Research Article

Viscosity Method for Hierarchical Fixed Point Problems with an Infinite Family of Nonexpansive Nonself-Mappings

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Received 4 January 2013; Accepted 22 March 2013

Academic Editor: Filomena Cianciaruso

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A viscosity method for hierarchical fixed point problems is presented to solve variational inequalities, where the involved mappings are nonexpansive nonself-mappings. Solutions are sought in the set of the common fixed points of an infinite family of nonexpansive nonself-mappings. The results generalize and improve the recent results announced by many other authors.

1. Introduction and Preliminaries

Let $X$ be a real Banach space and $J$ be the normalized duality mapping from $X$ into $2^{X^*}$ given by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}$$

(1)

for all $x \in X$, where $X^*$ denotes the dual space of $X$ and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between $X$ and $X^*$. If $X = H$ is a Hilbert space, then $J$ becomes the identity mapping on $H$.

A point $x \in C$ is a fixed point of $T : C \subset X \to X$ provided $Tx = x$. Denote by $F(T)$ the set of fixed points of $T$; that is, $F(T) = \{x \in C : Tx = x\}$.

Let $X$ be a normed linear space with dim $X \geq 2$. The modulus of smoothness of $X$ is the function $\rho_X : [0, +\infty) \to [0, +\infty)$ defined by

$$\rho_X(\tau) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}.$$  

(2)

The space $X$ is said to be smooth if $\rho_X(\tau) > 0$, for all $\tau > 0$. It is well known that if $X$ is smooth then $J$ is single valued. A Banach space $X$ is said to be strictly convex if $\|x\| = \|y\| = 1, x \neq y$, implies $\|x + y\|/2 < 1$.

Let $C$ be a nonempty closed convex subset of a real Banach space $X$. Recall the following concepts.

Definition 1. (i) A mapping $f : C \to C$ is a $\rho$-contraction if $\rho \in [0, 1)$ and if the following property is satisfied

$$\|f(x) - f(y)\| \leq \rho \|x - y\|, \ \forall x, y \in C.$$  

(3)

(ii) A mapping $T : C \to E$ is nonexpansive provided

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C.$$  

(4)

(iii) A mapping $S : C \to X$ is

(a) accretive if for any $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Sx - Sy, j(x - y) \rangle \geq 0;$$  

(5)

(b) $\beta$-strongly accretive if for any $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Sx - Sy, j(x - y) \rangle \geq \beta \|x - y\|^2,$$  

(6)

for some real constant $\beta > 0$.

Noting that if $S : C \to X$ is nonexpansive, then $I - S$ is accretive; if $f : C \to C$ is a $\rho$-contraction, then $I - f$ is $(1 - \rho)$-strongly accretive. Particularly, if $X = H$ is a Hilbert space, then (strongly) accretive mappings become (strongly) monotone mappings.
Definition 2. Let $C$ and $D$ be nonempty subsets of a Banach space $X$ such that $C$ is nonempty closed convex and $D \subset C$.

(i) A mapping $Q : C \to D$ is called sunny, if $Q(Qx + t(x - Qx)) = Qx$ for each $x \in C$ and $t \geq 0$ with $Q(Qx + t(x - Qx)) \in C$.

(ii) A mapping $Q : C \to D$ is called a retraction from $C$ to $D$ if $Q$ is continuous and $F(Q) = D$.

(iii) A subset $D$ of $C \subset E$ is said to be a sunny nonexpansive retraction of $C$ if there exists a sunny nonexpansive retraction $Q$ of $C$ onto $D$. For details, see [1–3].

Note that if $X = H$ is a Hilbert space, $Q$ becomes the projection on $C$, denoted by $P_C$.

