$ Block Matrix of Real Quaternions with a Partial Specified Inverse

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This paper considers a completion problem of a nonsingular $2 \times 2$ block matrix over the real quaternion algebra $\mathbb{H}$. Let $m_1, m_2, n_1, n_2$ be nonnegative integers, $m_1 + m_2 = n_1 + n_2 = n > 0$, and $A_{12} \in \mathbb{H}^{m_1 \times n_2}, A_{21} \in \mathbb{H}^{m_2 \times n_1}, A_{22} \in \mathbb{H}^{m_2 \times n_2}, B_{11} \in \mathbb{H}^{n_1 \times m_1}$ be given. We determine necessary and sufficient conditions so that there exists a variant block entry matrix $A_{11} \in \mathbb{H}^{m_1 \times n_1}$ such that $A = (A_{11} A_{12})$ is nonsingular, and $B_{11}$ is the upper left block of a partitioning of $A^{-1}$. The general expression for $A_{11}$ is also obtained. Finally, a numerical example is presented to verify the theoretical findings.

1. Introduction

The problem of completing a block-partitioned matrix of a specified type with some of its blocks given has been studied by many authors. Fiedler and Markham [1] considered the following completion problem over the real number field $\mathbb{R}$. Suppose $m_1, m_2, n_1, n_2$ are nonnegative integers, $m_1 + m_2 = n_1 + n_2 = n > 0$, $A_{11} \in \mathbb{R}^{m_1 \times n_1}, A_{12} \in \mathbb{R}^{m_1 \times n_2}, A_{21} \in \mathbb{R}^{m_2 \times n_1},$ and $B_{22} \in \mathbb{R}^{n_2 \times m_2}$. Determine a matrix $A_{22} \in \mathbb{R}^{m_2 \times n_2}$ such that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad (1)$$

is nonsingular and $B_{22}$ is the lower right block of a partitioning of $A^{-1}$. This problem has the form of

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & ? \end{pmatrix}^{-1} = \begin{pmatrix} ? & ? \\ ? & B_{22} \end{pmatrix}, \quad (2)$$

and the solution and the expression for $A_{22}$ were obtained in [1]. Dai [2] considered this form of completion problems with symmetric and symmetric positive definite matrices over $\mathbb{R}$.

Some other particular forms for $2 \times 2$ block matrices over $\mathbb{R}$ have also been examined (see, e.g., [3]), such as

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & ? \end{pmatrix}^{-1} = \begin{pmatrix} B_{11} & ? \\ ? & ? \end{pmatrix}, \quad (3)$$

The real quaternion matrices play a role in computer science, quantum physics, and so on (e.g., [4–6]). Quaternion matrices are receiving much attention as witnessed recently (e.g., [7–9]). Motivated by the work of [1, 10] and keeping such applications of quaternion matrices in view, in this paper we consider the following completion problem over the real quaternion algebra:

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \mid i^2 = j^2 = k^2 = ijk = -1 \text{ and } a_0, a_1, a_2, a_3 \in \mathbb{R} \}.$$

Problem 1. Suppose $m_1, m_2, n_1, n_2$ are nonnegative integers, $m_1 + m_2 = n_1 + n_2 = n > 0$, and $A_{12} \in \mathbb{H}^{m_1 \times n_2},$
$A_{21} \in \mathbb{H}^{m\times n_1}$, $A_{22} \in R^{m\times n_2}$, $B_{11} \in \mathbb{H}^{n_1\times n_1}$. Find a matrix $A_{11} \in \mathbb{H}^{n_1\times n_1}$ such that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{H}^{m\times n}$$

is nonsingular, and $B_{11}$ is the upper left block of a partitioning of $A^{-1}$. That is

$$\begin{pmatrix} ? & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} B_{11} & ? \\ ? & ? \end{pmatrix},$$

where $\mathbb{H}^{m\times n}$ denotes the set of all $m \times n$ matrices over $\mathbb{H}$ and $A^{-1}$ denotes the inverse matrix of $A$.

