Fixed Points of Closed and Compact Composite Sequences of Operators and Projectors in a Class of Banach Spaces

M. De la Sen

Faculty of Science and Technology, University of the Basque Country, Campus of Leioa (Bizkaia), Apartado 644, Bilbao, 48940 Leioa, Spain

Correspondence should be addressed to M. De la Sen; manuel.delasen@ehu.es

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1. Introduction

Some results on fixed points related to the contractive compositions of bounded operators in a class of complete metric spaces which can be also considered as Banach’s spaces are discussed through the paper. The class of composite operators under study can include, in particular, sequences of projection operators under, in general, oblique projective operators. In this paper we are concerned with composite operators which include sequences of pairs of contractive operators involving, in general, oblique projection operators. The results are generalized to sequences of, in general, nonconstant bounded closed operators which can have bounded, closed, and compact limit operators, such that the relevant composite sequences are also compact operators. It is proven that in both cases, Banach contraction principle guarantees the existence of unique fixed points under contractive conditions.

2. Some Results on Contractive Mappings and Fixed Points under Projection Operators

Let \( \{T_k\} \) be a sequence \( T_k : X \to X \) of self-mappings on \( X \), where \( (X,d) \) is a metric space and consider a sequence \( \{P_k\} \) of (non-necessarily orthogonal) projection operators on \( X \) of respective ranges \( M_k \) which are then closed subspaces of \( X \). We can then consider a sequence of projection operators \( \{P_M\} \) with \( P_M : X \to M_k \) such that \( P_k = P_M \) so that \( X = \text{im} P_k + \ker P_k \) and \( z = P_k x \in \text{im} P_k \) for any \( x \in X \) and \( z = (I - P_k) x \in \ker P_k \) for \( k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Now, consider sequences \( \{x_k\} \) in \( X \) and \( \{z_k\} \) in \( M_k \) with \( z_k = P_k x_k \) such that the identities

\[
\begin{align*}
x_{k+1} &= T_k x_k = P_{k+1} x_{k+1} + (I - P_{k+1}) x_{k+1} \\
&= z_{k+1} + (I - P_{k+1}) x_{k+1} \\
&= T_k P_k x_k + T_k (I - P_k) x_k
\end{align*}
\]

hold by construction for \( k \in \mathbb{N}_0 \). The subsequent result holds.

**Theorem 1.** Assume that \( (X,d) \) is a complete metric space with the metric \( d : X \times X \to \mathbb{R}_0^+ \) being homogeneous
and translation-invariant and $0 \in X; k \in \mathbb{N}_0$. The following properties hold.

(i) If all the self-mappings on $X$ in the sequence $\{T_k\}$ are nonexpansive and the sequence of projection operators $\{P_k\}$ from $X$ to the sequence of subspaces $\{M_k\}$ is uniformly bounded then $d(z_{k+2}, z_{k+1}) < \infty; k \in \mathbb{N}_0$.

(ii) Assume that the self-mappings on $X$ in the subsequence $\{T_k\}_{k \geq n_0}$ are contractive for some $n_0 \in \mathbb{N}_0$, that the sequence of operators $\{T_k\}$ converges to $T : X \to X$, and that the projection operator $P : X \to M \subset X$ is constant and bounded (i.e., if it is not orthogonal, i.e., it is oblique, then its norm exceeds one and it is finite) then Property (i) holds. Furthermore, $\sup_{k \to \infty} d(z_{k+2}, z_{k+1}) = 0$ and $\{z_{k+1} = PT_k x_k\}$ is a Cauchy sequence which converges to some unique limit point $z (= PTx) \in M$ for any initial iterate $x_0 \in X$, where $x (= Tx) \in X$ is the unique fixed point of $T : X \to X$.

(iii) Assume that there is a strictly sequence of nonnegative integers $\{j_k\}$ such that the difference sequence $\{j_{k+1} - j_k\}$ is uniformly bounded and has a limit $J \in \mathbb{N}_0; \lim k \to \infty$. Assume also that the associate sequence of composite self-mappings $\tilde{T}(j_{k+1} + 1, j_k) = T_{j_{k+1}} \cdots T_{j_k} T_0$ is contractive and that the sequence of projection operators $\{P_k\}$ from $X$ to the sequence of subspaces $\{M_k\}$ is uniformly bounded and has a set of subsequences such converging to a set $\{P_k\} \to \tilde{P}$ of projectors from $X$ to $\{M_k\}$. Then, $\sup_{k \to \infty} \sup \{d(z_{j_{k+1}}, z_{j_k}) = 0\}$, and there is at most a finite number $\mu$ of distinct Cauchy subsequences $\{z_{j_k}\}$ with distinct limit points $\{\tilde{z}_1, \ldots, \tilde{z}_\mu\}$ in $X$.

Since the metric is homogeneous and translation-invariant, the norm is an induced-metric norm, then $\sup \{\|x_k\| = d(x_k, 0) \leq \tilde{d}(T_k(k, 0), x_0, 0)\}$, where $x_k \in M_k$, and the self-mappings in the sequence $\{T_k\}$ on $X$ are all non-expansive; $k \in \mathbb{N}_0$, and the sequence of projection operators $\{P_k\}$ from $X$ to the sequence of subspaces $\{M_k\}$ is uniformly bounded with $\sup \|P_k\| \leq \mu < \infty$. Then, one has from (2)

$$d(z_{k+2}, z_{k+1}) \leq 4\mu \|x_0\| < \infty; k \in \mathbb{N}_0,$$  

where $\mu = 1$ if all the projections are orthogonal and $\mu > 1$, otherwise. Hence, Property (i). If $P_k = P \in \mathbb{P}$ for $k \in \mathbb{N}_0$ is a constant bounded projection from $X$ to $M = \{M_k\} \subset X$ being constant for $k \in \mathbb{N}_0$ and all the self-mappings on $X$ of the sequence $\{T_k\}$ are contractive then one gets from (2) for the real constant $K = \sup_{k \in \mathbb{N}_0} \|P_k\|$ such that $K \in [0, 1)$ that Property (i) holds according to the relation

$$d(z_{k+2}, z_{k+1}) \leq \sup \|T_k\| \|x_k - x_{k+1}\| + \sup \|T_k\| \|x_{k+1} - x_k\| \leq \sup \|T_k\| \|x_{k+1} - x_k\| + \sup \|T_k\| \|x_k - x_{k+1}\|,$$

$$\leq \sup \|T_k\| \|x_{k+1} - x_k\| + \sup \|T_k\| \|x_k - x_{k+1}\| \leq \sup \|T_k\| \|x_{k+1} - x_k\| + \sup \|T_k\| \|x_k - x_{k+1}\|.$$
\[ T: X \to X \]

is a fixed point of the self-mapping \( T \) and some point in \( X \). Thus, if the subsequence \( \{ T_k \}_{k \geq n_0} \) is contractive in \( X \) then its limit \( T \): \( X \to X \) is also contractive. Now, since \( T \): \( X \to X \) is contractive then its fixed point is unique since \((X,d)\) is complete. It is clear that \( z = PTx \) is a limit point in \( M \) of any iterated sequence. Assume that it is not unique so that there are two limit points \( z = PTx, \hat{z} = PT\hat{x}(\neq z) \in M \) for some \( \hat{x}(\neq x) \in X \) which is not trivially a fixed point of \( T: X \to X \) (since the fixed point \( x \in X \) of the contractive self-mapping \( T: X \to X \) is unique if \((X,d)\) is complete). Thus, from Banach contraction principle and since \((X,d)\) is complete, one has

\[ 0 \leftarrow d(PT\hat{x}, PTx) = \left\| PT\hat{x} - PTx \right\| \leq \left\| T\hat{x} - Tx \right\| \leq \left\| \frac{K}{1 - K} ||Tx|| \right\| < \infty; \]

\[ k \in N_0 \]