Let $P : C \to C$ a nonexpansive self-mapping on $C$ and \{$T_n \}$ be a countable family of nonexpansive nonself-mappings of $C$ into $X$ such that $\mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then we consider the following problem: find hierarchically a common fixed point of the infinite family $\{T_n\}$ with respect to a nonexpansive mapping $P$; namely, find $x^* \in \mathcal{F}$, such that

$$ \langle x^* - Px^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{7} $$

which was studied by Zhang et al. [5]. If $X = H$ is a Hilbert space and $\{T_n\}$ is an infinite family of nonexpansive self-mappings, Problem (7) reduces to the following problem: finding hierarchically a common fixed point of $\{T_n\}$ with respect to a nonexpansive mapping $P$; namely, find $x^* \in F(T)$ such that

$$ \langle x^* - Px^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \tag{8} $$

Recently, Marino and Xu [12] introduced the following explicit hierarchical fixed point algorithm in Hilbert spaces:

$$ x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n) (\alpha_n Vx_n + (1 - \alpha_n) T x_n), \quad \forall n \geq 0, \tag{11} $$

where $f$ is a contraction on $C$ and $V, T$ are two nonexpansive mappings of $C$ into itself and proved that the sequence $\{x_n\}$ generated by (11) converges strongly to a solution of problem (9).

Very recently, Zhang et al. [5] introduced the following iterative algorithm in order to find hierarchically a fixed point of Problem (8):

$$ x_0 \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \quad y_n = \beta_n P(x_n) + (1 - \beta_n) T x_n, \tag{12} $$

where $f : C \to C$ is a contraction, $P : C \to C$ is a nonexpansive mapping, $\{T_n\} : C \to C$ is a countable family of nonexpansive mappings, and $T : C \to C$ is a mapping defined by

$$ T = \sum_{n=1}^{\infty} \lambda_n T_n, \quad \lambda_n \geq 0 \ (n = 1, 2, \ldots) \quad \text{with} \quad \sum_{n=1}^{\infty} \lambda_n = 1. \tag{13} $$

Under suitable conditions on parameters $\{\alpha_n\}$ and $\{\beta_n\}$, they established some strong and weak convergence theorems. Note that, in [5], $\{T_n\}$ is an infinite family of self-mappings and $P$ is also a self-mapping. And they obtained the results in the setting of Hilbert spaces.

Motivated and inspired by the above researches, in a reflexive Banach space which admits a weakly sequentially continuous duality mapping $J$, we propose and analyze an iteration process for a countable family of nonexpansive nonself-mappings $\{T_n\} : C \to X$ and $S : C \to X$ is a nonexpansive nonself-mapping as follows:

$$ x_0 \in C, \quad x_{n+1} = Q(\alpha_n f(x_n) + (1 - \alpha_n) y_n), \tag{14} $$

$$ y_n = \beta_n S x_n + (1 - \beta_n) T x_n, \quad n \geq 0, $$

where $Q$ is a sunny nonexpansive retraction of $X$ onto $C$ and establishes a convergence theorem. Particularly, if $X = H$ is a Hilbert space, we obtain some convergence results.

To prove the main results, we need the following lemmas.

Lemma 3 (see [1]). Let $C$ be a nonempty and convex subset of a smooth Banach space $X$, $D \subset C$, $J : X \to X^*$ the normalized duality mapping of $X$, and $Q : C \to D$ a retraction. Then the following conditions are equivalent:

(i) $\langle x - Qx, J(y - Qx) \rangle \leq 0$, for all $x \in C$ and $y \in D$;

(ii) $Q$ is both sunny and nonexpansive.

Lemma 4 (see [13, Lemma 3.1, 3.3]). Let $X$ be a real smooth and strictly convex Banach space and $C$ a nonempty closed and
Lemma 5 (see [1]). Let $X$ be a real Banach space and $J : X \to 2^X^*$ the normalized duality mapping. Then for any $x, y \in X$, the following hold:

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle j(x), y\rangle$, for all $j(x) \in J(x)$;

(ii) $\|x\|^2 + 2\langle j(x), y\rangle \leq \|x + y\|^2$, for all $j(x) \in J(x)$.

Lemma 6 (see [14]). Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying

$$
\sum_{n=0}^{\infty} b_n < \infty, \quad a_{n+1} \leq a_n + b_n, \quad n = 0, 1, 2, \ldots.
$$

Then $\lim_{n \to \infty} a_n$ exists.