Throughout, over the real quaternion algebra $\mathbb{H}$, we denote the identity matrix with the appropriate size by $I$, the transpose of $A$ by $A^T$, the rank of $A$ by $r(A)$, the conjugate transpose of $A$ by $A^* = (A)^T$, a reflexive inverse of a matrix $A$ over $\mathbb{H}$ by $A^+$ which satisfies simultaneously $AA^+A = A$ and $A^+AA^+ = A^+$. Moreover, $L_A = I - A^*A$, $R_A = I - AA^*$, where $A^+$ is an arbitrary but fixed reflexive inverse of $A$. Clearly, $L_A$ and $R_A$ are idempotent, and each is a reflexive inverse of itself. $\mathcal{R}(A)$ denotes the right column space of the matrix $A$.

The rest of this paper is organized as follows. In Section 2, we establish some necessary and sufficient conditions to solve Problem 1 over $\mathbb{H}$, and the general expression for $A_{11}$ is also obtained. In Section 3, we present a numerical example to illustrate the developed theory.

## 2. Main Results

In this section, we begin with the following lemmas.

**Lemma 1** (singular-value decomposition [9]). Let $A \in \mathbb{H}^{m\times n}$ be of rank $r$. Then there exist unitary quaternion matrices $U \in \mathbb{H}^{m\times m}$ and $V \in \mathbb{H}^{n\times n}$ such that

$$UAV = \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix},$$

where $D_r = \text{diag}(d_1, \ldots, d_r)$ and the $d_i$'s are the positive singular values of $A$.

Let $\mathbb{H}_c^n$ denote the collection of column vectors with $n$ components of quaternions and $A$ be an $m \times n$ quaternion matrix. Then the solutions of $Ax = 0$ form a subspace of $\mathbb{H}_c^n$ of dimension $n(A)$. We have the following lemma.

**Lemma 2.** Let

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

be a partitioning of a nonsingular matrix $A \in \mathbb{H}^{m\times n}$, and let

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

be the corresponding (i.e., transpose) partitioning of $A^{-1}$. Then $n(A_{11}) = n(B_{22})$.

**Proof.** It is readily seen that

$$\begin{pmatrix} B_{22} & B_{21} \\ B_{12} & B_{11} \end{pmatrix}, \quad \begin{pmatrix} A_{22} & A_{21} \\ A_{12} & A_{11} \end{pmatrix}$$

are inverse to each other, so we may suppose that $n(A_{11}) < n(B_{22})$.

If $n(B_{22}) = 0$, necessarily $n(A_{11}) = 0$ and we are finished. Let $n(B_{22}) = c > 0$, then there exists a matrix $F$ with $c$ right linearly independent columns, such that $B_{22}F = 0$. Then, using

$$A_{11}B_{12} + A_{12}B_{22} = 0,$$

we have

$$A_{11}B_{12}F = 0.$$  \hfill (12)

From

$$A_{21}B_{12} + A_{22}B_{22} = I,$$

we have

$$A_{21}B_{12}F = F.$$  \hfill (14)

It follows that the rank $r(B_{12}F) \geq c$. In view of (12), this implies

$$n(A_{11}) \geq r(B_{12}F) \geq c = n(B_{22}).$$  \hfill (15)

Thus

$$n(A_{11}) = n(B_{22}).$$  \hfill (16) \hfill \Box

**Lemma 3** (see [10]). Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{p\times q}$, $D \in \mathbb{H}^{p\times q}$ be known and $X \in \mathbb{H}^{m\times p}$ unknown. Then the matrix equation

$$AXB = D$$

is consistent if and only if

$$AA^*DB^* = D.$$  \hfill (18)

In that case, the general solution is

$$X = A^*DB^* + L_AX_1 + Y_2R_B,$$

where $X_1, Y_2$ are any matrices with compatible dimensions over $\mathbb{H}$.

