ON THE STABILITY OF STATIONARY SOLUTIONS OF A LINEAR INTEGRO-DIFFERENTIAL EQUATION

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In this paper the following two connected problems are discussed. The problem of the existence of a stationary solution for the abstract equation

$$
\epsilon x''(t) + x'(t) = Ax(t) + \int_{-\infty}^{t} E(t-s) x(s) ds + \xi(t), t \in \mathbb{R}
$$

(1)

containing a small parameter $\epsilon$ in Banach space $B$ is considered. Here $A \in \mathcal{L}(B)$ is a fixed operator, $E \in C([0, +\infty), \mathcal{L}(B))$ and $\xi$ is a stationary process. The asymptotic expansion of the stationary solution for equation (1) in the series on degrees of $\epsilon$ is given.

We have proved also the existence of a stationary with respect to time solution of the boundary value problem in $B$ for a telegraph equation (6) containing the small parameter $\epsilon$. The asymptotic expansion of this solution is also obtained.

Key words: Stationary Solutions, Singular Perturbations, Telegraph Equation, Time-Stationary Solutions, Asymptotic Expansions.

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1. Introduction

Let $(B, \| \cdot \|)$ be a complex Banach space, $\overline{0}$ the zero element in $B$, and $\mathcal{L}(B)$ the Banach space of bounded linear operators on $B$ with the operator norm, denoted also by the symbol $\| \cdot \|$. For a $B$-valued function, continuity and differentiability refer to continuity and differentiability in the $B$-norm. For an $\mathcal{L}(B)$-valued function, continuity is the continuity in the operator norm. For operator $A$, the sets $\sigma(A)$ and $\rho(A)$ are its spectrum and resolvent set, respectively.
In the following, we will consider random elements on the same complete probability space \((\Omega, \mathcal{F}, P)\). The uniqueness of a random process that satisfies an equation, is its uniqueness up to stochastic equivalence. We consider only \(B\)-valued random functions which are continuous with a probability of one. All equalities with random elements in this article are always equalities with a probability one. For a given equation, we consider only solutions which are measurable with respect to the right-hand side random process.

It is well known that the stationary solutions of difference and differential equations are steady with respect to various perturbations of the right-hand side and perturbation of coefficients. For example, see [5]. In the present work, it is shown that stability has a place with respect to perturbations such as degeneracy of the equation.

In the first part of this paper, we consider the following equation

\[
\epsilon x''(t) + x'(t) = Ax(t) + \int_{-\infty}^{t} E(t-s)x(s)ds + \xi(t), t \in \mathbb{R}
\]  

(1)

containing a parameter \(\epsilon\) in \(B\). Here \(A \in \mathcal{L}(B)\) is fixed operator, \(\xi\) is a stationary process in \(B\) and \(E \in C([0, +\infty), \mathcal{L}(B))\) is a function satisfying the condition

\[
a_0 = \int_{0}^{+\infty} \| E(s) \| \, ds < +\infty.
\]

We suppose that the following condition

\[
\sigma(A) \cap i\mathbb{R} = \emptyset
\]  

(2)

holds. Under condition (2) the function

\[
G(t) = \begin{cases} 
- e^{At}P_+, & t < 0; \\
e^{At}P_-, & t > 0
\end{cases}
\]

satisfies the inequality

\[
b_0 = \int_{\mathbb{R}} \| G(s) \| \, ds < +\infty.
\]

Here \(P_-\) and \(P_+\) are Riesz spectral projectors corresponding to the spectral sets \(\sigma(A) \cap \{z \mid \text{Re}z < 0\}\) and \(\sigma(A) \cap \{z \mid \text{Re}z > 0\}\), respectively.