Lemma 7 (see [15]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying

$$
a_{n+1} \leq (1 - \lambda_n) a_n + \lambda_n b_n + c_n, \quad \forall n \geq 0,
$$

where $\{\lambda_n\}$, $\{b_n\}$ and $\{c_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} \lambda_n = \infty$ or, equivalently, $\Pi_{n=0}^{\infty} (1 - \lambda_n) = 0$;

(ii) $\lim \sup_{n \to \infty} b_n \leq 0$;

(iii) $c_n \geq 0 \quad (n \geq 0)$, $\sum_{n=0}^{\infty} c_n < \infty$.

Then $\lim_{n \to \infty} a_n = 0$.

If Banach space $X$ admits sequentially continuous duality mapping $J$ from weak topology to weak * topology, then by [16, Lemma 1] we get that duality mapping $J$ is single-valued. In this case, duality mapping $J$ is also said to be weakly sequentially continuous, that is, for each $\{x_n\} \subset X$ with $x_n \rightharpoonup x$, then $J(x_n) \to Jx$ [16, 17].

Recall that a Banach space $X$ is said to be satisfying Opial’s condition if for any sequence $\{x_n\}$ in $E$, $x_n \rightharpoonup x \quad (n \to \infty)$ implies that

$$
\lim_{n \to \infty} \|x_n - x\| < \lim_{n \to \infty} \|x_n - y\|, \quad \forall y \in E, \text{ with } y \neq x.
$$

By [16, Lemma 1], we know that if $X$ admits a weakly sequentially continuous duality mapping, then $X$ satisfies Opial’s condition.

In the sequel, we also need the following lemmas.

Lemma 8 (see [17]). Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $X$ which satisfies Opial’s condition and $T : C \to X$ a nonexpansive mapping. Then the mapping $I - T$ is demiclosed at zero, that is,

$$
x_n \rightharpoonup x, \quad x_n - Tx_n \to 0
$$

implies $x = Tx$.

Let $C$ be a nonempty and convex subset of a Banach space $X$. Then for $x \in C$, one defines the inward set $I_C(x)$ as follows [2, 3]:

$$
I_C(x) = \{y \in X : y = x + \lambda(z - x), z \in C, \lambda \geq 0\}.
$$

A mapping $T : C \to X$ is said to satisfy the inward condition if $Tx \in I_C(x)$ for all $x \in C$. $T$ is also said to satisfy the weakly inward condition if for each $x \in C$, $Tx \in \overline{I_C(x)}(I_C(x))$ is the closure of $I_C(x))$. Clearly $C \subset I_C(x)$ and it is not hard to show that $I_C(x)$ is a convex set if $C$ does.

Lemma 9 (see [18, Theorem 2.4]). Let $X$ be a reflexive Banach space which admits a weakly sequentially continuous duality mapping $J$ from $X$ to $X^*$. Suppose $C$ is a nonempty closed convex subset of $X$ which is also a sunny nonexpansive retract of $X$, and $T : C \to X$ is a nonexpansive mapping satisfying the weakly inward condition and $F(T) \neq \emptyset$. Let $\{u_n\}$ be defined by

$$
u_0 \in C, \quad u_{n+1} = Q\left(\alpha_n f(u_n) + (1 - \alpha_n) Tu_n\right),
$$

where $Q$ is a sunny nonexpansive retract of $X$ onto $C$ and $\alpha_n \in (0, 1)$ satisfy the following conditions:

(i) $\alpha_n \to 0$, as $n \to \infty$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) either $\lim_{n \to \infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n \to \infty} (\alpha_n/\alpha_{n+1}) = 1$.