By Lemma 1, let the singular value decomposition of the matrix $A_{22}$ and $B_{11}$ in Problem 1 be

$$A_{22} = Q\begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}R^*,$$

$$B_{11} = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}V^*.$$  \hfill (20)
where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_s)$ is a positive diagonal matrix, $\lambda_i \neq 0$ ($i = 1, \ldots, s$) are the singular values of $A_{22}$, $s = r(A_{22})$, $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$ is a positive diagonal matrix, $\sigma_i \neq 0$ ($i = 1, \ldots, r$) are the singular values of $B_{11}$ and $r = r(B_{11})$.

$$Q = (Q_1 \ Q_2) \in \mathbb{H}^{m \times m_2}, \ R = (R_1 \ R_2) \in \mathbb{H}^{m \times m_2},$$

$$U = (U_1 \ U_2) \in \mathbb{H}^{n \times m}, \ V = (V_1 \ V_2) \in \mathbb{H}^{n \times m},$$

are unitary quaternion matrices, where $Q_i \in \mathbb{H}^{m \times x}$, $R_i \in \mathbb{H}^{m \times x}$, $U_i \in \mathbb{H}^{n \times x}$, and $V_i \in \mathbb{H}^{n \times x}$.

**Theorem 4.** Problem 1 has a solution if and only if the following conditions are satisfied:

(a) $r(A_{12}) = n_2$,

(b) $n_2 - r(A_{22}) = m_1 - r(B_{11})$, that is $n_2 - s = m_1 - r$,

(c) $\mathcal{R}(A_{21}B_{11}) \subset \mathcal{R}(A_{22})$,

(d) $\mathcal{R}(A_{12}B_{11}) \subset \mathcal{R}(A_{12})$.

In that case, the general solution has the form of

$$A_{11} = B_{11}^+ + A_{12}R \begin{pmatrix} \Lambda^{-1}Q_1^*A_{21}U_1\Sigma & 0 \\ H & -(V_2^*A_{12}R_2)^{-1} \end{pmatrix}^{-1} \times V^*B_{11}^+ + Y - YB_{11}B_{11}^+,$$

where $H$ is an arbitrary matrix in $\mathbb{H}^{(n_2-s) \times r}$ and $Y$ is an arbitrary matrix in $\mathbb{H}^{m \times (n_2-s)}$.

**Proof.** If there exists an $m_1 \times n_1$ matrix $A_{11}$ such that $A$ is nonsingular and $B_{11}$ is the corresponding block of $A^{-1}$, then (a) is satisfied. From $AB = BA = I$, we have that

$$A_{21}B_{11} + A_{22}B_{21} = 0,$$

$$B_{11}A_{11} + B_{12}A_{22} = 0,$$

so that (c) and (d) are satisfied.

By (11), we have

$$r(A_{22}) + n(A_{22}) = n_2, \quad r(B_{11}) + n(B_{11}) = m_1. \quad (24)$$

From Lemma 2, Notice that $\left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix} \right)$ is the corresponding partitioning of $B^{-1}$, we have

$$n(B_{11}) = n(A_{22}), \quad (25)$$

implying that (b) is satisfied.

Conversely, from (c), we know that there exists a matrix $K \in \mathbb{H}^{m \times m_1}$ such that

$$A_{21}B_{11} = A_{22}K. \quad (26)$$

Let

$$B_{21} = -K. \quad (27)$$

From (20), (21), and (26), we have

$$A_{21}U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^* = Q \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} R^* K. \quad (28)$$

It follows that

$$Q^*A_{21}U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^* = Q^*Q \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} R^* K. \quad (29)$$

This implies that

$$\begin{pmatrix} Q_1^*A_{21}U_1^* & Q_1^*A_{21}U_2^* \\ Q_2^*A_{21}U_1^* & Q_2^*A_{21}U_2^* \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R_1^*KV_1 & R_1^*KV_2 \\ R_2^*KV_1 & R_2^*KV_2 \end{pmatrix}. \quad (30)$$

