INVARIANT MEASURES FOR CHEBYSHEV MAPS

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Let \( T_\lambda(x) = \cos(\lambda \arccos x), \) \(-1 \leq x \leq 1, \) where \( \lambda > 1 \) is not an integer. For a certain set of \( \lambda \)'s which are irrational, the density of the unique absolutely continuous measure invariant under \( T_\lambda \) is determined exactly. This is accomplished by showing that \( T_\lambda \) is differentially conjugate to a piece-wise linear Markov map whose unique invariant density can be computed as the unique left eigenvector of a matrix.

**Key words:** Chebyshev Map, Absolutely Continuous Invariant Measure, Markov Map.

**AMS subject classifications:** 37A05, 37E05.

1. Chebyshev Maps

The function \( T_n(x) = \cos(n \arccos x), \) \(-1 \leq x \leq 1, \) \( n = 0, 1, 2, \ldots \) defines the \( n \)th Chebyshev polynomial, which is a solution of the differential equation

\[
(1 - x^2)y'' - xy' + n^2y = 0. \tag{1}
\]

Polynomial \( T_n, n \geq 2, \) transforms each of the intervals \([\frac{i}{n-1}, \frac{i+1}{n-1}], i = -(n-1), \ldots, (n-1)-1, \) onto \([-1, 1]. \) It is easy to show (see [1]) that the unique absolutely continuous invariant measure for all \( T_n \)'s is

\[
d\mu(x) = \frac{1}{\pi \sqrt{1 - x^2}} dx.
\]

In this note, we consider the family of Chebyshev maps \( T_\lambda(x) = \cos(\lambda \arccos x), \) \(-1 \leq x \leq 1, \) where \( \lambda > 1 \) is not an integer. \( T_\lambda \) is a solution of the same differential equation (1) with \( n \) replaced by \( \lambda, \) but \( T_\lambda \) is no longer a polynomial and we, therefore, refer to it as a Chebyshev map. The first monotonic branch of \( T_\lambda \) is not onto, but all the others are (see Figure 1).

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The main result of this note characterizes a set of $\lambda$'s, not integers, for which the unique absolutely continuous measure invariant under $T_\lambda$ can be determined exactly. This is accomplished by showing that $T_\lambda$ is differentially conjugate to a piecewise linear Markov map whose unique invariant density can be computed as the unique left eigenvector of a matrix.

A related problem of finding an absolutely continuous invariant measure for a piecewise linear map with two branches ($1 < \lambda < 2$) was considered in [3].

2. Differentiable Conjugacy

For $\lambda > 1$, let $\Lambda_\lambda: [0,1] \to [0,1]$ be a piecewise linear continuous map having slope $1/\lambda$ and defined by joining points $(0,0)$, $(\lambda^{-1},1)$, $(\lambda^{-2},0)$, $(\lambda^{-3},1)$, ... Depending on $\lambda$, either of the two situations shown in Figure 2 can occur, where $[\lambda]$ is the largest integer less than or equal to $\lambda$.

![Figure 2: Two possible shapes of $\Lambda_\lambda$.](image)
Let \( h:[0,1] \rightarrow [-1,1] \) be defined by \( h(x) = \cos(\pi x) \). Both \( h \) and \( h^{-1} \) are continuous differentiable.

**Proposition 1:** For any \( \lambda > 1 \), we have \( \Lambda_{\lambda} = h^{-1} \circ T_{\lambda} \circ h \), i.e., \( T_{\lambda} \) is differentially conjugated to the triangle map \( \Lambda_{\lambda} \).