Then $\{u_n\}$ converges strongly to a fixed point $p$ of $T$ such that $p$ is the unique solution in $F(T)$ to the following variational inequality:

$$
\langle (I - f) p, (p - u) \rangle \leq 0, \quad \forall u \in F(T).
$$

Remark 10. If a Banach space $X$ admits a sequentially continuous duality mapping $J$ from weak topology to weak star topology, from Lemma 1 of [16] it follows that $X$ is smooth. So for Lemma 9, if $X$ is a reflexive and strictly convex Banach space which admits a weakly sequentially continuous duality mapping $J$, by Lemma 4, the weakly inward condition of $T$ can be removed.

2. Main Results

Theorem 11. Let $X$ be a reflexive and strictly convex Banach space which admits a weakly sequentially continuous duality mapping $J : X \to X^*$ and $C$ a nonempty, closed and convex subset of $X$ which is also a sunny nonexpansive retract of $X$. Let $S : C \to X$ be a nonexpansive nonself-mapping, $f : C \to C$ condition and $T : C \to X$ a nonexpansive mapping. Then the mapping $I - T$ is demiclosed at zero, that is,
a contractive mapping with a contractive constant $\rho \in (0,1)$ and $T_i : C \to X$ ($i = \{1,2,\ldots\}$) an infinite family of nonexpansive self-mappings such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $T : C \to X$ be defined by (13) and $Q$ a sunny nonexpansive retraction of $X$ onto $C$. Let $\{x_n\}$ be the sequence generated by (14), and $\{\alpha_n\}$ and $\{\beta_n\}$ the sequences in $(0,1)$ satisfying the following conditions:

(i) $\alpha_n \to 0$ ($n \to \infty$), $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n \to \infty} (\beta_n/\alpha_n) = 0$;

(iii) $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ converges strongly to some point $x^* \in F(T) = \bigcap_{i=1}^{\infty} F(T_i)$, which is the unique solution to the following variational inequality:

$$
\langle (I - f)x^*, f(x - x^*) \rangle \geq 0, \quad \forall x \in F(T).
$$

**Proof.** From condition (ii), without loss of generality, we can assume that $\beta_n \leq \alpha_n$, for all $n \geq 0$.

First we prove that the sequence $\{x_n\}$ is bounded. In fact, for any $u \in F(T)$, we have

$$
\|x_{n+1} - u\|
= \|Q(\alpha_n f(x_n) + (1 - \alpha_n) y_n) - Q(u)\|
\leq \alpha_n \|f(x_n) - u\| + (1 - \alpha_n) \|\beta_n S x_n + (1 - \beta_n) T x_n - u\|
\leq \alpha_n (\rho \|x_n - u\| + \|f(u) - u\|)
+ (1 - \alpha_n) (\beta_n \|S x_n - u\| + (1 - \beta_n) \|T x_n - u\|)
\leq (1 - \alpha_n (1 - \rho)) \|x_n - u\| + \alpha_n \|f(u) - u\|
+ (1 - \alpha_n) \beta_n \|S u - u\|
\leq \alpha_n (\|f(u) - u\| + \|S u - u\|)
\leq \max \left\{ \|x_n - u\|, \frac{\|f(u) - u\| + \|S u - u\|}{1 - \rho} \right\}.
$$

By induction,

$$
\|x_{n+1} - u\| \leq \max \left\{ \|x_0 - u\|, \frac{\|f(u) - u\| + \|S u - u\|}{1 - \rho} \right\}.
$$

Therefore, $\{x_n\}$ is bounded, so $\{S x_n\}$ and $\{T x_n\}$ are also bounded.

Next we prove that $\|x_n - u_n\| \to 0$, as $n \to \infty$, where the sequence $\{u_n\}$ is defined by

$$
u_0 = x_0 \in C,

u_{n+1} = Q(\alpha_n f(u_n) + (1 - \alpha_n) T u_n).
$$

By Lemma 9 and Remark 10, $\{u_n\}$ converges strongly to some point $x^* \in F(T)$, which is the unique solution to the following variational inequality:

$$
\langle (I - f)x^*, f(x - x^*) \rangle \leq 0, \quad \forall x \in F(T).
$$

Furthermore, we obtain

$$
\begin{align*}
\|x_{n+1} - u_{n+1}\|
& \leq \|Q(\alpha_n f(x_n) + (1 - \alpha_n) y_n) - Q(\alpha_n f(u_n) + (1 - \alpha_n) T u_n)\|
\leq \|\alpha_n (f(x_n) - f(u_n)) + (1 - \alpha_n) (y_n - T u_n)\|
\leq \alpha_n \rho \|x_n - u_n\| + (1 - \alpha_n)
\times (\beta_n \|S x_n - T u_n\| + (1 - \beta_n) \|T x_n - T u_n\|)
\leq (1 - \alpha_n (1 - \rho)) \|x_n - u_n\| + (1 - \alpha_n) \beta_n M
\leq (1 - \alpha_n (1 - \rho)) \|x_n - u_n\| + \beta_n M,
\end{align*}
$$

where $M = \sup_{u \in C} \|S x_n - T u_n\|$. It follows from conditions (i)-(ii) and Lemma 7 that we have $\lim_{n \to \infty} \alpha_n \|x_n - u_n\| = 0$. Since as $n \to \infty$, $u_n \to x^* \in F(T)$, we get $x_n \to x^*(n \to \infty)$, which is the unique solution to the variational inequality (22).

**Remark 12.** Theorem 11 extends Theorem 2.1 in [5] from the following aspects: (i) from Hilbert spaces to reflexive and strictly convex Banach spaces which admits a weakly sequentially continuous duality mapping; (ii) for the infinite family of mappings $\{T_i\}$ from self-mappings to nonself-mappings. In addition, the existence of the sunny nonexpansive retraction has been proved in [19, Theorem 3.10].

**Remark 13.** If we take

$$
\begin{align*}
\alpha_n &= \frac{1}{(1 + n)\alpha},
\beta_n &= \frac{1}{(1 + n)\beta},
\end{align*}
$$

$$
0 < \alpha < \beta < 1,
$$

then since $|\alpha_{n+1} - \alpha_n| \approx 1/n^{\alpha+1}$ and $|\beta_{n+1} - \beta_n| \approx 1/n^{\beta+1}$ (as $n \to \infty$), it is not hard to find that the conditions (i)-(iii) are satisfied. For details, see [12, Remark 3.2].

In the sequel, we consider the result in the setting of Hilbert spaces.

**Theorem 14.** Let $H$ be a Hilbert space and $C$ a nonempty, closed and convex subset of $H$. Let $S : C \to H$ be a nonexpansive nonself-mapping, $f : C \to C$ a contractive mapping with a contractive constant $\rho \in (0,1)$, and $T_i : C \to H$ ($i = \{1,2,\ldots\}$) an infinite family of nonexpansive self-mappings such that $F(T) = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by (14) and $\{\alpha_n\}$ and $\{\beta_n\}$ the sequences in $(0,1)$ satisfying the following conditions:

(i) $\alpha_n \to 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n \to \infty} (\beta_n/\alpha_n) = \tau \in (0, +\infty)$;

(iii) $\lim_{n \to \infty} ([\beta_n - \beta_{n-1} + |\alpha_n - \alpha_{n-1}|]/\alpha_n \beta_n) = 0$;

(iv) there exists a constant $K > 0$ such that $1/\alpha_n \left| (1/\beta_n) - (1/\beta_{n-1}) \right| \leq K$ for all $n > 0$. 

Furthermore, we obtain

$$
\begin{align*}
\|x_{n+1} - u_{n+1}\|
& \leq \|Q(\alpha_n f(x_n) + (1 - \alpha_n) y_n)
- Q(\alpha_n f(u_n) + (1 - \alpha_n) T u_n)\|
\leq \|\alpha_n (f(x_n) - f(u_n)) + (1 - \alpha_n) (y_n - T u_n)\|
\leq \alpha_n \rho \|x_n - u_n\| + (1 - \alpha_n)
\times (\beta_n \|S x_n - T u_n\| + (1 - \beta_n) \|T x_n - T u_n\|)
\leq (1 - \alpha_n (1 - \rho)) \|x_n - u_n\| + (1 - \alpha_n) \beta_n M
\leq (1 - \alpha_n (1 - \rho)) \|x_n - u_n\| + \beta_n M,
\end{align*}
$$