Comparing corresponding blocks in (30), we obtain

$$Q_2^*A_{21}U_1 = 0. \quad (31)$$

Let $R^*KV = \tilde{K}$. From (29), (30), we have

$$\tilde{K} = \begin{pmatrix} \Lambda^{-1}Q_1^*A_{21}U_1\Sigma & 0 \\ H & K_{22} \end{pmatrix}, \quad (32)$$

$$H \in \mathbb{H}^{(n_2-s) \times r}, \quad K_{22} \in \mathbb{H}^{(n_2-s) \times (m_1-r)}.$$

In the same way, from (d), we can obtain

$$V_2^*A_{12}R_2 = 0. \quad (33)$$

Notice that $\left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix} \right)$ in (a) is a full column rank matrix. By (20), (21), and (33), we have

$$\begin{pmatrix} 0 & \quad Q^* \\ V^* & \quad 0 \end{pmatrix} \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} R = \begin{pmatrix} \quad \Lambda & 0 \\ 0 & 0 \\ V_1^*A_{12}R_1 & V_1^*A_{12}R_2 \\ V_2^*A_{12}R_1 & V_2^*A_{12}R_2 \end{pmatrix}, \quad (34)$$

so that

$$n_2 = r\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} = r\begin{pmatrix} \quad 0 & \quad Q^* \\ V^* & \quad 0 \end{pmatrix} \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} R = r\begin{pmatrix} \Lambda & 0 \\ 0 & 0 \\ V_1^*A_{12}R_1 & V_1^*A_{12}R_2 \\ V_2^*A_{12}R_1 & V_2^*A_{12}R_2 \end{pmatrix} = r\begin{pmatrix} \Lambda \quad 0 \\ 0 & 0 \\ V_1^*A_{12}R_2 \\ V_2^*A_{12}R_2 \end{pmatrix} \quad (35)$$

It follows from (b) and (35) that $V_2^TA_{12}R_2$ is a full column rank matrix, so it is nonsingular.

From $AB = I$, we have the following matrix equation:

$$A_{11}B_{11} + A_{12}B_{21} = I, \quad (36)$$

that is

$$A_{11}B_{11} = I - A_{12}B_{21}, \quad I \in \mathbb{H}^{m \times m_1}. \quad (37)$$
where \( B_{11}, A_{12} \) were given, \( B_{21} = -K \) (from (27)). By Lemma 3, the matrix equation (37) has a solution if and only if
\[
(I - A_{12}B_{21}) B_{11}^+ B_{11} = I - A_{12}B_{21}.
\]
(38)
By (21), (27), (32), and (33), we have that (38) is equivalent to:
\[
(I + A_{12}K) V \begin{pmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^* U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^* = I + A_{12}K.
\]
(39)
We simplify the equation above. The left hand side reduces to
\[
(I + A_{12}K) (V_1 V_1^*) + B_{11} B_{11}^* = I + A_{12}K.
\]
(40)
So,
\[
A_{12} R \tilde{K} (V_1 V_1^*) = (V_1 V_1^*) - (V_1 V_1^*).
\]
(41)
This implies that
\[
A_{12} R \tilde{K} (V_1 V_1^*) = V_2 V_2^*,
\]
(42)
so that
\[
A_{12} R \tilde{K} \begin{pmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^* U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V_2 V_2^* = V_2 V_2^*.
\]
(43)
Finally, we obtain
\[
A_{12} R_2 K_{22} V_2^* = -V_2 V_2^*.
\]
(46)
Multiplying both sides of (46) by \( V^* \) from the left, considering (33) and the fact that \( V_2^* A_{12} R_2 \) is nonsingular, we have
\[
K_{22} = -(V_2^* A_{12} R_2)^{-1}.
\]
(47)
From Lemma 3, (38), (47), Problem 1 has a solution and the general solution is
\[
A_{11} = B_{11}^+ + A_{12} R \begin{pmatrix} \Lambda^{-1} Q_1^* A_{21} U_1 \Sigma & 0 \\ H & K_{22} \end{pmatrix} \begin{pmatrix} 0 \\ V_2^* \end{pmatrix} \begin{pmatrix} \Lambda^{-1} Q_1^* A_{21} U_1 \Sigma & 0 \\ H & K_{22} \end{pmatrix}^{-1} \times V^* B_{11}^+ + Y - Y B_{11}^+ B_{11},
\]
(48)
where \( H \) is an arbitrary matrix in \( \mathbb{C}^{(n_2-x) \times r} \) and \( Y \) is an arbitrary matrix in \( \mathbb{C}^{m \times n_1} \).

