MULTIOBJECTIVE DUALITY WITH $\rho - (\eta, \theta)$-INVEXITY

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Under $\rho - (\eta, \theta)$-invexity assumptions on the functions involved, weak, strong, and converse duality theorems are proved to relate properly efficient solutions of the primal and dual problems for a multiobjective programming problem.

1. Introduction

The notion of $\eta$-invexity was originally introduced by Hanson [6] who showed that, for a nonlinear programming problem whose objective and constrained functions are $\eta$-invex (all with respect to the same $\eta$), the Karush-Kuhn-Tucker necessary optimality conditions are also sufficient. The term invex (for invariant convex) was coined by Craven [2] to signify the fact that the invexity property of a function is invariant under certain types of coordinate transformations. Evidently, convex functions, in general, do not possess this property.

Various properties, extensions, and applications of $\eta$-invex functions are discussed in [1, 2, 7] among others. Later the concept of $\rho - (\eta, \theta)$-invexity has been introduced by Zalmai [11], which generalizes the notion of invexity.


In the present paper, duality results (weak, strong, and converse duality theorems) are proved for multiobjective programming problem under $\rho - (\eta, \theta)$-invexity assumptions on the functions involved.
2. Preliminaries

In [5], Geoffrion considered the following multiobjective programming problem:

\[(VP)\]

\[
\text{Minimize } f(x), \\
\text{subject to } x \in X \tag{2.1}
\]

where \( f : X \to \mathbb{R}^p \) and \( X \) is an open subset of \( \mathbb{R}^n \), and minimization means obtaining efficient solutions in the following sense.

A point \( \bar{x} \in X \) is an efficient solution for \( f = (f_1, f_2, \ldots, f_p) \) if there is no \( x \in X \) such that \( f(x) \leq f(\bar{x}) \) and \( f(x) \neq f(\bar{x}) \).

An efficient solution \( \bar{x} \in X \), for which there exists a scalar \( M > 0 \) such that for each \( i = 1, 2, \ldots, p \), we have

\[
\frac{f_i(x) - f_i(\bar{x})}{f_j(\bar{x}) - f_j(x)} \leq M \tag{2.2}
\]

for some \( j \) such that \( f_j(x) > f_j(\bar{x}) \) and \( f_i(x) < f_i(\bar{x}) \) for \( x \in X \), is called a properly efficient solution of \( (VP) \) (see Geoffrion [5]). Geoffrion [5] proved the following results.

**Lemma 2.1.** If for fixed \( 0 < \lambda \in \mathbb{R}^p \), \( \bar{x} \) is an optimal solution of the parametric programming problem \( (P_{\lambda}) \)

\[
\text{Minimize } \lambda^T f(x), \\
\text{subject to } x \in X \tag{2.3}
\]

where \( 0 < \lambda \in \mathbb{R}^p \) is a vector, then \( \bar{x} \) is a properly efficient solution of the multiobjective problem \( (VP) \).

**Lemma 2.2.** If \( X \) is convex and \( f_i, i = 1, 2, \ldots, p \), are all convex functions, then \( \bar{x} \) is a properly efficient solution for \( (VP) \) if and only if \( \bar{x} \) is an optimal solution of the parametric programming problem \( (P_{\lambda}) \) for some \( \lambda \in \mathbb{R}^p \) with strictly positive components.

Recently Hanson and Mond [7] have generalized Lemma 2.2 to invex functions. They have shown that if \( f_i, i = 1, 2, \ldots, p \), are differentiable invex functions with respect to the same \( \eta(x, u) \) (n-dimensional) for \( x \in X, u \in X \), then \( \bar{x} \) is properly efficient solution in the multiobjective programming problem \( (VP) \) if and only if \( \bar{x} \) is an optimal solution of the parametric programming problem \( (P_{\lambda}) \) for some \( \lambda \in \mathbb{R}^p \) with strictly positive components.