where $M = \sup_{u \in C} \|S x_n - T u_n\|$. It follows from conditions (i)-(ii) and Lemma 7 that we have $\lim_{n \to \infty} \alpha_n \|x_n - u_n\| = 0$. Since as $n \to \infty$, $u_n \to x^* \in F(T)$, we get $x_n \to x^*(n \to \infty)$, which is the unique solution to the variational inequality (22).
Then \(\{x_n\}\) converges strongly to some point \(x^* \in F(T)\), which is the unique solution to the following variational inequality:

\[ \left\langle \frac{1}{\tau} (I - f) x^* + (I - S) x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in F(T). \]  

(29)

\[ \text{Proof. By condition (ii), without loss of generality, we can assume that } \beta_n \leq (\tau + 1)\alpha_n, \text{ for all } n \geq 0. \text{ Similar to the proof of (24), for any } u \in F(T), \text{ we have} \]

\[ \|x_{n+1} - u\| \leq \max \left\{ \|x_0 - u\|, \frac{(r + 1)(\| f(u) - u \| + \|Su - u\|)}{1 - \rho} \right\}. \]

(30)

Thus \(\{x_n\}\) is bounded. Furthermore, \(\{f(x_n)\}, \{Tx_n\}, \{y_n\}, \{Sx_n\}\) are all bounded. Put \(u_n = \alpha_n f(x_n) + (1 - \alpha_n) y_n\) and \(M = \sup_{n \geq 0} \|f(x_n)\| + \|y_n\|, \|Tx_n\| + \|Sx_n\|\). So \(\{u_n\}\) and \(\{P_C(u_n)\}\) are also bounded.

Step 1. We prove that \(\|x_{n+1} - x_n\| \to 0 \, (n \to \infty)\).

From (14), we obtain

\[ \|x_{n+1} - x_n\| = \|P_C(u_n) - P_C(u_{n-1})\| \leq \|u_n - u_{n-1}\| \leq \alpha_n \| f(x_n) - f(x_{n-1}) \| + (1 - \alpha_n) \| y_n - y_{n-1} \| + (1 - \alpha_n) \| y_n - y_{n-1} \| \leq \alpha_n \rho \| y_n - y_{n-1} \| + (1 - \alpha_n) \| y_n - y_{n-1} \| \leq \| y_n - y_{n-1} \| + (1 - \alpha_n) M, \]

\[ \| y_n - y_{n-1} \| \leq \beta_n \| Sx_n - Sx_{n-1} \| + (1 - \beta_n) \| Tx_n - Tx_{n-1} \| \]

(31)

Substituting (32) into (31), we have

\[ \|x_{n+1} - x_n\| \leq (1 - \alpha_n (1 - \rho)) \| x_n - x_{n-1} \| + \frac{(1 - \alpha_n) M}{\alpha_n}. \]

(32)

By conditions (i), (iii), and Lemma 7, we have \(\|x_{n+1} - x_n\| \to 0 \, (n \to \infty)\).

Step 2. We prove that \(\omega_n(x_n) \subset F(T)\), where \(\omega_n(x_n)\) is the \(\omega\)-limit point set of \(\{x_n\}\) in the weak topology:

\[ \|x_{n+1} - QTx_n\| \leq \alpha_n \| f(x_n) \| + \beta_n \| Sx_n \| + (\alpha_n + \beta_n + \alpha_n \beta_n) \| Tx_n \|. \]

(34)

Noting that \(\alpha_n \to 0\) and \(\beta_n \to 0\), we have \(\|x_{n+1} - QTx_n\| \to 0 \, (n \to \infty)\). Then from Step 1 we have \(\|x_n - QTx_n\| \to 0 \, (n \to \infty)\). Furthermore, it follows from Lemmas 4 and 8 that \(\omega_n(x_n) \subset F(QT) = F(T)\), where \(Q = P_C\).