**Proof:** It is enough to show that \( T_{\lambda} \circ h = h \circ \Lambda_{\lambda} \). Let us consider \( x \) in the intervals \( \left[\frac{k}{\lambda}, \frac{k+1}{\lambda}\right] \), for \( 0 \leq k \leq \lfloor \lambda \rfloor - 1 \) and \( \left[\frac{\lfloor \lambda \rfloor}{\lambda}, 1\right] \), for \( k = \lfloor \lambda \rfloor \). For \( k \) even, we have \( \Lambda_{\lambda}(x) = \lambda(x - \frac{k}{\lambda}) \). Thus

\[
h(\Lambda_{\lambda}(x)) = \cos(\pi \lambda x - \pi k) = \cos(\pi \lambda x).
\]

For \( k \) odd, we have \( \Lambda_{\lambda}(x) = 1 - \lambda(x - \frac{k}{\lambda}) \). Therefore,

\[
h(\Lambda_{\lambda}(x)) = \cos(\pi - \pi \lambda x + \pi k) = \cos(\pi \lambda x).
\]

On the other hand, we always have

\[
T_{\lambda}(h(x)) = \cos(\lambda \arccos(\cos(\pi x))) = \cos(\lambda \pi x).
\]

This proves the claimed conjugation. \( \square \)

**Remark 1:** For \( \lfloor \lambda \rfloor \) even, we have \( \Lambda_{\lambda}(1) = \lambda - \lfloor \lambda \rfloor \). For odd \( \lfloor \lambda \rfloor \), we have \( \Lambda_{\lambda}(1) = 1 - (\lambda - \lfloor \lambda \rfloor) \).

### 3. The Measure Invariant Under a Chebyshev Map

**Definition:** Let \( \tau:[0,1] \rightarrow [0,1] \) be a piecewise monotonic map for which there exist points \( 0 = a_0 < a_1 < \ldots < a_{n-1} < a_n = 1 \) such that for \( i = 0, 1, \ldots, n-1 \), \( \tau \mid (a_i, a_{i+1}) \) is a homeomorphism onto some interval \( (a_j(i), a_k(i)) \). Then \( \tau \) is called Markov with respect to the partition \( (a_1, a_2, \ldots, a_n) \).

The main tool in exploring absolutely continuous invariant measures of \( \tau \) is the Frobenius-Perron operator \( P_\tau \) on the space of Lebesgue integrable functions \( L^1([0,1]) \):

\[
(P_\tau f)(x) = \sum_{i=1}^{n} \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|},
\]

where \( \tau_i^{-1}, i = 1, \ldots, n \), are inverse branches of \( \tau \). A function \( f \) satisfies \( P_\tau f = f \) if and only if \( f \) is the density of an absolutely continuous \( \tau \)-invariant measure. For more detailed information on \( P_\tau \), see [1].

Let \( \mathcal{P} \) be the partition \( (0, \frac{1}{\lambda}, \frac{2}{\lambda}, \ldots, \frac{\lfloor \lambda \rfloor}{\lambda}, 1) \). Clearly, if \( \lambda \) is a positive integer, \( \Lambda_{\lambda} \) is a Markov map with respect to \( \mathcal{P} \). Below we characterize the set of \( \lambda \)'s, not integers, for which \( \Lambda_{\lambda} \) is a Markov map with respect to \( \mathcal{P} \). It is easy to see that \( \Lambda_{\lambda} \) is Markov if and only if \( \Lambda_{\lambda}(1) = \frac{m}{\lambda} \), where \( m \) is an integer satisfying \( 1 \leq m \leq \lfloor \lambda \rfloor \).

**Proposition 2:** If \( \lambda \) is not an integer, then \( \Lambda_{\lambda} \) is Markov with respect to \( \mathcal{P} \) if and only if:

**Case 1:**

\[
\lambda = n + \sqrt{n^2 + m}, \text{ for } \lfloor \lambda \rfloor = 2n \text{ and } 1 \leq m \leq 2n;
\]
Case 2:

\[ \lambda = (n + 1) + \sqrt{(n + 1)^2 - m}, \text{ for } [\lambda] = 2n + 1 \text{ and } 1 \leq m \leq 2n. \]  

(3)

In both cases, \( \lambda \) is irrational. In (3), \( m \) cannot be equal to \( 2n + 1 \) since then \( \lambda \) would be an integer.