(8)

so that \( \exists \lim_{k \to \infty} d(z_{k+1}, z_{k+1}) = 0 \) from (4) for any initial value \( x_0 \in X \) of the iteration since \( K^{k-n_0} \to 0 \) and \( \| T_k - T_{k+1} \| \to 0 \) as \( k \to \infty \) since \( \{ T_k \} \to T \). Then, \( \{ z_k \} \) is Cauchy sequence which has a limit \( z \in M \), since \( M \) is closed, [4]. It is now proven that \( z(\in M) = Px \) is the unique limit point in \( M \) of any sequence of iterates, where \( x(\in Tx) \in X \) is a fixed point of the self-mapping \( T: X \to X \) which is unique from Banach contraction principle. It is now proven that \( T: X \to X \) is contractive. Assume not so that one has if \( T: X \to X \) is not contractive:

\[ d(x_{k+1}, x_k) \]

\[ \leq d(Tx_{k+1}, Tx_k) \]

\[ = d(T_k x_{k+1}, Tx_k + T_k x_k + T_k x_{k+1} - Tx_{k+1}) \]

\[ \leq d(T_k x_{k+1}, Tx_k) + d(T_k x_{k+1}, T_k x_k) + d(T_k x_{k+1}, T_k x_{k+1} - T_k x_k) \]

\[ = d(T_k x_{k+1}, Tx_k) + d(0, T_k x_k - T_k x_k + T_k x_{k+1} - T_k x_k) \]

\[ = d(T_k x_{k+1}, Tx_k) + d(T_k x_{k+1}, T_k x_k + T_k x_{k+1} - T_k x_k + T_k x_{k+1}) \]

\[ \leq d(T_k x_{k+1}, Tx_k) + \left\| T_k - T \right\| \left( \left\| x_k \right\| + \left\| x_{k+1} \right\| \right) \]

\[ \leq K_k d(x_{k+1}, x_k) + \left\| T_k - T \right\| \left( \left\| x_k \right\| + \left\| x_{k+1} \right\| \right); \]

\[ k \geq n_0 \]

(5)

for nonzero \( x_k \) and \( x_{k+1} \) since \( T_k: X \to X \) is contractive for \( k \geq n_0 \). Then, one gets, since \( \| T_k - T \| \to 0 \) as \( k \to \infty \), that

\[ \lim sup_{k \to \infty} \left\{ (1 - K_k) \left( \left\| x_k \right\| + \left\| x_{k+1} \right\| \right) \right\} = 0 \]

(6)

\[ \left\{ (1 - K_k) \left( \left\| x_k \right\| + \left\| x_{k+1} \right\| \right) \right\} \leq 0 \]

which is a contradiction since \( K_k < 1 \) for \( k \geq n_0 \) unless \( \{ x_k \} \) converges to zero. If \( \{ x_k \} \) converges to zero then there are \( n_1(\geq n_0) \in N_0 \) and \( 0 < \lambda = \lambda(n_1) < 1 - \sup_{k \geq n_0} K_k \) such that \( \| T_k - T \| \leq \lambda \) for all \( k \geq n_1 \) since \( \| T_k - T \| \to 0 \) as \( k \to \infty \) and some \( k_1 \geq n_1 \) such that \( \| x_k \| + \| x_{k+1} \| > 0 \) that yields the contradiction

\[ 0 < (1 - K_{k_1}) \lambda (\left\| x_k \right\| + \left\| x_{k+1} \right\|) \leq 0. \]

(7)

Thus, if the subsequence \( \{ T_k \}_{k \geq n_0} \) is contractive in \( X \) then its limit \( T: X \to X \) is also contractive. Now, since \( T: X \to X \) is contractive then its fixed point is unique since \((X,d)\) is complete. It is clear that \( z = PTx \) is a limit point in \( M \) of any iterated sequence. Assume that it is not unique so that there are two limit points \( z = PTx, \hat{z} = PT\hat{x}(\neq z) \in M \) for some \( \hat{x}(\neq x) \in X \) which is not trivially a fixed point of \( T: X \to X \) (since the fixed point \( x \in X \) of the contractive self-mapping \( T: X \to X \) is unique if \((X,d)\) is complete). Thus, from Banach contraction principle and since \((X,d)\) is complete, one has

\[ 0 \leftarrow d(PT\hat{x}, PTx) = \left\| PT\hat{x} - PTx \right\| \leq \left\| T\hat{x} - Tx \right\| \leq \left\| \frac{K}{1 - K} ||Tx|| \right\| < \infty; \]

\[ k \in N_0 \]

(8)

as \( k \to \infty \) since \( \{ T_k \} \) converges, there is a limit self-mapping \( T \) on \( X \):

\[ \hat{z} \leftarrow PT\hat{x} \to PTx \to PTX = Px = z. \]

Thus, \( PT_1x \to \hat{z} = z \) as \( k \to \infty \). Hence a contradiction to \( \hat{z} \neq z \) and then \( z \) in \( M \) is the unique limit point of \( T: X \to M \) even in the event that there is \( \hat{x}(\neq x = Tx) \in X \) such that \( PT\hat{x} = PTx = Px = z \). Property (ii) has been proven.

