TERMINAL VALUE PROBLEMS OF IMPULSIVE
INTEGRO-DIFFERENTIAL EQUATIONS
IN BANACH SPACES

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This paper uses cone theory and the monotone iterative technique to investigate the existence of minimal nonnegative solutions of terminal value problems for first order nonlinear impulsive integro-differential equations of mixed type in a Banach space.

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1. Introduction

The theory of impulsive differential equations has become an important area of investigation. Initial value problems of such equations have been discussed in detail in recent years (see [3]). In this paper, we shall use cone theory and the monotone iterative technique to investigate the existence of a minimal nonnegative solution of the terminal value problem (TVP) for a first order nonlinear impulsive integro-differential equation of mixed type in a Banach space.

2. Preliminaries

Let $E$ be a real Banach space and $P$ be a cone in $E$ which defines a partial order in $E$: $x \leq y$ if and only if $y - x \in P$. $P$ is said to be normal if there exists a positive constant $N$ such that $\theta \leq x \leq y$ implies $\|x\| \leq N \|y\|$, where $\theta$ denotes the zero element of $E$. $P$ is said to be regular (or fully regular) if $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \leq y$ (or $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots$ with $\sup_n \|x_n\| < \infty$) implies $\|x_n - x\| \to 0$ as $n \to \infty$ for some $x \in E$. The full regularity of $P$ implies the regularity of $P$, and the regularity of $P$ implies the normality of $P$ (see [2], Theorem 1.2.1). Moreover, if $E$ is weakly complete (in particular, reflexive), then the normality of $P$ implies the regularity of

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Consider the TVP in $E$:

$$\begin{align*}
    x' &= f(t,x,Tx,Sx), \\
    \Delta x \mid_{t=t_m} &= I_m(x(t_m)), \\
    x(\infty) &= x^*,
\end{align*}$$

where $J = [0, \infty)$, $f \in C(J \times P \times P \times P, -P)$, $0 < t_1 < \ldots < t_m < \ldots$, $t_m \to \infty$ and $m \to \infty$, $I_m \in C(P, -P)$ ($m = 1, 2, 3, \ldots$), $x^* \in P$, $x(\infty) = \lim_{t \to \infty} x(t)$, and

$$\begin{align*}
    (Tx)(t) &= \int_0^t k(t,s)x(s)ds, \\
    (Sx)(t) &= \int_0^\infty h(t,s)x(s)ds,
\end{align*}$$

$k \in C(D, R_+)$, $D = \{(t,s) \in J \times J: t \geq s\}$, $h \in C(J \times J, R_+)$. $\Delta x \mid_{t=t_m} = x(t_m^+) - x(t_m^-)$ which denotes the jump of $x(t)$ at $t = t_m$. Here $x(t_m^+)$ and $x(t_m^-)$ represent the right- and left-sided limits of $x(t)$ at $t = t_m$, respectively.

Let $PC(J,E) = \{x: x$ is a map from $J$ into $E$ such that $x(t)$ is continuous at $t \neq t_m$ and left continuous at $t = t_m$ and $x(t_m^+)$ exists for $m = 1, 2, 3, \ldots\}$, $BPC(J,E) = \{x \in PC(J,E): \sup \|x(t)\| < \infty\}$ and $TPC(J,E) = \{x \in PC(J,E): x(\infty) = \lim_{t \to \infty} x(t) \text{ exists}\}$. Evidently, $TPC(J,E) \subset BPC(J,E)$, and $BPC(J,E)$ is a Banach space with norm $\|x\| = \sup \|x(t)\|$. Let $BPC(J,P) = \{x \in BPC(J,E): x(t) \geq 0 \text{ for } t \in J\}$, $TPC(J,P) = \{x \in TPC(J,E): x(t) \geq 0 \text{ for } t \in J\}$ and $J' = J \setminus \{t_1, \ldots, t_m, \ldots\}$. A map $x \in TPC(J,P) \cap C^1(J',E)$ is called a non-negative solution of TVP(1) if it satisfies (1).