Step 3. We show that \(\|x_{n+1} - x_n\|/\beta_n \to 0 \, (n \to \infty)\).

It follows from (31) and (33) that

\[ \|x_{n+1} - x_n\|/\beta_n \leq \frac{\|u_n - u_{n-1}\|}{\beta_n} \leq (1 - \alpha_n (1 - \rho)) \| x_n - x_{n-1} \|/\beta_{n-1} + (1 - \alpha_n (1 - \rho)) \| x_n - x_{n-1} \| \| 1 - \| 1 - \beta_n \| + (1 - \alpha_n) M/\alpha_n \beta_n \]

(35)

\[ \|x_{n+1} - x_n\|/\beta_n \to 0 \, (n \to \infty). \]

(36)

Thus from (35), we get

\[ \|u_n - u_{n-1}\|/\beta_n \to 0 \, (n \to \infty). \]

(37)

Step 4. We show that \(\{x_n\}\) converges strongly to some point \(x^* \in F(T)\), which is the unique solution of (29).

Setting \(W_n = \beta_n S + (1 - \beta_n) T\), we have

\[ x_{n+1} = P_C(u_n) - u_n + \alpha_n f(x_n) + (1 - \alpha_n) W_n x_n. \]

(38)

Then

\[ x_n - x_{n-1} = u_n - P_C(u_n) + \alpha_n (I - f) x_n + (1 - \alpha_n) (I - W_n) x_n. \]

(39)

Letting \(v_n = (x_n - x_{n-1})/(1 - \alpha_n)\beta_n\), from condition (i) and (36), we have \(v_n \to 0 \, (n \to \infty)\). Noting that \(I - W_n\) is
Thus we have
\[
\langle v_n, x_n - x^* \rangle
= \frac{1}{(1 - \alpha_n) \beta_n} \langle u_n - P_C(u_n), x_n - x^* \rangle
+ \frac{\alpha_n}{(1 - \alpha_n) \beta_n} \langle (I - f)x_n, x_n - x^* \rangle
+ \frac{1}{\beta_n} \langle (I - W_n)x_n, x_n - x^* \rangle
= \frac{1}{(1 - \alpha_n) \beta_n} \langle u_n - P_C(u_n), x_n - x^* \rangle
+ \frac{\alpha_n}{(1 - \alpha_n) \beta_n} \langle (I - f)x_n, x_n - x^* \rangle
+ \frac{1}{\beta_n} \langle (I - W_n)x_n - (I - W_n)x^*, x_n - x^* \rangle
+ \frac{1}{\beta_n} \langle (I - W_n)x^*, x_n - x^* \rangle
\]
\[
\geq \frac{1}{(1 - \alpha_n) \beta_n} \langle u_n - P_C(u_n), P_C(u_{n-1}) - P_C(u_n) \rangle
+ \frac{\alpha_n}{(1 - \alpha_n) \beta_n} \| x_n - x^* \|^2
+ \frac{\alpha_n}{(1 - \alpha_n) \beta_n} \langle (I - f)x^*, x_n - x^* \rangle
+ \langle (I - S)x^*, x_n - x^* \rangle.
\]  
Combining condition (ii), \( v_n \to 0 \) \( (n \to \infty) \), (41), and (42), every weak cluster point of \( \{x_n\} \) is also a strong cluster point.