**Proof:** We use the values of \( \Lambda_{\lambda}(1) \) from Remark 1. In Case 1, we have \( \Lambda_{\lambda}(1) = \lambda - 2n = \frac{m}{\lambda} \), or \( \lambda^2 - 2\lambda n - m = 0 \), whose positive solution is given in (2). In Case 2, we have \( \Lambda_{\lambda}(1) = 1 - (\lambda - 2n - 1) = \frac{m}{\lambda} \), or \( \lambda^2 - 2\lambda(n + 1) + m = 0 \). The only positive solution is given in (3). \( \square \)

**Remark 2:** There are no non-integer solutions to (3) for \([\lambda] = 1\). Thus, the smallest \([\lambda] \) we can actually consider is \([\lambda] = 2\).

When \( \Lambda_{\lambda} \) is Markov with respect to \( \Phi \), then its Frobenius-Perron operator restricted to the piecewise constant functions on the partition \( \Phi \), can be represented by the \( \lambda_1 \times \lambda_1 \) matrix \( M \), where \( \lambda_1 = [\lambda] + 1 \). For \([\lambda] = 2n \) and \( \lambda - [\lambda] = \frac{m}{\lambda} \) (Case 1), we have

\[
M = \begin{bmatrix}
\frac{1}{\lambda} & \frac{1}{\lambda} & \cdots & \cdots & \frac{1}{\lambda} & \frac{1}{\lambda} \\
\frac{1}{\lambda} & \frac{1}{\lambda} & \cdots & \cdots & \frac{1}{\lambda} & \frac{1}{\lambda} \\
& & & & \vdots & \vdots \\
\frac{1}{\lambda} & \frac{1}{\lambda} & \cdots & \cdots & \frac{1}{\lambda} & \frac{1}{\lambda} \\
\frac{1}{\lambda} & \frac{1}{\lambda} & \cdots & \frac{1}{\lambda} & 0 & \cdots & 0 
\end{bmatrix},
\]

where the upper \( \lambda_1 - 1 \) rows consists of \( \frac{1}{\lambda} \)'s and there are exactly \( m \) \( \frac{1}{\lambda} \)'s at the beginning of the last row. If \([\lambda] = 2n + 1 \) and \( 1 - (\lambda - [\lambda]) = \frac{m}{\lambda} \) (Case 2), the matrix \( M \) is similar, except that the last row starts with \( m \) zeros and ends with \( (\lambda_1 - m) \frac{1}{\lambda} \)'s.

Let \( f = (f_1, f_2, \ldots, f_{\lambda_1}) \) represent a piecewise constant function on the partition \((0, 1, \frac{1}{\lambda}, \ldots, \frac{[\lambda]}{\lambda}, 1)\). We consider the equation \( fM = f \) with the normalizing condition which makes \( f \) a density of a probability measure:

\[
\sum_{i=1}^{[\lambda]} f_i \frac{1}{\lambda} + f_{\lambda_1} (1 - \frac{[\lambda]}{\lambda}) = 1.
\]

(4)

**Case 1:** \([\lambda] = 2n \) and \( \lambda - [\lambda] = \frac{m}{\lambda} \). Then, \( fM = f \) reduces to

\[
\begin{align*}
f_1 &= f_2 = \ldots = f_m \\
f_{m+1} &= f_{m+2} = \ldots = f_{\lambda_1} \\
f_1 &= \frac{m}{\lambda} f_1 + \frac{\lambda - m}{\lambda} f_{\lambda_1}
\end{align*}
\]
Since $\lambda = [\lambda] = \frac{m}{\lambda}$, the last two equations coincide and we obtain:

$$f_1 = \frac{\lambda - m}{\lambda - m} f_{\lambda_1}.$$  

Substituting into (4), we get

$$\frac{m}{\lambda} \frac{\lambda - m}{\lambda - m} f_{\lambda_1} + \frac{[\lambda] - m}{\lambda - m} f_{\lambda_1} + (1 - \frac{[\lambda]}{\lambda}) f_{\lambda_1} = 1,$$

which gives

$$f_{\lambda_1} = \frac{\lambda(\lambda - m)}{m(\lambda - m) + (\lambda - m)^2} \text{ and } f_1 = \frac{\lambda(\lambda - m)}{m(\lambda - m) + (\lambda - m)^2}.$$  

(5)

**Case 2:** $[\lambda] = 2n + 1$ and $1 - (\lambda - [\lambda]) = \frac{m}{\lambda}$. Considerations analogous to that of Case 1 lead to:

$$f_{\lambda_1} = \frac{\lambda([\lambda] - m)}{m([\lambda] - m) + (\lambda - m)^2} \text{ and } f_1 = \frac{\lambda([\lambda] - m)}{m([\lambda] - m) + (\lambda - m)^2}.$$  

(6)

We have proved the following theorem.