### 3. Main Results

Let us list some conditions.

\begin{itemize}
    \item[(H1)] $k^* = \sup_{t \in J} \int_0^t k(t,s)ds < \infty$, $h^* = \sup_{t \in J} \int_0^\infty h(t,s)ds < \infty$ and
    \[\lim_{t' \to t} \int_0^t |h(t',s) - h(t,s)| ds = 0, \quad t \in J.\]
    \item[(H2)] $\|f(t,x,y,z)\| \leq p(t) + q(t)(a \|x\| + b \|y\| + c \|z\|)$, $t \in J$, $x,y,z \in P$, and
    \[\|I_m(x)\| \leq a_m + b_m \|x\|, \quad x \in P(m = 1, 2, 3, \ldots),\]
    where $p,q \in C(J,R_+)$ and $a \geq 0$, $b \geq 0$, $a_m \geq 0$, $b_m \geq 0$ ($m = 1, 2, 3, \ldots$) satisfying
    \[p^* = \int_0^\infty p(t)dt < \infty, \quad q^* = \int_0^\infty q(t)dt < \infty, \quad a^* = \sum_{m=1}^{\infty} a_m < \infty, \quad b^* = \sum_{m=1}^{\infty} b_m < \infty.\]
    \item[(H3)] $f(t,x,y,z)$ is nonincreasing in $x,y,z \in P$ and $I_m(x)$ is nonincreasing in $x \in P$ ($m = 1, 2, 3, \ldots$), i.e.
\end{itemize}
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\[ f(t, x, y, z) \leq f(t, \bar{x}, \bar{y}, \bar{z}), \quad t \in J, \quad x \geq \bar{x} \geq \theta, \quad y \geq \bar{y} \geq \theta, \quad z \geq \bar{z} \geq \theta \]

and

\[ I_m(x) \leq I_m(\bar{x}), \quad x \geq \bar{x} \geq \theta \quad (m = 1, 2, 3, \ldots). \]

It is easy to see that when (H₁) is satisfied, T and S, defined by (2), are bounded linear operators from BPC(J, E) into BPC(J, E).

**Lemma 1:** If conditions (H₁) and (H₂) are satisfied, then for any \( x \in BPC(J, P) \), the integral

\[ \int_0^\infty f(t, x(t), (Tx)(t), (Sx)(t))dt \]

and the series

\[ \sum_{m=1}^\infty I_m(x(t_m)) \]

are convergent.

**Proof:** Let \( x \in BPC(J, P) \). By virtue of (H₁) and (H₂), it is easy to see that

\[ \int_0^\infty \| f(s, x(s), (Tx)(s), (Sx)(s)) \| ds \leq \int_0^\infty p(s)ds + (a + bk^* + ch^*) \| x \| B \int_0^\infty q(s)ds < \infty \]

and

\[ \sum_{m=1}^\infty \| I_m(x(t_m)) \| \leq \sum_{m=1}^\infty a_m + \| x \| B \sum_{m=1}^\infty b_m < \infty, \]

so, integral (3) and series (4) are convergent.

**Lemma 2:** Let conditions (H₁) and (H₂) be satisfied. Then \( x \in TPC(J, P) \cap C^1(J', E) \) is a solution of TVP(1) if and only if \( x \in BPC(J, P) \) is a solution to the following impulsive integral equation

\[ x(t) = x^* - \int_t^{\infty} f(s, x(s), (Tx)(s), (Sx)(s))ds - \sum_{t \leq t_m < \infty} I_m(x(t_m)), \quad t \in J. \]

**Proof:** Let \( x \in TPC(J, P) \cap C^1(J', E) \) be a solution of TVP(1). We first establish the following formula:

\[ x(t) = x(0) + \int_0^t x'(s)ds + \sum_{0 < t_m < t} [x(t_m^+) - x(t_m)], \quad t \in J. \]

In fact, let \( t_m \leq t \leq t_{m+1} \). Then

\[ x(t) - x(0) = \int_0^{t_1} x'(s)ds, \quad x(t_1) - x(0) = \int_{t_1}^{t_2} x'(s)ds, \]

\[ \dot{\cdots}, \dot{\cdots}, \dot{\cdots}\]

\[ x(t_m) - x(t_{m-1}) = \int_{t_{m-1}}^{t_m} x'(s)ds, \quad x(t) - x(t_m^+) = \int_{t_m}^{t} x'(s)ds. \]
Summing up these equations, we get

$$x(t) = x(0) + \int_0^t f(s, x(s), (Tx)(s), (Sx)(s))ds + \sum_{0 < m < t} I_m(x(t_m)), \quad t \in J. \tag{7}$$

By Lemma 1, integral (3) and series (4) are convergent, hence, from (1) and (7) we get

$$x^* = x(0) + \int_0^\infty f(s, x(s), (Tx)(s), (Sx)(s))ds + \sum_{m = 1}^{\infty} I_m(x(t_m)). \tag{8}$$

Solving $x(0)$ from (8) and substituting it into (7), we find that $x(t)$ satisfies equation (5).