From (40), we obtain
\[
\langle (I - f)x_n, x_n - x^* \rangle
= \frac{(1 - \alpha_n) \beta_n}{\alpha_n} \| v_n \|^2
\]
\[
- \frac{1}{\alpha_n} \langle u_n - P_C(u_n), x_n - x^* \rangle
- \frac{1}{\alpha_n} \langle (I - S)x^*, x_n - x^* \rangle
- \frac{1}{\alpha_n} \langle (I - f)x^*, x_n - x^* \rangle
- \frac{1}{\alpha_n} \langle (I - W_n)x^*, x_n - x^* \rangle.
\]  

Thus we have
\[
\| x_n - x^* \|^2
\leq \frac{(1 - \alpha_n) \beta_n}{\alpha_n (1 - \rho)} \langle v_n, x_n - x^* \rangle
\]
Replacing \( F(T) \), and

\[
\langle (I-S)x^*, x_n - x^* \rangle 
\]

\[
\leq \frac{(1-\alpha_n)}{\alpha_n} \|v_n\| \|x_n - x^*\| 
+ \frac{1}{\alpha_n} \|u_n - P_C(u_n)\| P_C(u_{n-1}) - P_C(u_n)\| 
- \frac{(1-\alpha_n)}{\alpha_n} \langle (I-S)x^*, x_n - x^* \rangle 
\]

\[
\leq \frac{(1-\alpha_n)}{\alpha_n} \|v_n\| \|x_n - x^*\| 
+ \frac{1}{\alpha_n} \|u_n - P_C(u_n)\| P_C(u_{n-1}) - P_C(u_n)\| 
- \frac{(1-\alpha_n)}{\alpha_n} \langle (I-S)x^*, x_n - x^* \rangle. 
\]

(43)

Note that the sequence \( \{x_n\} \) is bounded; thus there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) converging to a point \( x' \in H \). From Step 2, we have \( x' \in F(T) \). Then it follows from the above inequality, (42), and \( v_n \to 0 \) \( (n \to \infty) \) that

\[
\langle (I-f)x', x' - x^* \rangle 
\leq -\tau \langle (I-S)x^*, x' - x^* \rangle, \quad \forall x^* \in F(T). 
\]

Replacing \( x^* \) with \( x' + \mu(x^* - x') \), where \( \mu \in (0, 1) \) and \( x^* \in F(T) \), we have

\[
\langle (I-f)x', x' - x^* \rangle 
\leq -\tau \langle (I-S)(x' + \mu(x^* - x') \rangle , x' - x^* \rangle, \quad (44) \]

\[
\forall x^* \in F(T). 
\]

Letting \( \mu \to 0 \), we have

\[
\langle (I-f)x', x' - x^* \rangle 
\leq -\tau \langle (I-S)x', x' - x^* \rangle, \quad (45) \]

\[
\forall x^* \in F(T). 
\]

If there exists another subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) converging to a point \( x'' \in H \). From Step 2, we also have \( x'' \in F(T) \). Then from (46) we obtain

\[
\langle (I-f)x', x' - x'' \rangle \leq -\tau \langle (I-S)x', x' - x'' \rangle \quad (47) \]

and, via interchanging \( x' \) and \( x'' \),

\[
\langle (I-f)x'', x'' - x' \rangle \leq -\tau \langle (I-S)x'', x'' - x' \rangle. \quad (48) \]

Adding up these two inequalities yields

\[
(1-\rho) \|x'-x''\|^2 \leq \langle (I-f)x' - (I-f)x'', x' - x'' \rangle \leq 0, \quad (49) \]

which implies \( x' = x'' \). Then \( \{x_n\} \) converges strongly to \( x' \in F(T) \), which is the solution to the following variational inequality:

\[
\langle \frac{1}{\tau} (I-f)x' + (I-S)x', x-x' \rangle \geq 0, \quad \forall x \in F(T). 
\]

(50)

Since \( I-f \) is \((1-\rho)\)-strongly monotone and \( I-S \) is monotone, it is easy to see that the above variational inequality has a unique solution.

**Remark 15.** Theorem 14 extends Theorem 3.2 in [12] from the following aspects: (i) from a nonexpansive mapping \( T \) to an infinite family of nonexpansive mappings \( \{T_i\} \); (ii) from self-mappings to nonself-mappings.

**Acknowledgments**

The author is extremely grateful to the referees for their useful suggestions that improved the content of the paper. Supported by the China Postdoctoral Science Foundation Funded Project (no. 2012M51928).

**References**