On the other hand, if the sequence of operators is uniformly bounded then \( \sup_{k \in N_0} \| P_k \| \leq \mu < \infty \) and, if furthermore, the sequence of composite mappings \( \{ \tilde{T}(j_k, 1) \} \) is contractive with some constant \( \tilde{K} \in [0, 1) \) given by \( \tilde{K} = \sup_{k \in N_0} (\prod_{j \in j_k} K_j) \), \( k \in N_0 \), where \( \{ j_k \} \) is a strictly increasing sequence of natural numbers such that the sequence \( \{ j_k \} \) is uniformly bounded, one has directly from (1)-(2):

\[ d(z_{j_k+1}, z_{j_k}) \leq \mu d(x_{j_k}, \tilde{T}(j_k+1, j_k) x_{j_k}) \]

\[ \leq \mu (K_{j_k}^j) [d(x_1, x_0) + 2 \| x_0 \|] \]

\[ < \infty; \]

\[ k \in N_0, \]

(9)

\[ d(z_{j_k+1}, z_{j_k}) \leq \mu d(x_{j_k}, \tilde{T}(j_k+1, j_k) x_{j_k}) \]

\[ \leq \mu (K_{j_k}^{j_k+1}) (\tilde{K}_{j_k}^j [d(x_1, x_0) + 2 \| x_0 \|]) \]

\[ < \infty \]

(10)

for \( k \in N_0 \), where \( \{ i_k \} \) is a sequence of finite sets of natural numbers satisfying \( j \in i_k \Rightarrow 1 \leq j < (j_{k+1} - j_k) + 1 \) for \( k \in N_0 \). Thus, one gets from (10):

\[ \exists \lim_{j_k \to \infty} d(z_{j_k+1}, z_{j_k}) = \lim_{j_k \to \infty} d(z_{j_k+1}, z_{j_k}) = 0 \]

(11)

for any \( j \in i_k \) since \( j_k \to \infty \) and \( k \to \infty \) and. Since \( \{ j_k \} \) is uniformly bounded with existing limit \( J \in N \) as
Lemma 3. Let \( x \in X \). The proof of Theorem 1(ii) often happens. For instance if \( k \to \infty \), then there is a natural number \( n_1 \) such that \( \lim_{k \to \infty} \hat{T}(j_k+1,0)X_0 \to 0 \) holds.

Remark 2. The existence of some \( \varepsilon \in \mathbb{R} \) is equivalent to a complete metric space \((X,d)\) being equivalent to a complete metric space \((x,\|\|)\), where \( \varepsilon = (1/2)(\sqrt{(p+t)^2+4\delta} - p - t) \leq \sqrt{\delta}/2 \). Since \( \|P_n\| \to P \) and \( \|T_n\| \to t \), there are finite natural numbers \( n_1 = n_1(\varepsilon) \) such that \( n_2 = n_2(\varepsilon) > n_1(\varepsilon) \) for any integer \( n > n_1 \). If \( PT : X \to X \) is contractive then \( pt < K < 1 \) for some real constant \( K \). Thus, there is \( n_3 = n_3(\varepsilon) \) such that \( pt \leq K \leq \|P_nT_n\| < K < 1 \) for any given real \( \delta < K - 1 \) and \( \varepsilon \leq (1/2)(\sqrt{(p+t)^2+4\delta} - p - t) \leq \sqrt{\delta}/2 \), then the sequence \( P_nT_n \) is asymptotically contractive.

Note that Lemma 3 holds irrespective of the fact that one of the operators be a projection.

3. Results on Contractive Mappings of Sequences of Composite Bounded Operators

The results of Theorem 1 are now extended to the study of contractive compositions of linear operators belonging to two sequences of bounded operators \( \{T_n\} \) and \( \{P_n\} \), which is equivalent to a complete metric space \((X,\|\|)\) with a homogeneous and translation-invariant metric induced-norm \( d : X \times X \to \mathbb{R}_+ \) for any real \( a \) and any \( x, y \in X \). The subsequent result refers to the asymptotic distances in sequences involving a convergent composite sequence of bounded linear operators.

Lemma 4. Consider a Banach space \((X,\|\|)\), being equivalent to a complete metric space \((x,\|\|)\) with a homogeneous and translation-invariant metric induced-norm \( d : X \times X \to \mathbb{R}_+ \), such that \( \|x\| = d(x,0) = d(x+ay,ay) \) for any real \( a \) and any \( x, y \in X \). The subsequent result refers to the asymptotic distances in sequences involving a convergent composite sequence of bounded linear operators.

Proof. Direct calculations yield

\[
\|P_nT_n\| = \sup_{x \in X} \|PTx + P(T_n - T)x + (P_n - P)T_nx\| \\
\leq \|PT\| + \|P\|\|T_n - T\| + \|P_n - P\|\|T_n\| \\
\leq \|PT\| + \|P\|\|T_n - T\| \\
+ \|P_n - P\|\|T\| + \|P_n - P\|\|T_n - T\| \\
\text{and for any given } \varepsilon \in \mathbb{R}_+, \text{ there are } n_1 = n_1(\varepsilon) \in \mathbb{N}_0 \text{ for } i = 1, 2 \text{ such that } \|T_n - T\| < \varepsilon \text{ for all integer } n > n_1 \text{ and } \|P_n - P\| < \varepsilon \text{ for all integer } n > n_2. \text{ Thus, if } n_3 = \max(n_1,n_2) \text{ then one has for any } n > n_3:\n
\|P_nT_n\| \leq \|PT\| + (\|P\| + \|T\| + \varepsilon) \varepsilon \leq pt + (p + t + \varepsilon) \leq pt + \delta.
\]

The last inequality holds if and only if \( \varepsilon^2 + (p + t)\varepsilon - \delta < 0 \) for any given positive real constant \( \delta \) and any positive real constant \( \varepsilon = \varepsilon(\delta) \) being sufficiently small to satisfy \( \varepsilon \leq (1/2)(\sqrt{(p+t)^2+4\delta} - p - t) \leq \sqrt{\delta}/2 \). Since \( \|P_n\| \to P \) and \( \|T_n\| \to t \), there are finite natural numbers \( n_1 = n_1(\delta) \) such that \( n_2 = n_2(\delta) > n_1(\delta) \) such that \( \hat{T}(j_k+1,0)X_0 \to 0 \) holds.

(i) Assume that \( \{T_n\} \to T_1(i = 1, 2) \). Then, \( \lim_{k \to \infty} d(T_iX, T_{2k}T_1X) = 0 \) for all \( x \in \text{Dom}(T_{2k}) \) for all \( k \in \mathbb{N}_0 \).

(ii) Assume that \( \{T_{1k}\} \to T_1(i = 1, 2) \). Then, \( \lim_{k \to \infty} d(T_{1k}X, T_{2k}T_1X) = 0 \) for all \( x \in \text{Dom}(T_{2k}) \) for all \( k \in \mathbb{N}_0 \).

(iii) Assume that \( \{T_{2k}\} \to T_2(i = 1, 2) \). Then, \( \lim_{k \to \infty} d(T_{1k}X, T_{2k}T_1X) = 0 \) for all \( x \in \text{Dom}(T_{2k}) \) for all \( k \in \mathbb{N}_0 \).

(iv) Define the operator composite sequence \( \{\hat{T}(k + i + 1, k)\} \) of operators as \( \hat{T}(k + i + 1, k) = T_{ki+1} \cdots T_{k+1}T_k \)
Properties (i)–(iii) hold for any \( \lim_{k \to \infty} H \), hence Property (i). Now, assume that only \( T_{2k} \) has a limit. Then,

\[
T_k x = T_{2k} T_{1k} x
\]

which is of the form \( T_{2k} T_{1k} x \).