### 3. An Example

In this section, we give a numerical example to illustrate the theoretical results.

**Example 5.** Consider Problem 1 with the parameter matrices as follows:

\[
A_{12} = \begin{pmatrix} 2 + j & \frac{1}{2}k \\ -k & 1 + \frac{1}{2}j \end{pmatrix},
\]
(49)
\[
A_{21} = \begin{pmatrix} \frac{3}{2} + \frac{1}{2}i & -\frac{3}{2}j - \frac{1}{2}k \\ \frac{1}{2}j + \frac{1}{2}k & \frac{3}{2} + \frac{1}{2}i \end{pmatrix},
\]
\[
A_{22} = \begin{pmatrix} 2 & i \\ 2j & k \end{pmatrix}, \quad B_{11} = \begin{pmatrix} 1 \\ i \end{pmatrix}.
\]

It is easy to show that (c), (d) are satisfied, and that
\[
n_2 = r \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} = 2,
\]
(50)
\[
n_2 - r(A_{22}) = m_1 - r(B_{11}) = 0,
\]
so (a), (b) are satisfied too. Therefore, we have
\[
B_{11}^* = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}j \\ \frac{1}{2} & -\frac{1}{2}k \end{pmatrix},
\]
(51)
\[
A_{22} = Q \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} R^*, \quad B_{11} = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^*,
\]
where
\[
Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ j & k \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 2 \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix},
\]
\[
R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ j & k \end{pmatrix},
\]
\[
\Sigma = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

We also have
\[
Q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ j & k \end{pmatrix}, \quad R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]
\[
U_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ j & k \end{pmatrix}, \quad V_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
(53)
By Theorem 4, for an arbitrary matrices $Y \in \mathbb{H}^{2 \times 2}$, we have
\[
A_{11} = B_{11}^* + A_{12} R \left( A_{11}^{-1} \Sigma^* A_{21}^* U_1 \right) V^* B_{11}^* + Y - Y B_{11}^* B_{11}
\]
\[
= \begin{pmatrix}
\frac{3}{2} + \frac{1}{4} j + \frac{1}{4} k & \frac{3}{4} + \frac{1}{4} i - \frac{3}{2} j \\
\frac{1}{2} - i + \frac{1}{4} j - \frac{1}{4} k & \frac{1}{4} - \frac{3}{4} i - \frac{1}{4} j - \frac{1}{4} k
\end{pmatrix},
\]
(54)
it follows that
\[
A = \begin{pmatrix}
\frac{3}{2} + \frac{1}{4} j + \frac{1}{4} k & \frac{3}{4} + \frac{1}{4} i - \frac{3}{2} j + 2 + j & \frac{1}{2} \\
\frac{1}{2} - i + \frac{1}{4} j - \frac{1}{4} k & \frac{1}{4} - \frac{3}{4} i - \frac{1}{4} j - k - k + 1 + \frac{1}{2} j \\
\frac{3}{2} + \frac{1}{2} j & \frac{1}{2} - \frac{1}{2} j - \frac{1}{2} k & 2 + i \\
\frac{1}{2} + \frac{1}{2} j & \frac{3}{2} + \frac{1}{2} j & 2 i - k
\end{pmatrix},
\]
\[
A^{-1} = \begin{pmatrix}
1 & i & -1 & -1 \\
0 & j & 0 & -1 \\
-1 & 0 & \frac{3}{4} & \frac{1}{2} - \frac{3}{4} j \\
-1 & -1 & \frac{1}{2} & \frac{1}{2} - i - \frac{1}{2} j - k
\end{pmatrix}.
\]
(55)
The results verify the theoretical findings of Theorem 4.

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