Let \(S\) be the class of all stationary \(B\)-valued processes \(\{\xi(t) : t \in \mathbb{R}\}\) which possess continuous derivatives of all orders on \(\mathbb{R}\) with a probability one and such that, for some numbers \(L = L_\xi > 0\), \(C = C_\xi > 0\), \(\delta > 0\), the following inequalities

\[
\forall n \geq 0: E\left\{ \sup_{0 \leq s \leq \delta} \| \xi^{(n)}(s) \| \right\} \leq LC^n
\]

hold. The notations \(\xi \in S(L, C, \delta)\) and \(\xi \in S\) will be used. Then we have the following result.
Theorem 1: Let $A \in \mathcal{L}(B)$ be an operator satisfying (2). Suppose that $\xi \in S$ and $ab < 1$. Then there exists $\epsilon_0 > 0$ such that for every $\epsilon$ with $|\epsilon| < \epsilon_0$, the equation (1) has a stationary solution $x_\epsilon \in S$, which for every bounded subset $J$ of $\mathbb{R}$, satisfies

$$E\left( \sup_{s \in J} \| x_\epsilon(s) - y_0(s) \| \right) \to 0, \quad \epsilon \to 0,$$

where $y_0$ is a unique stationary solution of the equation

$$x'(t) = Ax(t) + \int_{-\infty}^{t} E(t-s)x(s)ds + \xi(t), \quad t \in \mathbb{R}. \quad (3)$$

The process $x_\epsilon$ is a unique solution of (1) in the class of all stationary connected processes in $S$.

This theorem is proved in Section 2. The method of proof uses a modification of the proof of Theorem 1 in [7] about the stability of stationary solutions for equation (1) with $E \equiv 0$.

Remark 1: The asymptotic expansion for a stationary solution of (1) is obtained.

Remark 2: The assumption (2) is equivalent to the existence of a unique stationary solution $\{x(t) \mid t \in \mathbb{R}\}$ with $E \| x(0) \| < +\infty$ of the equation

$$x'(t) = Ax(t) + \xi(t), \quad t \in \mathbb{R}$$

for every stationary process $\{\xi(t) \mid t \in \mathbb{R}\}$ with $E \| \xi(0) \| < +\infty$, see [3, pp. 201-202].

Remark 3: The general approach to the analysis of the Cauchy problem for deterministic differential equations containing a small parameter leads to the appearance of boundary layer summands in the asymptotic expansion of solution [10]. These summands are absent in the asymptotic expansion of the stationary solution in the considered problem.

Remark 4: The problem of the existence of stationary solutions for difference and differential stochastic equations has been investigated by many authors. See, for example, monograph [1], surveys [2, 4] and article [6].

Corollary 1: Let $A \in \mathcal{L}(B)$ be an operator satisfying (1). Suppose that $\xi \in S$. Then there exists $\epsilon_0 > 0$ such that for every $\epsilon$ with $|\epsilon| < \epsilon_0$, the equation

$$\epsilon x''(t) + x'(t) = Ax(t) + \xi(t), \quad t \in \mathbb{R} \quad (4)$$

has a unique stationary solution $x_\epsilon \in S$, which, for every bounded subset $J$ of $\mathbb{R}$, satisfies

$$E\left( \sup_{s \in J} \| x_\epsilon(s) - x_0(s) \| \right) \to 0, \quad \epsilon \to 0,$$

where $x_0$ is a unique stationary solution of the equation

$$x'(t) = Ax(t) + \xi(t), \quad t \in \mathbb{R}.$$

The second part of this paper deals with the asymptotic expansion of the stationary with respect to time solution of a boundary value problem containing a small parameter. The following definition is necessary.
Definition 1: A $B$-valued random function $u$ defined on $Q: = \mathbb{R} \times [0, \pi]$ is time-stationary if
\[
\forall t \in \mathbb{R} \forall n \in \mathbb{N} \forall \{(t_1, x_1), \ldots, (t_n, x_n)\} \subset Q \forall \{D_1, \ldots, D_n\} \subset \mathcal{B}(B):

P \left( \bigcap_{k=1}^{n} \{ \omega: u(\omega; t_k + t, x_k) \in D_k \} \right) = P \left( \bigcap_{k=1}^{n} \{ \omega: u(\omega; t_k, x_k) \in D_k \} \right),
\]
where $\mathcal{B}(B)$ is the Borel $\sigma$-algebra of $B$.