A differentiable function \( f(x) \) is said to be invex (see [1, 6]) at a point \( u \in X \) over \( X \) if there exists \( \eta(x, u) \in \mathbb{R}^n \) such that

\[
f(x) - f(u) \geq \eta(x, u)^T \nabla f(u) \quad \forall x \in X. \tag{2.4}
\]

Here \( \nabla f \) denotes the gradient of \( f \) and the subscript “\( T \)” stands for the transpose of a vector. Later the concept of \( \rho - (\eta, \theta) \)-invexity has been studied by Zalmai (see [11]), which generalizes the notion of invexity function.
Definition 2.3. A differentiable function \( h: X \rightarrow \mathbb{R} \) is called \( \rho - (\eta,\theta) \)-invex with respect to vector-valued functions \( \eta \) and \( \theta \) if there exists some real number \( \rho \) such that for all \( x,u \in X \),

\[
 h(x) - h(u) \geq \eta^T (x,u) \nabla h(u) + \rho \|\theta(x,u)\|^2. \tag{2.5}
\]

If \( \rho > 0 \), then \( f(x) \) is called strongly \( \rho - (\eta,\theta) \)-invex, if \( \rho = 0 \), we obviously get the usual notion of invexity, and if \( \rho < 0 \), then \( f(x) \) is called weakly \( \rho - (\eta,\theta) \)-invex. It is clear that

\[
\text{strongly }\rho - (\eta,\theta) \text{-invex } \Rightarrow \text{invex } \Rightarrow \text{weakly }\rho - (\eta,\theta) \text{-invex}. \tag{2.6}
\]

Definition 2.4. \( h \) is said to be \( \rho - (\eta,\theta) \)-pseudoinvex with respect to vector-valued functions \( \eta \) and \( \theta \), if there exists some real number \( \rho \) such that for all \( x,u \in X \)

\[
\eta^T (x,u) \nabla h(u) \geq -\rho \|\theta(x,u)\|^2 \Rightarrow h(x) \geq h(u). \tag{2.7}
\]

Definition 2.5. \( h \) is said to be \( \rho - (\eta,\theta) \)-quasi-invex with respect to vector-valued functions \( \eta \) and \( \theta \), if there exists some real number \( \rho \) such that for all \( x,u \in X \),

\[
h(x) \leq h(u) \Rightarrow \eta^T (x,u) \nabla h(u) \leq -\rho \|\theta(x,u)\|^2. \tag{2.8}
\]

3. Duality

Consider the following multiobjective programming problems:

(PVP) Minimize \( x \in X \) \( f(x) \) subject to \( g(x) \leq 0 \),
(DVP) Maximize \( x \in X,\lambda,\gamma \) \( f(u) + y^T g(u) e \) subject to \( \nabla \lambda^T f(u) + \nabla y^T g(u) = 0 \), \( y \geq 0 \), \( \lambda \geq 0 \), \( \lambda^T e = 1 \),

where \( f: X \rightarrow \mathbb{R}^p \), \( g: X \rightarrow \mathbb{R}^m \), \( y \in \mathbb{R}^m \), \( \lambda \in \mathbb{R}^p \), and \( e \) is \( p \)-tuple of 1’s. Thus parametric (scalar) programming problems corresponding (PVP) and (DVP) are

(PC\( \lambda \)) Minimize \( x \in X \) \( \lambda^T f(x) \) subject to \( g(x) \leq 0 \),
(DC\( \lambda \)) Maximize \( x \in X,\lambda,\gamma \) \( \lambda^T f(u) + y^T g(u) \) subject to \( \nabla \lambda^T f(u) + \nabla y^T g(u) = 0 \), \( y \geq 0 \), \( \lambda \geq 0 \), respectively. In programming problems (PC\( \lambda \)) and (DC\( \lambda \)), the vector \( 0 < \lambda \in \mathbb{R}^p \) is predetermined. In [4], Egoddo and Hanson proved weak and strong duality theorems between (PVP) and (DVP) for invex functions. We prove the following duality theorems.