Hence, Property (ii). Finally, assume that only \( T_{2k} \) has a limit. Then,

\[
T_k x = T_{2k} T_{1k} x
\]

for some \( k \), \( i = 1, 2 \), and \( T_{2k} = T_1 \). Hence, Property (iii).

Hence, Property (iv).

Thus, Property (iv) is direct from Properties (i)–(iii) and the associative property of composition of operators since for any \( k \), \( x \in \text{Dom}(T_k) \), \( x \) has a limit. Then,

\[
\lim_{k \to \infty} d ((T_k - T_{2k}) x, 0) = 0;
\]

\[
\forall x \in \text{Dom}(T_k), \forall i \in \mathbb{N}_0.
\]
Proof. Note the following:

(1) \( \{ T_n x_m^{(n)} \} \rightarrow T_n x_m \) as \( n, m \rightarrow \infty \) for any bounded sequence \( \{ x_m^{(n)} \} \subset \text{Dom}(T_n) \) since \( \lim_{n, m \rightarrow \infty} \| (T_n - T)x_m^{(n)} \| \leq \lim_{n, m \rightarrow \infty} \| T_n - T \| \| x_m^{(n)} \| = 0 \) since \( \{ T_n \} \rightarrow T \). Furthermore,

\[
\| T x \| \leq \| T_n x \| + \| (T_n - T) x \| \leq \| K_n + \tilde{K}_n \| \| x \| \quad \forall x \in X
\]

Thus, we have the following result using Lemma 5.

(2) If the bounded sequence \( \{ x_m^{(n)} \} \subset \text{Dom}(T_n) \) converges to \( x^{(n)} \subset \text{Dom}(T_n) \) for any \( n \in N_0 \) then \( \{ T_n x_m^{(n)} \} \rightarrow T_n x^{(n)} \) for any \( n \in N_0 \) as \( m \rightarrow \infty \) since \( \{ T_n \} \) is a closed operator for any \( n \in N_0 \) so that

\[
\exists \lim_{m \rightarrow \infty} \| T_n x_m^{(n)} - T_n x^{(n)} \| \leq \lim_{m \rightarrow \infty} \| T_n (x_m^{(n)} - x^{(n)}) \| \leq \| T_n \| \lim_{m \rightarrow \infty} \| x_m^{(n)} - x^{(n)} \| = 0.
\]

(3) One gets combining the above points (1)-(2) that:

\[
\{ T_n x_m^{(n)} \} \rightarrow T_n x_m \rightarrow T_n x - T x \quad \text{as } n, m \rightarrow \infty \text{ if } \{ x_m^{(n)} \} \rightarrow x^{(n)} \rightarrow x \text{ for any } n \in N_0 \text{ where } \{ x_m^{(n)} \} \rightarrow \{ x^{(n)} \} \subset \text{Dom}(T_n) \text{ as } m \rightarrow \infty \text{ and then } \{ x_m^{(n)} \} \rightarrow \{ x^{(n)} \} \subset \text{Dom}(T_n) \text{ as } m \rightarrow \infty \text{ for any } n \in N_0 \text{ and } \{ x_m^{(n)} \} \rightarrow \{ x^{(n)} \} \rightarrow x \in \text{Dom}(T) \text{ as } n, m \rightarrow \infty \text{ since } \{ T_n \} \text{ is a sequence of closed operators which converges. Thus, the limit of bounded converging sequences belongs to the domain of the limit operator. Furthermore, one has for any bounded sequence } \{ x_n \} \text{ converging to } x \in \text{Dom}(T):
\]

\[
\| T_n x_n - T x \| = \| T x_n + (T_n - T) x_n - T x \| \\
\leq \| T \| \| x_n - x \| + \| T_n - T \| \| x_n \| \rightarrow 0
\]

as \( k \rightarrow \infty \) then \( T_n x_n \rightarrow T x \) strongly so that \( T : \text{Dom}(T) \subset X \rightarrow \text{Im}(T) \subset X \) is a closed operator as a result.

The above result can be extended to sequences of operators not all of them being bounded provided that each of such sequences of operators can be decomposed as a composition of subsequences of composite operators such that each of such a composite sequence is bounded. The above result can be applied to sequences of operators not all of them being bounded. It is well-known that a sequence of linear operators on a Hilbert space [5, 6] is bounded if and only if they are closed and their domain is the whole vector space \( X \), [1, 4]. Thus, we have the following result using Lemma 5.

Lemma 6. Consider a sequence of linear bounded operators \( \{ T_n \} \) defined by \( T_n : X \rightarrow X \) in a Banach space \( (X, \| \|) \) which converge to a limit operator \( T : X \rightarrow X \). Then, such a limit is a bounded linear closed operator.

Proof. Since the operators are all bounded then their domain is \( X \), their range is in \( X \) and are all closed. The conditions of Lemma 5 hold with

\[
X \equiv \text{Im}(T_n) \subset X \equiv \text{Dom}(T_{n+1})
\]

with \( \text{Im}(T_n) \cap \text{Dom}(T_{n+1}) = X \neq \emptyset. \)

Then, the limit \( T : X \rightarrow X \) of \( \{ T_n \} \) is also bounded and closed from Lemma 5.

The subsequent result is concerned with the limit operator of a sequence of linear operators being compact if all the operators in the sequence are bounded and at least one of them is compact.

Lemma 7. The following properties hold.

(i) Consider a sequence of bounded compact linear operators \( \{ T_n \} \) defined by \( T_n : \text{Dom}(T_n) \subset X \rightarrow \text{Im}(T_n) \subset X \) in a Banach space \( (X, \| \|) \), such that \( \text{Im}(T_n) \subset \text{Dom}(T_{n+1}) \) with \( \text{Im}(T_n) \cap \text{Dom}(T_{n+1}) = X \neq \emptyset \), which converge to a limit operator \( T : \text{Dom}(T) \subset X \rightarrow \text{Im}(T) \subset X \). Then, such a limit is a compact operator.

(ii) Assume that the sequence \( \{ T_n \} \) of bounded operators satisfies that there is at least one compact operator within all subsequences \( \{ T_{j_k}, T_{j_{k+1}}, \ldots, T_{j_{k+1}} \} \) being subject to \( \max_{n \in N_0} (j_{k+1} - j_k) \leq c_j < \infty \) for some subsequence \( \{ j_k \} \subset N_0 \) for any \( n \in N_0 \). Then, the composed operator \( T(n,m) \) is compact as it is its limit provided that it exists.