Let $c: \mathbb{N} \rightarrow c(0) \subset (0, 1, 2} \subset \mathbb{R}$.

Theorem 2: Let $A \in L(B)$ be an operator satisfying the following condition
\[
\{ k^2 + ia \} \subset \rho(A). \tag{5}
\]
Suppose that $g \in C^3_0$ and $\xi \in S$ with a number $\delta > 0$ and $ab < 1$. Then there exists $\epsilon_0 > 0$ such that for every $\epsilon$ with $|\epsilon| < \epsilon_0$, the boundary value problem
\[
\begin{align*}
\epsilon u''_t(t, x; \epsilon) + u'_t(t, x; \epsilon) - u''_{xx}(t, x; \epsilon) \\
&= Au(t, x; \epsilon) + g(x)\xi(t), \quad t \in \mathbb{R}, \quad x \in [0, \pi]
\end{align*}
\]
has a unique time-stationary solution $u(\cdot, \cdot; \epsilon)$ with
\[
E \left( \sup_{0 \leq s \leq \delta, 0 \leq x \leq \pi} \| u(s, x; \epsilon) \| \right) + E \left( \sup_{0 \leq s \leq \delta, 0 \leq x \leq \pi} \| u'_t(s, x; \epsilon) \| \right) < +\infty,
\]
which, for every $t \in \mathbb{R}$, satisfies
\[
E \left( \sup_{t \leq s \leq t + \delta, 0 \leq x \leq \pi} \| u(t, x; \epsilon) - v(t, x) \| \right) \rightarrow 0, \quad \epsilon \rightarrow 0,
\]
where $v$ is the unique time-stationary solution of the following boundary value problem for a heat equation
\[
\begin{align*}
\nu'_t(t, x) - \nu''_{xx}(t, x) &= Av(t, x) + g(x)\xi(t), \quad t \in Q \\
v(t, 0) &= v(t, \pi) = 0, \quad t \in \mathbb{R}
\end{align*}
\]
with
\[
\sup_{0 \leq x \leq \pi} E \| v(0, x) \| < +\infty.
\]

This theorem is proved in Section 3.

Remark 5: Condition (5) is a necessary and sufficient condition of the existence of a time-stationary solution for boundary value problem (7) [8].

Remark 6: Note that, if $\epsilon > 0$, problem (6) is a boundary value problem for a hyperbolic equation and that, if $\epsilon = 0$, we have a boundary value problem for a parabolic equation.

Remark 7: The study of the asymptotic behavior of a solution $u(\cdot, \cdot; \epsilon)$ of the telegraph equation from (6) as $\epsilon \rightarrow 0 +$ has also physical sense [9].
2. Asymptotic Expansion of the Stationary Solution of Equation (1)

In order to prove Theorem 1, a few lemmas will be needed.

**Lemma 1:** Let $A \in \mathcal{L}(B)$ be an operator satisfying (2). Suppose that $\xi \in S$. Then the equation

$$x'(t) = Ax(t) + \xi(t), t \in \mathbb{R}$$

has a unique stationary solution $x \in S$, which can be presented in the form

$$x(t) = \int_{\mathbb{R}} G(t-s)\xi(s)ds = \int_{\mathbb{R}} G(s)\xi(t-s)ds, t \in \mathbb{R}.$$ 

**Proof:** This is the corollary of Theorem 1 in [3, pp. 201-202].

**Lemma 2:** Let $A \in \mathcal{L}(B)$ be an operator satisfying (2). Suppose that $\xi \in S$. The following two statements are equivalent:

(i) A stationary process $x \in S$ is a unique stationary solution of the equation (3).