Theorem 3.1 (weak duality). Let \( S \) be the feasible region for the primal problem (PVP), that is, \( S = \{ x \in X, g(x) \leq 0 \} \). Let \( (u,\lambda,\gamma) \) be a feasible point in the dual problem (DVP) such that \( \lambda^T f \) is \( \rho - (\eta,\theta) \)-invex at \( u \in S \) and \( y^T g \) is \( \rho_1 - (\eta,\theta) \)-invex at \( u \in S \) with \( \rho + \rho_1 \geq 0 \). Then

\[
\lambda^T f(x) \geq \lambda^T f(u) + y^T g(u) \quad \forall x \in S. \tag{3.1}
\]

**Proof.** Since \( \lambda^T f \) is \( \rho - (\eta,\theta) \)-invex at \( u \) over \( S \) and \( y^T g \) is \( \rho_1 - (\eta,\theta) \)-invex at \( u \) over \( S \), we have

\[
\lambda^T f(x) - \lambda^T f(u) \geq \eta^T (x,u) \nabla (\lambda^T f(u)) + \rho \|\theta(x,u)\|^2, \tag{3.2}
\]

\[
y^T g(x) - y^T g(u) \geq \eta^T (x,u) \nabla (y^T g(u)) + \rho_1 \|\theta(x,u)\|^2, \tag{3.2}
\]

\[
\lambda^T f(x) - \lambda^T f(u) \geq \eta^T (x,u) \nabla (\lambda^T f(u)) + \rho \|\theta(x,u)\|^2, \tag{3.2}
\]

\[
y^T g(x) - y^T g(u) \geq \eta^T (x,u) \nabla (y^T g(u)) + \rho_1 \|\theta(x,u)\|^2, \tag{3.2}
\]

\[
\lambda^T f(x) - \lambda^T f(u) \geq \eta^T (x,u) \nabla (\lambda^T f(u)) + \rho \|\theta(x,u)\|^2, \tag{3.2}
\]

\[
y^T g(x) - y^T g(u) \geq \eta^T (x,u) \nabla (y^T g(u)) + \rho_1 \|\theta(x,u)\|^2, \tag{3.2}
\]
Since a constraint qualification [8] (also see Ben-Israel and Mond [1]) is satisfying problem (dual programming problem (3.4)),

\[
\lambda^T f(x) \geq \lambda^T f(u) + y^T g(u) + (\rho + \rho_1) \|	heta(x,u)\|^2.
\]

(3.3)

Now since \((u,\lambda,\theta)\) is feasible in \((DVP)\), we have \(\eta^T(x,u)[\nabla(\lambda^T f(u)) + y^T g(u)] = 0\) and \(y^T g(x) \leq 0\) for all \(x \in S\), therefore inequality (3.3) reduces to

\[
\lambda^T f(x) \geq \lambda^T f(u) + y^T g(u) + (\rho + \rho_1) \|	heta(x,u)\|^2.
\]

(3.4)

Again \(\rho + \rho_1 \geq 0\), so

\[
\lambda^T f(x) \geq \lambda^T f(u) + y^T g(u) \quad \forall x \in S.
\]

(3.5)

**Theorem 3.2** (strong duality). Let \(\bar{x}\) be a properly efficient solution of the multiobjective programming problem \((PVP)\) at which a constraint qualification is satisfied. Then there exists \((\bar{\lambda}, \bar{\theta})\) such that \((\bar{x}, \bar{\lambda}, \bar{\theta})\) is a feasible solution in the programming problem \((DVP)\) and \(y^T g(\bar{x}) = 0\). If also for each feasible \((u,\bar{\lambda},\bar{\theta})\) in the dual programming problem \((DVP)\), \(\bar{\lambda}^T f\) is \(\rho - (\eta,\theta)\)-invex and \(y^T g\) is \(\rho_1 - (\eta,\theta)\)-invex at \(u\) over the primal feasible region \(S = \{x \mid x \in X : g(x) \leq 0\}\) with \(\rho + \rho_1 \geq 0\), then \((\bar{x}, \bar{\lambda}, \bar{\theta})\) is a properly efficient solution of the dual programming problem \((DVP)\) and the objective values are equal.