Proof. We have to prove that if \( \{ x_n \} \) is bounded then \( \{ T x_n \} \) is convergent. Note that for given bounded sequences \( \{ x_n^{(i)} \} \) and \( \{ x_n^{(j)} \} \); \( i, j, n \in N_0 \) that

\[
\left\| T x_n^{(i)} - T x_n^{(j)} \right\| \\
= \| (T_n - T) x_n^{(i)} + T_n x_n^{(i)} - T_n x_n^{(j)} - (T_n - T) x_n^{(j)} \| \\
\leq \| (T_n - T) x_n^{(i)} \| + \| (T_n - T) x_n^{(j)} \| + \| T_n \| \| x_n^{(i)} - x_n^{(j)} \|
\]

and, one gets by taking subsequences \( \{ z_j \} \subset \{ x_n^{(i)} \}, \{ z_j \} \subset \{ x_n^{(j)} \}

\[
\left\| T z_j - T z_j \right\| \leq \| T - T_n \| (z_j \| + \| z_j \|) + \| T_n z_j - T_n z_j \|.
\]
Since \( \{T_n\} \to T \), we can find \( n_0, i_0 \in \mathbb{N}_0 \) such that for \( n \geq n_0 \), \( \min(i,j) > i_0 \), we have
\[
\|T - T_n\| < \frac{\varepsilon}{4c},
\]
\[
\max(\|z_j\|, \|z_j\|) \leq K_z \leq 2c < \infty,
\]
\[
\|T_n z_j - T_n z_j\| < \frac{\varepsilon}{2}
\]
for any given \( c \) and \( \varepsilon = \varepsilon(c) \in \mathbb{R}_+ \), since \( \{z_j\} \) and \( \{z_j\} \) are bounded subsequences, and \( \{T_n z_j\} \) converges, so that it is a Cauchy sequence, since \( \{T_n\} \) contains at least one compact operator. As a result, \( \|T z_i - T z_i\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \) is arbitrarily small for \( \varepsilon \) being sufficiently small. Thus, \( \{T z_i\} \) is convergent. Property (i) has been proven. Property (ii) follows from Property (i) and the fact that any operator composite sequence of bounded operators is a compact operator if there is at least one which is compact.

Now, define the composite operator \( \hat{T}(k + i + 1, k) : X \to X \) for all \( i, k \geq i \in \mathbb{N}_0 \) by
\[
\hat{T}(k + i + 1, k) = T_{k+i+1} \cdots T_{k+2} T_k
\]
\[
= (T_{j_k+i+1} \cdots T_{j_k+2} T_{j_k+1}) \cdots (T_{j_k+1} \cdots T_{j_k+1} T_{j_k+1}) ; \quad (31)
\]
\[\forall x \in \text{Dom}(T_{10}) \; \forall k \in \mathbb{N}_0 .\]

Define also the sequence \( \{\hat{T}^0(k + i + 1, k)\} \) of composite operators as \( \hat{T}^0(k + i + 1, k) = T_{k+i+1}^0 \cdots T_{k+1}^0 T_k^0 \), for all \( k \in \mathbb{N}_0 \), where \( T_{k+i+1}^0 \) replaces each composite operator \( T_{k+i+1} \) by its limit when such a limit exists. A result is now given based on the existence of the following limit:
\[
\lim_{k \to \infty} d \left( \hat{T}(k + i + 1, i), \hat{T}^0(k + i + 1, i) \right) = 0 ; \quad (32)
\]
\[\forall x \in \text{Dom}(T_{10}) .\]

The following result is obtained from Lemmas 4–7.

**Theorem 8.** Consider the operator composite sequence \( \hat{T}(k + i + 1, k) \); for all \( k, i \in \mathbb{N}_0 \) of composite bounded operators in (31) on a Banach space \( (X, \|\|) \), subject to \( \text{Im}(T_{j_k}^0) \neq \emptyset \subset \text{Dom}(T_{j_k+1}) \) for \( j \in \mathbb{N}_0 \), \( \text{Im}(T_{j_k}) \neq \emptyset \subset \text{Dom}(T_{j_k+1}) \) for \( 1 \leq j_k \leq j \in \mathbb{N}_0 \), the sequence of composite operators \( \{\hat{T}^0(k + i + 1, k)\} \) of defined in the same way as \( \{\hat{T}(k + i + 1, k)\} \) as \( k \to \infty \) by replacing each operator possessing a limit by such a limit. The following properties hold.

(i) Either the sequences \( \{\hat{T}(k+i+1, k)\} \) and \( \{\hat{T}^0(k+i+1, k)\} \) have limits and both limits coincide or none of them has a limit and, furthermore, and \( \hat{T}(k + i + 1, k) \to \hat{T}^0(k + i + 1, k) \) as \( k \to \infty \).

(ii) If the limits of Property (i) exist and are finite then the limits of the sequences of operators \( \{\hat{T}(k+i+1, k)\} \) and \( \{\hat{T}^0(k+i+1, k)\} \) as \( k \to \infty \); for all \( i \in \mathbb{N}_0 \)

(iii) Assume, in addition, that for some \( k \in \mathbb{N}_0 \), there is at least one compact operator in the composition operator \( \hat{T}(k + i + 1, k) \), and that \( \text{Im}(T_{j_k}^0) \neq \emptyset \subset \text{Dom}(T_{j_k+1}) \) for \( j \in \mathbb{N}_0 \), \( \text{Im}(T_{j_k}) \neq \emptyset \subset \text{Dom}(T_{j_k+1}) \) for \( 1 \leq j \leq j_k \in \mathbb{N}_0 \); and all the operators are closed. If Property (i) holds with \( \hat{T}(k + i + 1, k) \to \hat{T}^0(k + i + 1, k) \to \hat{T}^* \) as \( k \to \infty \); for some \( i \in \mathbb{N}_0 \) and \( \|\hat{T}^*\| \leq K < 1 \), then \( \hat{T}^* \) is contractive and

\[
\lim_{k \to \infty} \|\hat{T}(k + n(i + 1), i) x - \hat{T}^0(k + n(i + 1), i) y\| = \lim_{k \to \infty} \|\hat{T}(k + n(i + 1), i) x - \hat{T}^0(k + n(i + 1), i) y\| = \lim_{k \to \infty} \|\hat{T}^*(i+1) x - \hat{T}^*(i+1) y\| = 0 ; \quad \forall x, y \in \text{Dom}(T_{10}) . \quad (33)
\]

the sequences of composite operators \( \{\hat{T}(k + j, i)\} \) and \( \{\hat{T}^0(k + j, i)\} \) converge to zero as \( k \to \infty \), and \( \hat{T}^* : \text{Dom}(\hat{T}^*) \subset X \to \text{Im}(\hat{T}^*) \subset X \) is bounded, closed and compact and has a unique fixed point in \( \text{Dom}(\hat{T}^*) \cap \text{Im}(\hat{T}^*) \) to which all sequences with initial conditions in \( \text{Im}(\hat{T}^*) \) converge.