(ii) A stationary process $x \in S$ is a unique stationary solution of the equation

$$x(t) = \int_{\mathbb{R}} G(t-s)\int_{-\infty}^{s} E(s-u)x(u)du ds + \int_{\mathbb{R}} G(t-s)\xi(s)ds, t \in \mathbb{R}. \quad (8)$$

**Proof:** The result is a consequence of Lemma 1.

**Lemma 3:** Let $A \in \mathcal{L}(B)$ be an operator satisfying (2) and $ab < 1$. Suppose that $\xi$ is a stationary process in $B$, which, for some $\delta > 0$, satisfies

$$E\left( \sup_{0 \leq t \leq \delta} \| \xi(t) \| \right) < +\infty.$$ 

Then the equation (8) has a unique stationary solution $x$, which satisfies

$$E\left( \sup_{0 \leq t \leq \delta} \| x(t) \| \right) < +\infty. \quad (9)$$

**Proof:** Let $S_0$ be the class of all stationary connected $B$-valued processes $x$ which are stationary connected with $\xi$ and, for given $\delta > 0$, satisfy (9). Let us introduce the operator

$$(Tx)(t): = \int_{\mathbb{R}} G(t-s)\int_{-\infty}^{s} E(s-u)x(u)du ds + \int_{\mathbb{R}} G(t-s)\xi(s)ds, t \in \mathbb{R}. \quad (T x)(t): = \int_{\mathbb{R}} G(t-s)\int_{-\infty}^{s} E(s-u)x(u)du ds + \int_{\mathbb{R}} G(t-s)\xi(s)ds, t \in \mathbb{R}.$$ 

Then $Tx \in S_0$ and

$$E\left( \sup_{0 \leq t \leq \delta} \| (Tx)(t) - (Ty)(t) \| \right) \leq abE\left( \sup_{0 \leq t \leq \delta} \| x(t) - y(t) \| \right),$$

therefore $T$ is a continuous operator on $S_0$. Set

$$x_0(t): = \int_{\mathbb{R}} G(t-s)\xi(s)ds, \quad t \in \mathbb{R},$$

then $x_0 \in S_0$ and
Introduce the sequences of random processes

\[ x_0, x_1 = Tx_0, x_2 = Tx_1, \ldots, x_n = Tx_{n-1}, \ldots \]

It is clear that

\[ x_n \in S_0, n \in \mathbb{N}; \quad x_{n+1} = Tx_n, n \geq 0 \]

and for every \( t \in \mathbb{R} \)

\[
E \| x_{n+1}(t) - x_n(t) \| \leq E \left( \sup_{0 \leq s \leq t + \delta} \| x_{n+1}(s) - x_n(s) \| \right) \\
\leq a(ab)^n E \left( \sup_{0 \leq t \leq \delta} \| \xi(t) \| \right), \quad n \geq 0.
\]

Hence, the series

\[ x(t) = x_0(t) + [x_1(t) - x_0(t)] + \ldots + [x_n(t) - x_{n-1}(t)] + \ldots \]

converges with a probability one for every \( t \in \mathbb{R} \) and this convergence is uniform over the bounded subset of \( \mathbb{R} \) with a probability one. By continuity of \( T \) we have \( x = Tx \).

The solution \( x \) of (8) is unique.

**Lemma 4:** Let \( A \in \mathcal{L}(B) \) be an operator satisfying (2) and \( ab < 1 \). Suppose that \( \xi \) is a stationary process in \( B \), which, for some \( \delta > 0 \), satisfies

\[
E \left( \sup_{0 \leq t \leq \delta} \| \xi(t) \| \right) < +\infty.
\]

Then equation (3) has a unique stationary solution \( x \), which satisfies (9).

**Proof:** The result is an immediate consequence of Lemma 2 and Lemma 3.

Set \( c : = (1 - ab)^{-1} \).

**Lemma 5:** Let \( A \in \mathcal{L}(B) \) be an operator satisfying (2) and \( ab < 1 \). Suppose that \( \xi \in S(L, C, \delta) \). The equation (3) has a unique stationary solution \( x \in S(bcL, C, \delta) \).