**Theorem 1:** If $\lambda > 1$ is such that $\Lambda_\lambda$ is a Markov map with respect to $\mathcal{P}$, then the unique invariant density of $\Lambda_\lambda$ is:

- for $[\lambda] = 2n$: 
  $$f_{\lambda_1}(x) = \begin{cases} 
  \frac{\lambda(\lambda - m)}{m(\lambda - m) + (\lambda - m)^2} & \text{for } 0 \leq x < \frac{m}{\lambda}, \\
  \frac{\lambda(\lambda - m)}{m(\lambda - m) + (\lambda - m)^2} & \text{for } \frac{m}{\lambda} \leq x \leq 1;
  \end{cases}$$

- and for $[\lambda] = 2n + 1$: 
  $$f_{\lambda}(x) = \begin{cases} 
  \frac{\lambda([\lambda] - m)}{m([\lambda] - m) + (\lambda - m)^2} & \text{for } 0 \leq x < \frac{m}{\lambda}, \\
  \frac{\lambda(\lambda - m)}{m([\lambda] - m) + (\lambda - m)^2} & \text{for } \frac{m}{\lambda} \leq x \leq 1.
  \end{cases}$$

Now, we can use the differentiable conjugacy $h$ of Proposition 1 to find invariant densities for Markov Chebyshev maps. By Proposition 2 of [2], the $T_\lambda$-invariant density is

$$F_\lambda(x) = f_\lambda(h^{-1}(x)) | (h^{-1})' | = f_\lambda\left(\frac{1}{\pi} \arccos x\right) \frac{1}{\pi \sqrt{1 - x^2}}.$$  

Thus the following theorem holds.

**Theorem 2:** If $\lambda > 1$ is such that $T_\lambda$ is a Markov map with respect to $\mathcal{P}$, then the unique invariant density of $T_\lambda$ is
\[ F_\lambda(x) = \begin{cases} 
\frac{f_1}{\pi \sqrt{1 - x^2}} & \text{for } 0 \leq \frac{1}{\pi} \arccos x < \frac{m}{\lambda}, \\
\frac{f_{\lambda_1}}{\pi \sqrt{1 - x^2}} & \text{for } \frac{m}{\lambda} \leq \frac{1}{\pi} \arccos x \leq 1;
\end{cases} \]

where \( m \) satisfies \( \lambda - [\lambda] = \frac{m}{\lambda} \) for \( [\lambda] = 2n \) or \( 1 - (\lambda - [\lambda]) = \frac{m}{\lambda} \) for \( [\lambda] = 2n + 1 \) and constants \( f_1, f_{\lambda_1} \) are given by formulas (5) or (6), respectively.

Remark 3: For a non-integer \( \lambda \), we always have \( |T_\lambda'(1-t)| = O(\sqrt{1-t})^{-1} \), as \( t \to 0^+ \), which explains the lack of singularity of \( F_\lambda \) at \( \frac{m}{\lambda} \).

4. Examples

Example 1: Using Maple V, release 5, we have calculated some values of \( \lambda \) and corresponding values of \( f_1 \) and \( f_{\lambda_1} \). They are presented in the tables below.