Conversely, if $x \in BPC(J, P)$ is a solution of equation (5), direct differentiation of (5) implies that $x \in C^1(J', E)$ and $x(t)$ satisfies TVP(1).

Consider operator $A$ defined by

$$(Ax)(t) = x^* - \int_0^\infty f(s, x(s), (Tx)(s), (Sx)(s))ds - \sum_{t \leq t_m < \infty} I_m(x(t_m)). \tag{9}$$

**Lemma 3:** If conditions $(H_1)$ and $(H_2)$ are satisfied, then $A$ defined by (9) is an operator from $BPC(J, P)$ into $BPC(J, P)$.

**Proof:** Let $x \in BPC(J, P)$. Since $f \in C(J \times P \times P \times P, P)$, $I_m \in C(P, P)$ and $x^* \in P$, we see that $(Ax)(t) \geq 0$ for $t \in J$, and clearly $Ax \in PC(J, P)$. By $(H_1)$ and $(H_2)$, we have

$$\| (Ax)(t) \| \leq \| x^* \| + \int_0^\infty p(s)ds + (a + bk^* + ch^*) \| x \| B \int_0^\infty q(s)ds$$

$$+ \sum_{t \leq t_m < \infty} a_m + \| x \| B \sum_{t \leq t_m < \infty} b_m$$

$$\leq \| x^* \| + p^* + a^* + [b^* + (a + bk^* + ch^*)q^*] \| x \| B, \quad t \in J,$$

and therefore

$$\| Ax \| B \leq \| x^* \| + p^* + a^* + [b^* + (a + bk^* + ch^*)q^*] \| x \| B. \tag{10}$$

Hence $Ax \in BPC(J, P)$.

In the following, let $J_0 = [0, t_1], \ J_m = (t_m, t_{m+1}] (m = 1, 2, 3, \ldots)$.

**Theorem 1:** Let cone $P$ be fully regular and conditions $(H_1)$, $(H_2)$, $(H_3)$ be satisfied. Assume that

$$r = b^* + (a + bk^* + ch^*)q^* < 1, \tag{11}$$

where constants $k^*, h^*, a, b, c, q^*, b^*$ are defined by $(H_1)$ and $(H_2)$. There exists a nondecreasing sequence $\{x_n\} \subset TPC(J, P) \cap C^1(J', E)$ which converges on $J$ (uniformly in each $J_m$, $m = 0, 1, 2, \ldots$) to the minimal solution $x \in TPC(J, P) \cap C^1(J', E)$ of TVP(1) in $TPC(J, P) \cap C^1(J', E)$, i.e., for any solution $x \in$
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$TPC(J, P) \cap C^1(J', E)$ of $TVP(1)$, we have

$$x(t) \geq \bar{x}(t), \quad t \in J.$$  \hfill (12)

Moreover,

$$\bar{x}(t) \geq \bar{x}(t'), \quad 0 \leq t < t' < \infty,$$  \hfill (13)

and

$$\| \bar{x} \|_B \leq (1 - r)^{-1}(\| x^* \| + p^* + a^*),$$  \hfill (14)

where $r$ is given by (11) and $p^*, a^*$ are defined by $(H_2)$.

**Proof:** Let $x_0(t) = \theta$, $x_n(t) = (Ax_{n-1})(t)$ $(n = 1, 2, 3, \ldots)$, i.e.,

$$x_n(t) = x^* - \int_0^\infty f(s, x_{n-1}(s), (Tx_{n-1})(s), (Sx_{n-1})(s))ds$$

$$\leq \sum_{t \leq t_m < \infty} I_m(x_{n-1}(t_m)), \quad t \in J(n = 1, 2, 3, \ldots).$$  \hfill (15)

By Lemma 3, $x_n \in BPC(J, P)$ $(n = 0, 1, 2, \ldots)$ and $x_1(t) \geq \theta = x_0(t)$ for $t \in J$, so, (15) and $(H_3)$ imply that

$$0 = x_0(t) \leq x_1(t) \leq x_2(t) \leq \ldots \leq x_n(t) \leq \ldots, \quad t \in J.$$  \hfill (16)