**Proof:** We return to the proof of Lemma 3 where the stationary solution \( x \) for equation (3) was given. From the inclusion \( \xi \in S(L, C, \delta) \) and representation

\[ x_0(t) = \int_{\mathbb{R}} G(s)\xi(t-s)ds, t \in \mathbb{R} \]

it follows that

\[ x_0^{(k)}(t) = \int_{\mathbb{R}} G(s)\xi^{(k)}(t-s)ds, \quad t \in \mathbb{R} \]

for every \( k \geq 0 \) and \( x_0 \in S(bL, C, \delta) \). For the process \( x_1 - x_0 \), we have

\[ x_1(t) - x_0(t) = \int_{\mathbb{R}} G(u) \int_{0}^{+\infty} E(v)x_0(t-u-v)dudv, t \in \mathbb{R}. \]

Hence, for every \( k \geq 0 \), we have

\[ x_1^{(k)}(t) - x_0^{(k)}(t) = \int_{\mathbb{R}} G(u) \int_{0}^{+\infty} E(v)x_0^{(k)}(t-u-v)dudv, t \in \mathbb{R}, \]
and \((x_1 - x_0) \in S(ab^2L, C, \delta)\). By induction, we find

\[(x_n - x_{n-1}) \in S(b(ab)^nL, C, \delta), n \geq 1.\]

Therefore,

\[x \in S(bcL, C, \delta).\]

Lemma 5 is proved.

**Proof of Theorem 1:** Let \(\xi \in S(L, C, \delta)\). We shall construct the asymptotic expansion for a solution of (1) in the following way. From Lemma 5, equation (3) has a unique stationary solution \(y_0 \in S(bcL, C, \delta)\). Note that \(y'_0 \in S(bcLC^2, C, \delta)\).

Let \(y_1\) be a unique stationary solution for equation

\[y'_1(t) = Ay_1(t) + \int_{-\infty}^{t} E(t-s)y_1(s)ds - y'_0(t), t \in R.\]

This solution exists from Lemma 5 and

\[y_1 \in S(b^2c^2LC^2, C, \delta).\]

By analogy with \(y_1\), let \(y_2\) be a unique stationary solution for equation

\[y'_2(t) = Ay_2(t) + \int_{-\infty}^{t} E(t-s)y_2(s)ds - y'_1(t), t \in R.\]

For this solution, we have \(y_2 \in S(b^3c^3LC^4, C, \delta)\).

If the processes \(y_0, y_1, \ldots, y_{n-1}\) for \(n \geq 1\) are already constructed we will define process \(y_n\) as a unique stationary solution of the equation

\[y'_n(t) = Ay_n(t) + \int_{-\infty}^{t} E(t-s)y_n(s)ds - y'_{n-1}(t), t \in R,\]

which satisfies

\[y_n \in S(b^{n+1}c^{n+1}LC^{2n}, C, \delta).\]

It is clear that the processes \(y_n, n \geq 0\) are stationary connected [3].

Set

\[y_\epsilon(t) = \sum_{n=0}^{\infty} \epsilon^n y_n(t), t \in R.\]  \hspace{1cm} (10)

Since

\[\sum_{n=0}^{\infty} |\epsilon^n| E \left(\sup_{t \leq s \leq t + \delta} \|y_n(s)\|\right) \leq \sum_{n=0}^{\infty} |\epsilon| \frac{b^{n+1}LC^{2n}}{(1-ab)^{n+1}} \leq \frac{2bL}{1-ab}\]

for every \(t \in R\) and \(|\epsilon| \leq \epsilon_0 = (1-ab)/(2bC^2)\), the series for \(y_\epsilon\) converges uniformly on bounded subsets of \(R\) with a probability one. This shows that \(y_\epsilon\) is continuous on \(R\) with a probability one stationary process.
By exactly the same arguments as those used above, we claim that the series for $y'_\epsilon$, $y''_\epsilon$ are also absolutely and uniform convergent on bounded subsets of $\mathbb{R}$ with a probability one and we have

$$
\epsilon y''_\epsilon(t) + y'_\epsilon(t) = \sum_{n=0}^{\infty} \left( \epsilon^{n+1} y''_n(t) + \epsilon^n y'_n(t) \right)
$$

$$
= \sum_{n=0}^{\infty} \left[ \epsilon^{n+1} \left( A y_{n+1}(t) + \int_{-\infty}^{t} E(t-s)y_{n+1}(s)ds - y'_{n+1}(t) \right) + \epsilon^n y'_n(t) \right]
$$