**Case 1:**

<table>
<thead>
<tr>
<th>((n, m))</th>
<th>(\lambda)</th>
<th>(f_1)</th>
<th>(f_{\lambda_1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)</td>
<td>2.414213562</td>
<td>1.207106781</td>
<td>.8535533903</td>
</tr>
<tr>
<td>(5, 7)</td>
<td>10.65685425</td>
<td>1.030330086</td>
<td>.9419417381</td>
</tr>
<tr>
<td>(100, 107)</td>
<td>200.5335765</td>
<td>1.002319740</td>
<td>.9973462776</td>
</tr>
</tbody>
</table>

**Case 2:**

<table>
<thead>
<tr>
<th>((n, m))</th>
<th>(\lambda)</th>
<th>(f_1)</th>
<th>(f_{\lambda_1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)</td>
<td>3.732050808</td>
<td>.7886751346</td>
<td>1.077350270</td>
</tr>
<tr>
<td>(5, 7)</td>
<td>11.38516481</td>
<td>.9642383460</td>
<td>1.057086015</td>
</tr>
<tr>
<td>(100, 107)</td>
<td>201.4689006</td>
<td>.9976664394</td>
<td>1.002643103</td>
</tr>
</tbody>
</table>

For some \( \lambda > 1 \), the map \( \Lambda_\lambda \) may not be Markov with respect to the partition \( \mathcal{P} \) but will be Markov with respect to some finer partition. In all such cases, it is possible to find an invariant density of \( \Lambda_\lambda \) and then use it to find the invariant density of the Chebyshev map \( T_\lambda \). Below, we present two simple examples of such situations.

**Example 2:** Let us look for \( \lambda \) such that \( [\lambda] = 1 \) and \( \Lambda_\lambda(1) = \frac{1}{\lambda^2} \), see Figure 3a). Then \( \lambda(1 - \frac{1}{\lambda}) = 1 - \frac{1}{\lambda^2} \), or \( \lambda^2 - 2\lambda^2 + 1 = 0 \). The non-integer positive solution is \( \lambda = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618 \). It is easy to see that \( \Lambda_\lambda \) is Markov with respect to the partition \( \mathcal{Q} = (0, \frac{1}{\lambda^2}, \frac{1}{\lambda}, 1) \) and the corresponding Frobenius-Perron matrix is
Its left invariant vector is \( f = (f_1, f_2, f_3) = (0, 1, \lambda) \) and after the normalization the \( \Lambda_\lambda \)-invariant density is \( f = (0, \frac{\lambda+1}{2\lambda-1}, \frac{2\lambda+1}{2\lambda-2}) \approx (0, 1.171, 1.894) \).

\[
\mathcal{M} = \begin{bmatrix}
\frac{1}{\lambda} & \frac{1}{\lambda} & 0 \\
0 & 0 & \frac{1}{\lambda} \\
\frac{1}{\lambda} & \frac{1}{\lambda} & \frac{1}{\lambda}
\end{bmatrix}.
\]

Example 3: Let us look for \( \lambda \) such that \( [\lambda] = 2 \) and \( \Lambda_\lambda(1) = \frac{1}{2} \), see Figure 3b). Then, \( \lambda(1 - \frac{1}{\lambda^2}) = \frac{1}{\lambda^2} \), or \( \lambda^2 - 2\lambda^2 - 1 = 0 \). The only real solution is \( \lambda \approx 2.206 \). It is easy to see that \( \Lambda_\lambda \) is Markov with respect to the partition \( Q = (0, \frac{1}{\lambda^2}, \frac{1}{\lambda}, 1) \). The corresponding Frobenius-Perron matrix is

\[
\mathcal{M} = \begin{bmatrix}
\frac{1}{\lambda} & \frac{1}{\lambda} & 0 & 0 \\
0 & 0 & \frac{1}{\lambda} & \frac{1}{\lambda} \\
\frac{1}{\lambda} & \frac{1}{\lambda} & \frac{1}{\lambda} & \frac{1}{\lambda} \\
\frac{1}{\lambda} & 0 & 0 & 0
\end{bmatrix}.
\]

Its left invariant vector is \( f = (f_1, f_2, f_3, f_4) = (\lambda^2 - \lambda - 1, \lambda - 1, 1, 1) \) and after normalization, the \( \Lambda_\lambda \)-invariant density is \( f = \frac{1}{3\lambda - 4}(\lambda^2 - \lambda + 1, \lambda^2 - \lambda, \lambda, \lambda) \approx (1.398, 1.016, .843, .843) \).

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References