On the other hand, from (10) we know

$$\| x_n \|_B = \| Ax_{n-1} \|_B \leq d + r \| x_{n-2} \|_B, \quad (n = 1, 2, 3, \ldots),$$

where $d = \| x^* \| + p^* + a^*$ and $r$ is given by (11), thus

$$\| x_n \|_B \leq d + r(d + r \| x_{n-2} \|_B) \leq d + rd + r^2(d + r \| x_{n-3} \|_B)$$

$$\leq d + rd + \ldots + r^{n-1}d + r^n \| x_0 \|_B = d + rd + \ldots + r^{n-1}d = d(1 - r^n)(1 - r)^{-1}$$

$$\leq d(1 - r)^{-1}, \quad (n = 1, 2, 3, \ldots).$$  \hfill (17)

It follows from (16), (17), and the full regularity of $P$ that the following limit exists:

$$\lim_{n \to \infty} x_n(t) = \bar{x}(t), \quad t \in J.$$  \hfill (18)

Now we have, by (17),

$$\| f(s, x_{n-1}(s), (Tx_{n-1})(s), (Sx_{n-1})(s)) \| \leq p(s) + (a + bk^* + ch^*) \| x_{n-1} \|_B q(s)$$

$$\leq p(s) + (a + bk^* + ch^*)d(1 - r)^{-1}q(s), \quad s \in J \quad (n = 1, 2, 3, \ldots),$$  \hfill (19)

so, from (15) we know that functions $\{x_{mn}(t)\}$ $(n = 0, 1, 2, \ldots)$ are equicontinuous in $J_m$ $(m = 0, 1, 2, \ldots)$, where $J_m = [t_m, t_{m+1}]$ and

$$x_{mn}(t) = \begin{cases} x_n(t), & t_m < t \leq t_{m+1}; \\ x_n(t_m^+), & t = t_m. \end{cases}$$

Hence, observing (18) and using the Ascoli-Arzela theorem, we see that $\{x_{mn}(t)\}$ $(n = 0, 1, 2, \ldots)$ is compact in $C(J_m, E)$ $(m = 0, 1, 2, \ldots)$ and therefore, by diagonal method, $\{x_n(t)\}$ has a subsequence which converges to $\bar{x}(t)$ uniformly in each $J_m$ $(m = 0, 1, 2, \ldots)$. Since $P$ is also normal and $\{x_n(t)\}$ is nondecreasing, on account of (16), we conclude that the entire sequence $\{x_n(t)\}$ converges to $\bar{x}(t)$ uniformly in
each $J_m$ ($m = 0, 1, 2, ...$), hence, $\bar{x} \in \text{PC}(J, P)$. Moreover, from (17) we know that $\bar{x} \in B\text{PC}(J, P)$ and $\|\bar{x}\|_B \leq d(1 - r)^{-1}$, i.e., (14) holds.

From (18) and (19), we see that

$$
\lim_{n \to \infty} f(s, x_{n-1}(s), (T x_{n-1})(s), (S x_{n-1})(s)) = f(s, \bar{x}(s), (T \bar{x})(s), (S \bar{x})(s))
$$

as $n \to \infty$, $s \in J$, (20)

and

$$
\|f(s, x_{n-1}(s), (T x_{n-1})(s), (S x_{n-1})(s)) - f(s, \bar{x}(s), (T \bar{x})(s), (S \bar{x})(s))\|
\leq 2p(s) + 2(a + b k^* + ch^*)d(1 - r)^{-1}q(s), \quad s \in J \quad (n = 1, 2, 3, ...).
$$

In addition, (17), (18) and (H2) imply that

$$
I_m(x_{n-1}(tm)) \to I_m(\bar{x}(tm)) \text{ as } n \to \infty \quad (m = 1, 2, 3, ...),
$$

and

$$
\sum_{m = j}^{\infty} \|I_m(x_{n-1}(tm))\| \leq \sum_{m = j}^{\infty} a_m + d(1 - r)^{-1} \sum_{m = j}^{\infty} b_m \quad (n = 1, 2, 3, ...),
$$

and

$$
\sum_{m = j}^{\infty} \|I_m(\bar{x}(tm))\| \leq \sum_{m = j}^{\infty} a_m + d(1 - r)^{-1} \sum_{m = j}^{\infty} b_m.
$$

Observing (20)-(24) and taking limits in (15) as $n \to \infty$, we obtain by virtue of the dominated convergence theorem that

$$
\bar{x}(t) = x^* - \int_t^\infty f(s, \bar{x}(s), (T \bar{x})(s), (S \bar{x})(s)) ds - \sum_{t \leq t_m < \infty} I_m(\bar{x}(tm)), \quad t \in J,
$$

which by Lemma 2 implies that $\bar{x} \in T\text{PC}(J, P) \cap C^1(J', E)$ and $\bar{x}(t)$ is a solution of TVP(1). From (25) we see clearly that (13) holds.