$$
= \sum_{n=0}^{\infty} \epsilon^{n+1} A y_{n+1} + \sum_{n=0}^{\infty} \epsilon^n \int_{-\infty}^{t} E(t-s)y_{n+1}ds - \sum_{m=1}^{\infty} \epsilon^m y'_m + \sum_{n=0}^{\infty} \epsilon^n y'_n(t)
$$

$$
= A \left( \sum_{m=1}^{\infty} \epsilon^m y_m(t) \right) + \int_{-\infty}^{t} E(t-s) \left( \sum_{m=1}^{\infty} \epsilon^m y_m(s) \right) ds + y'_0(t)
$$

$$
= A y'_\epsilon(t) + \int_{-\infty}^{t} E(t-s)y'_\epsilon(s)ds - A y'_0(t) - \int_{-\infty}^{t} E(t-s)y_0(s)ds + y'_0(t)
$$

$$
= A y'_\epsilon(t) + \int_{-\infty}^{t} E(t-s)y'_\epsilon(s)ds + \xi(t), \; t \in \mathbb{R}.
$$

Moreover, for every $t \in \mathbb{R}$, we have

$$
E \left( \sup_{t \leq s \leq t+\delta} \| y'_\epsilon(s) - y_0(s) \| \right) \leq \sum_{m=1}^{\infty} | \epsilon | \frac{m^{-1} b^{m+1} L C^m}{(1-ab)^m+1} \leq \frac{2 b^2 L C^2}{(1-ab)^2},
$$

if $| \epsilon | \leq \epsilon_0$.

To complete the proof of Theorem 1 we need show only the uniqueness. It is sufficient to prove the following fact. If $z$ is stationary connected with the process $x_\epsilon$ solution of (1), which satisfies

$$
E \left( \sup_{0 \leq t \leq \delta} \| z(t) \| \right) < + \infty, \quad E \left( \sup_{0 \leq t \leq \delta} \| z'(t) \| \right) < + \infty,
$$

then $z = x_\epsilon$. We apply Lemma 4 in the following way. The difference $u := x_\epsilon - z$ is a stationary process which satisfies the equation

$$
\epsilon u''(t) + u'(t) = Ax(t) + \int_{-\infty}^{t} E(t-s)u(s)ds, \; t \in \mathbb{R} \tag{11}
$$

and
Let us consider a Banach space $B^2$ of two vectors equipped with term-by-term linear operations and with the norm which is equal to the sum of the norms of the coordinates. Let

$$u(t) = \begin{pmatrix} u' \\ u \end{pmatrix}, \quad \mathbb{A} = \begin{pmatrix} -\epsilon^{-1} & \epsilon^{-1}A \\ \Theta & \Theta \end{pmatrix}, \quad \mathbb{E} = \begin{pmatrix} \Theta & \epsilon^{-1}E \\ \Theta & \Theta \end{pmatrix},$$

where $\Theta$ and $I$ are the zero operator and identity operator on $B$, respectively. Then the following equation in $B^2$

$$u'(t) = \mathbb{A}u(t) + \int_{-\infty}^{t} \mathbb{E}(t-s)uds, \quad t \in \mathbb{R}$$

is equivalent to the equation (11) in $B$. By direct computation we obtain that condition

$$\sigma(\mathbb{A}) \cap i\mathbb{R} = \emptyset$$

is fulfilled if, for every $\alpha \in \mathbb{R}$, an operator $A - (i\alpha + \alpha^2\epsilon)I$ has a bounded inverse. For the justification of this assertion for all small $|\epsilon|$ it suffices to make use of condition (2) and the boundedness of operator $A$. Then, by Lemma 4, the equation (12) has a unique stationary solution and hence $u(t) = 0$, $t \in \mathbb{R}$ with the probability of one.

The proof is complete.

**Remark 8:** Let $B = \mathbb{R}$. It can be proven that the existence of expansion (10) for the solution of equation (4) leads to condition $\xi = C^\infty(\mathbb{R})$.