(iv) Assume that there is a \( (\text{general, nonunique}) \) strictly increasing sequence of nonnegative integers \( \{j_k\} \) with \( j_0 = 0 \) and \( 0 < j_{k+1} - j_k \leq m < \infty \) such that

\[
\|\hat{T}(j_{k+2}, j_k)\| \leq K(\hat{T}(j_{k+2}, j_{k+1})\|\hat{T}(j_{k+1}, j_k)\|) \leq K \|\hat{T}(j_{k+1}, j_k)\| ; \quad \forall k \in \mathbb{N}_0 . \quad (34)
\]

for some nonnegative real sequence \( \{\lambda_j(j_{k+1}, j_k)\} ; k \in \mathbb{N}_0 \) and some real constant \( K \in [0,1] \). Assume, in addition, that for some \( k \in \mathbb{N}_0 \), there is at least one compact operator in any composite operator \( \hat{T}(j_{k+1}, j_k) \) and that all the operators are closed. Then, the sequences of composite operators \( \{\hat{T}(k + j, i)\} \) and \( \{\hat{T}^0(k + j, i)\} \) converge to zero as \( k \to \infty \) for any finite \( j \in \mathbb{N}_0 \). Finally, assume that \( \hat{T}(j_{k+1}, j_k) \to \hat{T}^* \) as \( k \to \infty \). Then, \( \hat{T}^* \) is contractive, continuous, bounded, closed, and compact and has a unique fixed point in \( \text{Dom}(\hat{T}^*) \cap \text{Im}(\hat{T}^*) \) to which all sequences with initial conditions in \( \text{Im}(\hat{T}^*) \) converge.

**Proof.** Note from the definition of the sequences \( \{\hat{T}(k + i + 1, k)\} \) and \( \{\hat{T}^0(k + i + 1, k)\} \) that for any, since

\[
\|\hat{T}^0(k + i + 1, k)\| \\
\leq \|\hat{T}(k + i + 1, k)\| + \|\hat{T}^0(k + i + 1, k) - T(k + i + 1, k)\|,
\]


\[ \| \hat{T}(k + i + 1, k) \| \leq \| \hat{T}^0(k + i + 1, k) \| + \| \hat{T}^0(k + i + 1, k) - T(k + i + 1, k) \| \] 

so that \( \lim \sup_{k \to \infty} \| \hat{T}(k + i + 1, k) \| - \| \hat{T}^0(k + i + 1, k) \| \leq 0 \). Then, either both sequences of operators \( \{ \hat{T}(k + i + 1, k) \} \) and \( \{ \hat{T}^0(k + i + 1, k) \} \) have the same (finite or infinity) limit or none of them has a limit and \( \hat{T}(k + i + 1, k) \to \hat{T}^0(k + i + 1, k) \) as \( k \to \infty \). Hence, Property (i). Property (ii) follows trivially from Property (i) for the case \( \lim_{k \to \infty} \hat{T}(k + i + 1, k) = \hat{T}(k + i + 1, k) \) as \( k \to \infty \). Similarly, it is proven that \( \| \hat{T}(k + j, j) \| \to 0 \) as \( k \to \infty \).

On the other hand, note the following.

1. Any convergent sequence \( \{x_k\} \) for \( k \in \mathbb{N}_0 \) construct-ed from the composed operators

\[
\hat{T}(k + i + 1, k) = T_{k+i} \cdots T_{k+1} T_k
\]

for any \( k \in \mathbb{N}_0 \) as follows \( x_{ik} \in \text{Dom}(T_{k+1}), x_{ik} = T_{2k} x_{ik-1}, x_{ik} = T_{1k} x_{ik-1}, x_{ik} = T_{jk} x_{ik-1} \) converges to a point \( \bar{x} \) in \( \text{Dom}(\hat{T}^*\hat{T}) \), since all the operators in the above composite sequence of operators are closed and then the limit operator \( \hat{T}^* : \text{Dom}(\hat{T}^*) \subset X \to \text{Im}(\hat{T}^*) \subset \text{Im}(\hat{T}^*) \subset \text{Dom}(\hat{T}^*) \subset X \) is also bounded and closed (from Lemma 6 and the associative property of operator compositions), and \( \hat{T}^* \bar{x} \in \text{Im}(\hat{T}^*) \subset \text{Im}(\hat{T}^*) \) with \( \text{Im}(\hat{T}^*) \) being closed (i.e., \( \text{Im}(\hat{T}^*) \) is relatively compact) since all composite sequences of operators \( \hat{T}(k + j, j) \) are compact for any given \( j \in \mathbb{N}_0 \), \( k(> j) \in \mathbb{N} \) since at least one of the operators within any of such sequences is compact and all of them are bounded, \( [1, 3, 4] \).

2. Any convergent sequence \( \{\hat{T}^* x\} \) of elements in \( \text{Dom}(\hat{T}^*) \) with \( x \in \text{Dom}(T_{j_0}) \) converges to some point \( x^* \) in \( \text{Dom}(\hat{T}^*) \), which maps to \( \hat{T}^* x^* \) in \( \text{Im}(\hat{T}^*) \subset \text{Im}(\hat{T}^*) \subset \text{Dom}(\hat{T}^*) \) which is also the limit of the same convergent sequence. Such a limit \( \{\hat{T}^* x\} \) has a limit in \( \text{Dom}(\hat{T}^*) \) where \( \text{Im}(\hat{T}^*) \) is also the unique fixed point of \( \hat{T}^* \). Otherwise, if there were two distinct fixed points \( x^* \) and \( y^* \) then it would be \( x, y \in \text{Dom}(T_{j_0}) \) such that \( \lim_{n \to \infty} d(\hat{T}^* x - \hat{T}^* y) = d(x^*, y^*) > 0 \), then a contradiction and hence Property (iii).

To prove Property (iv), note that strictly increasing se-

sequence of nonnegative integers \( \{j_k\} \) with \( j_0 = 0 \) and \( 0 < j_{k+1} - j_k \leq m < \infty \) such that

\[
\| \hat{T}(j_{k+1}, 0) \| \leq K^h \| \hat{T}(j_k, 0) \| \to 0 \quad \text{as} \quad k \to \infty ;
\]

\[
\| \hat{T}(j_{k+1} + m_k, 0) \| \leq M K^h \| \hat{T}(j_1, 0) \| \to 0
\]