Finally, we prove the minimal property of $\bar{x}(t)$. Let $x \in T\text{PC}(J, P) \cap C^1(J', E)$ by any solution of TVP(1). By Lemma 2, $x(t)$ satisfies equation (5). We have $x(t) \geq \bar{x}(t)$ for $t \in J$. Assume that $x(t) \geq x_1(t)$ for $t \in J$. Then (15), (5) and (H3) imply that $x(t) \geq x_n(t)$ for $t \in J$. Hence, by induction, $x(t) \geq x_n(t)$ for $t \in J(n = 0, 1, 2, ...)$, and by taking the limit, we get $x(t) \geq \bar{x}(t)$ for $t \in J$, i.e., (12) holds. The proof is complete.

**Example 1:** Consider the TVP of infinite system for scalar nonlinear impulsive integro-differential equations

$$
x'_n = -\frac{e^{-t}}{2n + 3(1 + x_n + \sqrt{x_n + 1 + 2x_{2n+1}})} - \frac{e^{-t}}{3n} \left( \int_0^t e^{-(t+s)x_n(s)} ds \right)^{1/3},
\Delta x_n |_{t = m} = -\frac{1}{2n + m + 2} [x_n(m) + x_n + 2(m)], \quad (m = 1, 2, 3, ...),
\quad x_n(\infty) = \frac{1}{n^2}, \quad (n = 1, 2, 3, ...).
$$

**Corollary:** TVP(26) has a minimal, nonnegative and continuously differentiable on $[0, \infty) \setminus \{1, 2, 3, ...\}$ solution $\{x_n(t)\}$ ($n = 1, 2, 3, ...$) satisfying
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Proof: Let \( E = \ell^1 = \{ x = (x_1, \ldots, x_n, \ldots) : \sum_{n=1}^{\infty} |x_n| < \infty \} \), with norm \( \|x\| = \sum_{n=1}^{\infty} |x_n| \) and \( P = \{ x = (x_1, \ldots, x_n, \ldots) \in \ell^1 : x_n \geq 0, \ n = 1, 2, 3, \ldots \} \). Thus, \( P \) is a normal cone in \( E \). Since \( \ell^1 \) is weakly complete, we conclude that \( P \) is regular. We now prove that \( P \) is fully regular. Let \( x_k = (x_{k1}, \ldots, x_{kn}, \ldots) \in \ell^1 \) \( (k = 1, 2, 3, \ldots) \) satisfy \( x_1 \leq x_2 \leq \ldots \leq x_k \leq \ldots \) and \( M = \sup_{k} \|x_k\| < \infty \). Then, \( x_{1n} \leq x_{2n} \leq \ldots \leq x_{kn} \leq \ldots \leq M \ (n = 1, 2, 3, \ldots) \), so, \( \lim_{k \to \infty} x_{kn} = y_n \ (n = 1, 2, 3, \ldots) \) exist. For any positive integer \( i \), we have \( \sum_{n=1}^{i} |x_{kn}| \leq M \ (k = 1, 2, 3, \ldots) \), so, by letting \( k \to \infty \), we find \( \sum_{n=1}^{i} |y_n| \leq M \). Since \( i \) is arbitrary, it follows that \( \sum_{n=1}^{\infty} |y_n| \leq M < \infty \), and therefore \( y = (y_1, \ldots, y_n, \ldots) \in \ell^1 \). It is clear that \( x_1 \leq x_2 \leq \ldots \leq x_k \leq \ldots \leq y \), consequently, the regularity of \( P \) implies that \( \|x_k - x\| \to 0 \) as \( k \to \infty \) for some \( x \in \ell^1 \). Hence the full regularity of \( P \) is proven.