### 3. Time-Stationary Solutions of the Boundary Value Problem for PDE Containing a Parameter

**Proof of Theorem 2:** Let a process $\xi \in S(L,C,\delta)$ and a function $g \in C^3_0$ be given. Then, one can expand $g$ as

$$g(x) = \sum_{k=1}^{\infty} g_k \sin kx, \quad x \in [0,\pi]; \quad \{g_k: k \geq 1\} \subset C,$$

where the series on the right-hand side is uniformly convergent. Note that

$$g_k = \frac{2}{\pi} \int_{0}^{\pi} g(x) \sin kx \, dx, \quad k \geq 1.$$

Let $k \geq 1$ be fixed. From assumption (5) and Corollary 1, it follows that there is $\xi_k > 0$ such that for every $\epsilon$ with $|\epsilon| < \xi_k$, the equation

$$\epsilon v_k''(t;\epsilon) + v_k'(t;\epsilon) + k^2 v_k(t;\epsilon) = Av_k(t;\epsilon) + g_k(t), \quad t \in \mathbb{R}$$

(13)
has a unique stationary solution \( v_k(\cdot, \epsilon) \) such that

\[
E \left( \sup_{t \in J} \| v_k(t; \epsilon) - v_k(t) \| \right) \to 0, \quad \epsilon \to 0,
\]

where \( v_k \) is a unique stationary solution of the equation

\[
v'_k(t) + k^2 v_k(t) = Av_k(t) + g_k \xi(t), \quad t \in \mathbb{R},
\]

and \( J \) is a bounded subset of \( \mathbb{R} \). Moreover, for every \( t \in \mathbb{R} \), we have

\[
E \left( \sup_{t \leq s \leq t + \delta} \| v_k(s; \epsilon) - v_k(s) \| \right) \leq 2 \left| g_k \right| LL_{1,k},
\]

and

\[
E \left( \sup_{t \leq s \leq t + \delta} \| v_k(s; \epsilon) - v_k(s) \| \right) \leq 2 \left| g_k \right| LL_{2,k}^2 | \epsilon |,
\]

if \( | \epsilon | \leq \epsilon_k \), where

\[
L_{1,k}: = \int_{\mathbb{R}} \| G_k(s) \| \, ds < + \infty
\]

and \( G_k \) is Green’s function for operator \( A - k^2 I \); \( k \geq 1 \). It follows from the properties of \( G_k \) that

\[
L_{1,k}: = \int_{\mathbb{R}} \| G_k(s) \| \, ds < + \infty
\]

and a number \( L \) can be chosen to be independent of \( k \).

Now we shall remark, that by virtue of boundedness of an operator \( A \), the numbers \( \epsilon_k \), \( k \geq 1 \) are identifiable and not depending on \( k \). Really, let \( k_0 \) be the least natural number such that a spectrum of an operator \( A - (\alpha^2 \epsilon - k_0^2) I \) resides in the left half-plane. Then the spectrum of an operator \( A - (\alpha^2 \epsilon - k^2) I \), \( k > k_0 \) also resides in the left half-plane and it is possible to put \( \epsilon_0: = \min \{ \epsilon_1, \epsilon_2, \ldots, \epsilon_{k_0} \} > 0 \). Thus, for every \( \epsilon \), \( | \epsilon | < \epsilon_0 \), all equations (13) have a unique stationary solution.

Let us consider the series

\[
u(t, z; \epsilon): = \sum_{k=1}^{\infty} v_k(t; \epsilon) \sin kx, \quad (t, x) \in Q
\]

for \( | \epsilon | \leq \epsilon_0 \). It follows from (14) and (16) that

\[
\sum_{k=1}^{\infty} E \left( \sup_{t \leq s \leq t + \delta, 0 \leq x \leq \pi} \| v_k(t; \epsilon) \sin kx \| \right) \leq \sum_{k=1}^{\infty} 2 \left| g_k \right| LL_{1,k} < + \infty,
\]