for any sequence of nonnegative integers \( \{m_k\} \) subject to
\( 0 \leq m_k < j_{k+1} - j_k \leq m < \infty \); for all \( k \in \mathbb{N}_0 \) for some nonnegative real sequence \( \{K_k(j_{k+1}, j_k)\} \); \( k \in \mathbb{N}_0 \) and some real constant \( K \in [0, 1] \). As a result the sequences of
composite operators \( \{\overline{T}(k,0)\}, \{\overline{T}_0(k,j)\}, \) and \( \{\overline{T}_0(k + j, j)\} \) converge to zero as \( k \to \infty \) for any finite \( j \in \mathbb{N}_0 \). On the other hand, if \( \overline{T}(j_{k+1}, j_k) \to \overline{T}_g^* \) as, \( k \to \infty \) then \( \overline{T}_g^* \) is contractive. Otherwise, it would hold trivially that \( \lim \inf_{k \to \infty} ||\overline{T}(k,0)|| > 0 \), a contradiction. Thus, the limit operator \( \overline{T}_g^* \) is contractive and bounded then it is also continuous as a result. \( \overline{T}_g^* : \text{Dom}(\overline{T}_g^*) \subset X \to \text{Im}(\overline{T}_g^*) \subset \text{Im}(\overline{T}_g^*) \subset X \) is closed (from Lemma 6) and compact, since all the operators in the composite sequence of operators \( (\overline{T}(j_{k+1}, j_k)) \) are bounded and at least one of them is compact. Thus, \( \overline{T}_g^* \) has a unique fixed point in \( \text{Dom}(\overline{T}_g^*) \cap \text{Im}(\overline{T}_g^*) \) to which all sequences to which all the sequences \( \{x_{jk}\}; k \in \mathbb{N}_0 \) of the form \( x_{jk} \in \text{Dom}(T_{jk}), x_{2jk} = T_{2jk} x_{jk}, \ldots, x_{jk} = T_{j_{k+1}j_k} x_{jk-1,k+1} = T_{j_{k+1}j_k} x_{j_k,k+1} = \ldots \) with initial point in \( \text{Dom} \overline{T}_{10} \) converge. Hence, Theorem 8 is fully proven.

Remark 9. Note that the existence of the operator limits in Theorem 8(iii)-(iv) is not required for each operator within the composite sequence of operators but only for certain composite strips of such operators.

The subsequent result, whose proof is omitted, extends in a natural way Theorem 8 through the associative property of composite operators to the case that there are subsets of composite operators having limits although each individual operator is not requested to have a limit.

**Theorem 10.** Consider the composite operator below:

\[
\begin{align*}
\overline{T}^{*}_g(k + j, k) &= T_{j_{k+1}} \cdot \cdots \cdot T_{k+1} K_k \\
&= (T_{j_{k+1}} \cdot \cdots \cdot T_{j_{k+1}} K_{j_k}) \cdots (T_{j_k} \cdot \cdots \cdot T_{j_k} K_1),
\end{align*}
\]

on a Banach space (\( X, ||\cdot|| \)), subject to the following conditions.

1. The elements of the sequences of sets \( \{i_k\} \) are finite and each of those sets has a finite cardinal for all \( k \in \mathbb{N}_0 \), and \( i_k \to i^* (< \infty) k \to \infty \).
2. \( \text{Im}(T_{jk}) \neq \emptyset \subset \text{Im}(T_{jk}) \subset \text{Dom}(T_{j_{k+1}k}) \) for \( j \in j_k \), \( i \leq j_k \leq j_k + \epsilon \). 
3. \( \text{Im}(T_{jk}) \neq \emptyset \subset \text{Im}(T_{jk}) \subset \text{Dom}(T_{j_{k+1}k}) \) for \( 0 \leq \ell \leq j_k \).
4. All the operators in each of the sets \( \{T_{i_kj_k} \leq i_k \leq j_k \} \); for all \( k \in \mathbb{N}_0 \), are linear, bounded, and closed (so that all the operators are linear, bounded, and closed) and at least one of them in each set is compact.
5. The sequences of composite operators \( \{T_{i_kj_k} \} \) for \( 0 \leq i_k \leq j_k \); for all \( k \in \mathbb{N}_0 \) tend to respective limit operators \( T_g^* \) for \( 0 \leq j \leq i^* \) as \( k \to \infty \).

Then, \( \overline{T}^{*}_g(k + j, k) \to \overline{T}^{*}_g(k + i^*, k) \to T^{*}_g = T^{*}_1 \cdot \cdots \cdot T^{*}_j \) which is linear, continuous, bounded, closed, and compact. Furthermore, if \( ||T^{*}_g|| \leq K_j \) and \( K = \prod_{j=0}^{i^*} K_j < 1 \), then \( T^{*}_g \) has a unique fixed point in \( \text{Dom} \overline{T}^{*}_g \cap \text{Im} \overline{T}^{*}_g \) to which all sequences \( \{x_{jk}\}; k \in \mathbb{N}_0 \) of the form \( x_{jk} \in \text{Dom}(T_{jk}), x_{2jk} = T_{2jk} x_{jk}, \ldots, x_{jk} = T_{j_{k+1}j_k} x_{jk-1,k+1} = T_{j_{k+1}j_k} x_{j_k,k+1} = \ldots \) with initial point in \( \text{Dom} T_{10} \) converge.

Example II. This example discusses a way to use oblique projections to build composite operators with sequences of operators to take into account the approximation of the images in finite-dimensional spaces and also to take account of computing or measurement errors as well as connections with fixed point issues. Consider the complex pre-Hilbert space \( L^2_p(a) \) of square-integrable \( p \)-vector functions on \([0, a]\) endowed with an inner product defined by the complex number \( \langle x, y \rangle \); for all \( x, y \in L^2_p(a) \) with associate inner product induced norm \( ||x|| = (\langle x, x \rangle)^{1/2} \); for all \( x \in L^2_p(a) \).

Consider a bounded linear operator \( T : L^2_p(a) \to L^2_p(a) \), of norm \( ||T|| = \sup_{||x|| = 1} ||Tx|| \), represented by \( y(t) = (Tx)(t) = \sum_{i=1}^{\infty} \langle y, \varphi_i \rangle \varphi_i(t) \); for all \( t \in [0, a] \), where \( \varphi_i, \varphi_j : [0, a] \to L^2_p(a); i, j \in \mathbb{N} \) are sets of linearly independent functions which define mutually reciprocal basis \( \{\varphi_i : i \in \mathbb{N}\} \) and \( \{\varphi_j : j \in \mathbb{N}\} \), that is, \( \langle \varphi_i, \varphi_j \rangle = \delta_{ij} \). If such basis are orthogonal then they are identical leading to \( y(t) = (Tx)(t) = \sum_{i=1}^{\infty} \langle y, \varphi_i \rangle \varphi_i(t) \); for all \( t \in [0, a] \). We can decompose \( L^2_p(a) \) uniquely as a direct sum of orthogonal subspaces as follows as \( L^2_p(a) = M_n \oplus (L^2_p(a) \cap M_n) \) for each \( n \in \mathbb{N} \) where the orthogonal projection of \( T : L^2_p(a) \to L^2_p(a) \) is given by the composite operator \( P_nT_n : L^2_p(a) \to M_n \) of the orthogonal projection \( P_n : L^2_p(a) \to M_n \) represented by \( \overline{y}_n(t) = (P_n y_n)(t) \), where \( y_n(t) = (T_n x)(t) = \sum_{i=1}^{\infty} \langle y, \varphi_i \rangle \varphi_i(t) \); for all \( t \in [0, a] \) is defined through the truncated operator \( T_n : L^2_p(a) \to L^2_p(a) \) so that