Now, system (26) can be regarded as a TVP of the form (1), where

\[
f(t, x, y, z) = -e^{-(t+1)s} \left(1 + x_n + \sqrt{x_n + 1 + 2x_{2n} + 1} - e^{-2t/3}y_n^{1/3} - e^{-t/3}1/5\right)
\]

and \( t_m = m, \ I_m = (I_{m1}, \ldots, I_{mn}, \ldots) \) with

\[
I_{mn}(x) = -\frac{1}{2^n + m + 2}(x_n + x_n + 2), \ (m, n = 1, 2, 3, \ldots),
\]

and \( x^* = (1, \ldots, 1/2^n, \ldots) \in E \). Evidently, \( f \in C(J \times P \times P \times P, -P) \) and \( I_m \in C(P, -P) \ (m = 1, 2, 3, \ldots) \). \( (H_1) \) is obviously satisfied since

\[
k^* = \sup_{t \in J} \int_{0}^{t} e^{-(t+1)s} ds = \sup_{t \in J}(t+1)(1 - e^{-(t+1)t}) \leq 1,
\]

\[
h^* = \sup_{t \in J} \int_{0}^{\infty} \frac{ds}{1 + t + s^2} \leq \frac{\pi}{2},
\]

and

\[
\int_{0}^{\infty} \left| \frac{1}{1 + t' + s^2} - \frac{1}{1 + t + s^2} \right| ds = \int_{0}^{\infty} \frac{|t' - t|}{(1 + t' + s^2)(1 + t + s^2)} ds \leq \frac{\pi}{2} |t' - t| \to 0
\]
as \( t' \to t \). It is easy to verify the following scalar inequality:

\[
u^\alpha \leq 1 - \alpha + \alpha u, \ 0 \leq u < \infty, \ 0 < \alpha < 1,
\]

so, for \( t \in J, x, y, z \in P \),

\[
|f_n(t, x, y, z)|
\]
\[ \leq \frac{e^{-t}}{2^n + 3} (1 + x_n + \frac{1}{2} (x_{n+1} + 2x_{2n+1})) + \frac{e^{-2t}}{3^n} \left( \frac{2}{3} + \frac{1}{5} y_n \right) + \frac{e^{-t}}{4^n} \left( \frac{4}{5} + \frac{1}{5} z_{2n} \right) \]

and therefore,

\[ \| f(t, x, y, z) \| = \sum_{n=1}^{\infty} | f_n(t, x, y, z) | \leq e^{-t} \left( \sum_{n=1}^{\infty} \frac{1}{2^n + 3} + \frac{2}{3} \sum_{n=1}^{\infty} \frac{1}{3^n} + \frac{4}{5} \sum_{n=1}^{\infty} \frac{4}{5^n} \right) \]

\[ + e^{-t} \left( \| x \| \sum_{n=1}^{\infty} \frac{1}{2^n + 3} + \frac{1}{3} \| y \| \sum_{n=1}^{\infty} \frac{1}{3^n} + \frac{1}{5} \| z \| \sum_{n=1}^{\infty} \frac{4}{5^n} \right) \]

\[ \leq \frac{87}{120} e^{-t} + e^{-t} \left( \frac{1}{8} \| x \| + \frac{1}{6} \| y \| + \frac{1}{15} \| z \| \right) \]

In addition, we have, for \( x \in P, \)

\[ | I_{mn}(x) | \leq \frac{1}{2^n + m + 1} \| x \| , \]

and so

\[ \| I_m(x) \| = \sum_{n=1}^{\infty} | I_{mn}(x) | \leq \frac{1}{2^n + 1} \| x \| . \]

Hence \( (H_2) \) is satisfied for \( p(t) = (87/120) e^{-t}, \quad q(t) = e^{-t}, \quad a = 1/8, \quad b = 1/6, \]

\( c = 1/15, \quad a_m = 0 \quad \text{and} \quad b_m = 1/2^{m+1} \quad (m = 1, 2, 3, \ldots), \)

and therefore \( p^* = 87/120, \quad q^* = 1, \quad a^* = 0 \quad \text{and} \quad b^* = 1/2. \)

On the other hand, \( (H_3) \) is obviously satisfied, and

\[ r = b^* + (a + bk^* + ch^*) q^* \leq \frac{1}{2} + \left( \frac{1}{8} + \frac{1}{6} + \frac{1}{30} \right) < 1, \]

i.e., (11) holds. Hence the assertion follows from Theorem 1.

\[ \square \]

References