for every \( t \in \mathbb{R} \) and \( | \epsilon | \leq \epsilon_0 \). This implies that the series (17) converges absolutely and uniformly on \([t, t + \delta] \times [0, \pi]\) with the probability one and the random function \( u(\cdot, \cdot; \epsilon) \) is a continuous, time-stationary with respect of time variable, random functions. In addition,

\[
E \left( \sup_{0 \leq s \leq \delta, 0 \leq x \leq \pi} \| u(s, x; \epsilon) \| \right) < + \infty.
\]
Using the above-mentioned reasoning, the following equalities are installed

\[ u'_k(t, x; \epsilon) := \sum_{k=1}^{\infty} v'_k(t; \epsilon) \sin kx, \]

\[ u''_{tt}(t, x; \epsilon) := \sum_{k=1}^{\infty} v''_k(t; \epsilon) \sin kx, \]

\[ u'''_{xx}(t, x; \epsilon) := \sum_{k=1}^{\infty} (-k^2)v_k(t; \epsilon) \sin kx, \]  \hspace{1cm} (18)

for \((t, x) \in Q\) and uniform on \([t, t+\delta] \times [0, \pi]\) convergence with the probability one of an appropriate series for any \(t \in \mathbb{R}\) and \(|\epsilon| \leq \epsilon_0\). We have also

\[ E \left( \sup_{0 \leq s \leq \delta, 0 \leq x \leq \pi} |u'_k(s, x; \epsilon)| \right) < +\infty. \]

From (17), (18), and (13), it follows that

\[ \epsilon u''_{tt}(t, x; \epsilon) + u'_t(t, x; \epsilon) - u'''_{xx}(t, x; \epsilon) \]

\[ = \sum_{k=1}^{\infty} (\epsilon v''_k(t; \epsilon) + v'_k(t; \epsilon) + k^2 v_k(t; \epsilon)) \sin kx \]

\[ = \sum_{k=1}^{\infty} (Av_k(t; \epsilon) + g_k(t; \epsilon)) \sin kx \]

\[ = Au(t, x; \epsilon) + g(x) \xi(t), (t, x) \in Q. \]

Hence, the random function \(u(\cdot, \cdot; \epsilon)\) for \(\epsilon\) with \(|\epsilon| < \epsilon_0\) is a time-stationary solution of (6).

This solution is unique. To see this, we observe that for any \(t \in \mathbb{R}\), the elements \(\{v_k(t; \epsilon)\}\) are Fourier coefficients of \(u(t, \cdot; \epsilon) \in C^2([0, \pi], B)\) which determine \(u(t, \cdot; \epsilon)\) uniquely with the probability one. See, for example [3] for details. By Corollary 1, the solutions of (13) are also determined uniquely with a probability one.

Similarly, by repeating the above arguments, we conclude that random function

\[ v(t, x) := \sum_{k=1}^{\infty} v_k(t) \sin kx, \hspace{1cm} (t, x) \in Q \]

is a unique, stationary with respect to time variable, solution of (7) and

\[ E \left( \sup_{t \leq s \leq t + \delta, 0 \leq x \leq \pi} |v(s, x; \epsilon)| \right) < +\infty \]

for every \(t \in \mathbb{R}\). Note that the random functions \(u(\cdot, \cdot; \epsilon), |\epsilon| \leq \epsilon_0\) and \(v\) are time-stationary connected.

Finally, let us consider the difference \(u(\cdot, \cdot; \epsilon) - v(\cdot, \cdot)\) for \(|\epsilon| < \epsilon_0\). By Corollary 1, the following inequalities

\[ E \left( \sup_{t \leq s \leq t + \delta, 0 \leq x \leq \pi} \|u(t, x; \epsilon) - v(t, x)\| \right) \]
\[ \leq \sum_{k=1}^{\infty} E \left( \sup_{t \leq s \leq t + \delta} \| v(t; \varepsilon) - v_k(t) \| \right) \leq \sum_{k=1}^{\infty} 2L \| g_k \|_{L_{1,k}^2} | \epsilon | \]

hold.

Theorem 2 is proved.

References