\[
\overline{y}_n(t) = \sum_{i=1}^{n} \langle P_n T_n x, \varphi_i \rangle \varphi_i(t)
\]
(∑_{i=1}^{∞} (T^i x, φ_i) φ_i(t); for all t ∈ [0, a] (due, for instance, to computational or measurement errors) and defined by some relative uncertainty operator ̃T in T on L^2_p(a) belonging to a family ̃T = {̃T ∈ L^2_p(a) : ̃T ≤ t̃, some t̃ ∈ R^p} so that y_{mes}(t) = y(t) + ̃y(t) and its projected value on M_n, through the orthogonal projection M_n, is

\[ y_{mes_n}(t) = (P_n(T_n + ̃T_n) x)(t) = (P_nT_n x)(t) + (P_n ̃T_n x)(t) \]

= \̃y_n(t) + ̃y_n(t)

= \left( \sum_{i=1}^{n} (P_n (T + ̃T) x, φ_i) φ_i(t) \right) (t) = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} (P_n T φ_j + P_n ̃T φ_j, φ_i) α_j φ_i(t) \right) (t)

= \left( \sum_{i=1}^{n} \sum_{j=1}^{n} (P_n (I + ̃T) T φ_j, φ_i) α_j φ_i(t) \right) (t)

= \left( \sum_{i=1}^{n} \sum_{j=1}^{n} (P_n ̃T T φ_j, φ_i) α_j φ_i(t) \right) (t)

= \sum_{i=1}^{n} \left( β_i + ̃β_i \right) φ_i(t); \ \forall t \in [0, a],

where \{T_n\} → ̃T,

\[ β_i(t) = \sum_{j=1}^{n} (P_n ̃T T φ_j, φ_i) α_j(t) \]

= \sum_{j=1}^{n} (P_n ̃T T φ_j, φ_i)(x, φ_i)(t); \ \forall t \in [0, a],

i ∈ \bar{n} = \{1, 2, …, n\}

with P_m(̃T_n) = P_n(I + ̃T_n) is an oblique operator which depends on the particular uncertainty operator ̃T_n in the class ̃T which has necessarily a norm exceeding unity while the orthogonal operator P_n has unit norm. The (non-necessarily unique) worst case in a norm deviation sense of the measured projection of y on the subspace M_n is given by

\[ y_{mes_n}^w(t) = \sup_{T \in T} \sum_{i=1}^{n} \sum_{j=1}^{n} (P_m(̃T) T φ_j, φ_i) α_j φ_i(t) \] (46)

so that the maximum deviation amount of the projected vector is

\[ \|y_{mes_n}^w - ̃y_n\| = \sup_{T \in T} \sum_{i=1}^{n} \sum_{j=1}^{n} (P_m(̃T) T φ_j, φ_i) α_j φ_i(t) \] (47)

If the basis \{φ_i : i ∈ \bar{n}\} is orthonormal then it is autoreciprocal, then all its vector functions have unit norm and

\[ y_{mes_n}^w(t) = \sup_{T \in T} \sum_{i=1}^{n} \sum_{j=1}^{n} (P_m(̃T) T φ_j, φ_i) α_j φ_i(t), \]

\[ \|y_{mes_n}^w - ̃y_n\| = \sup_{T \in T} \sum_{i=1}^{n} \sum_{j=1}^{n} (P_m(̃T) T φ_j, φ_i) α_j φ_i(t), \]

The problem can be reformulated for the case a = ∞ for T : L^2_p → L^2_p being a bounded linear operator on the Hilbert (then complete) space L^2_p. Thus, T : L^2_p → L^2_p is closed, since bounded, and its domain is L^2_p and it is also guaranteed to be compact from of its representation. It is clear that the operators in the sequence \{T_n\} are bounded, closed, compact, of closed range so that their ranges have n-finite dimension and their domain is 1^2_p. The orthogonal and oblique operators involved in the above discussion are all bounded and of closed range. Then, all the composite operators of the forms \{P_n T\}, \{P_n ̃T\}, \{P_n ̃TT\} and the operators in the converging sequences \{P_n T_n\}, \{P_n ̃T_n\}, \{P_n ̃TT_n\} are all bounded, closed, and compact of domain L^2_p. If \|T\| ≤ K < 1 then for any given real ε ∈ (0, 1 − K) there is n_0 ∈ N such that \|T_n\| ≤ K + ε < 1 since \{T_n\} → T. Assume that the class of uncertainty operators T_n in the class ̃T on L^2 has the property \|P_m(̃T_n)\| = \|I + ̃T_n\| ≤ \|P_m\| ≤ 1/(K + ε) < 1; for all n > n_0. Thus, the composite operators P_m(̃T_n) are contractive if \|I + ̃T_n\| ≤ 1/(K + ε) < 1; for all n > n_0 and each of such composite operators has a unique fixed point, which depends on n; for all n > n_0 and which converges to the unique fixed point of the contractive operator (I + ̃T) as n → ∞ from Theorem 8 since (I + ̃T)T_n → (I + ̃T)T as n → ∞ so that d((I + ̃T_n)T_m z, (I + ̃T)T z) → 0 and d((I + ̃T)^n y, (I + ̃T)^n z) → 0 as n → ∞ for any y, z ∈ 1^2_p.

Remark 12. Some ideas in Example II combining uncertainties with projections both being described through "ad hoc" operators are useful in problems of Signal Theory and Control Systems Theory. [4]. Some related problems can be combined with stability and stabilization issues of dynamic systems subject to unmodeled dynamics and/or parametrical-type uncertainties by using Lyapunov stability theory and fixed
point analysis. See, for instance, [7–10]. Fixed point analysis can also be a useful technical tool when using iterative methods in numerical approaches. See, for instance, [11, 12] and references therein. It can be direct the extension of the results to a formalism concerning the replacement of fixed points by best proximity points of cyclic $p$-self-mappings [13–17] on unions of sets which do not intersect since best proximity points are also fixed points of certain strips of fixed length $p$ of companion composite self-mappings $T^p : \bigcup_{i \in p} A_i \to \bigcup_{j \in p} A_j$, with themselves, the sizes $p$ of such composite self-mappings being the number of disjoint sets $A_i \subset X, i \in \mathbb{P} := \{1, 2, \ldots, p\}$ in the cyclic disposal. The location of fixed points has also been approximated in some background bibliography on the field. See [18, 19] and references there in. In particular, approximated fixed points have been characterized for nonself mappings which do not possess fixed points. See, for instance, [19] and references therein.

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**References**