**Proof.** Since a constraint qualification [8] (also see Ben-Israel and Mond [1]) is satisfied at \(\bar{x}\), then, from Kuhn-Tucker necessary conditions [5], there exists \((\bar{\lambda}, \bar{\theta})\) such that \((\bar{x}, \bar{\lambda}, \bar{\theta})\) is a feasible solution in the programming \((DVP)\) and \(y^T g(\bar{x}) = 0\). Hence, objective function values are equal. Also since for each feasible \((u,\bar{\lambda},\bar{\theta})\) in the dual programming problem \((DVP)\), \(\bar{\lambda}^T f\) is \(\rho - (\eta,\theta)\)-invex and \(y^T g\) is \(\rho_1 - (\eta,\theta)\)-invex at \(u\) over \(S\), then

\[
\begin{align*}
\bar{\lambda}^T f(x) - \bar{\lambda}^T f(u) & \geq \eta^T(x,u) \nabla(\bar{\lambda}^T f(u)) + \rho \|	heta(x,u)\|^2, \\
y^T g(x) - y^T g(u) & \geq \eta^T(x,u) \nabla(y^T g(u)) + \rho_1 \|	heta(x,u)\|^2,
\end{align*}
\]

(3.6)

that is,

\[
\{\bar{\lambda}^T f(x) + y^T g(x) \geq \bar{\lambda}^T f(u) + y^T g(u) + \eta^T(x,u)[\nabla(\bar{\lambda}^T f(u)) + \nabla y^T g(u)] + (\rho + \rho_1) \|	heta(x,u)\|^2}\}.
\]

(3.7)

But \([\nabla(\bar{\lambda}^T f(u)) + \nabla y^T g(u)] = 0\), therefore

\[
\bar{\lambda}^T f(x) + y^T g(x) \geq \bar{\lambda}^T f(u) + y^T g(u) + (\rho + \rho_1) \|	heta(x,u)\|^2.
\]

(3.8)

Since \(y \geq 0\) and \(g(x) \leq 0\), for all \(x \in X\), we have \(y^T g(x) \leq 0\), for all \(x \in S\). Hence, for all feasible \((u,\bar{\lambda},\bar{\theta})\) in the dual programming problem \((DVP)\), we have

\[
\bar{\lambda}^T f(x) \geq \bar{\lambda}^T f(u) + y^T g(u) + (\rho + \rho_1) \|	heta(x,u)\|^2 \quad \forall x \in S.
\]

(3.9)
As $\rho + \rho_1 \geq 0$, so
\[
\lambda^T f(x) \geq \lambda^T f(\bar{u}) + y^T g(\bar{u}) \quad \forall x \in S.
\] (3.10)

By assumption, $\bar{x}$ is feasible in the primal ($PVP$) and we have shown that $(\bar{x}, \bar{\lambda}, \bar{y})$ in the dual ($DVP$) we have
\[
\lambda^T f(\bar{u}) + y^T g(\bar{u}) \leq \lambda^T f(\bar{x}).
\] (3.11)

Now (3.11) implies that for $\bar{\lambda}$, $(\bar{x}, \bar{y})$ solves the parametric problem ($DC_\lambda$). Since $\bar{\lambda} > 0$, from Geoffrion's [5] sufficient conditions, we conclude that $(\bar{x}, \bar{\lambda}, \bar{y})$ is properly efficient for the problem ($DVP$).

\[
\square
\]

4. Converse duality

In this section, we study the converse duality theorem.

**Theorem 4.1** (converse duality). Let $(\bar{x}, \bar{\lambda}, \bar{y})$ be a feasible solution for the dual problem ($DVP$) such that $\lambda f$ is $\rho - (\eta, \theta)$-invex at $\bar{u}$ and $\bar{y}^T g$ is $\rho_1 - (\eta, \theta)$-invex at $\bar{u}$ over the primal feasible region $S = \{x \mid x \in X : g(x) \leq 0\}$ with $\rho + \rho_1 \geq 0$. Suppose there exists $\bar{x} \in S$ such that $\lambda^T f(\bar{x}) = \lambda^T f(\bar{u}) + \bar{y}^T g(\bar{u})$. Then $\bar{x}$ is properly efficient solution of ($PVP$). If also for each feasible $(u, \bar{\lambda}, y)$ in the dual programming problem ($DVP$), $\lambda^T f$ is $\rho - (\eta, \theta)$-invex at $u$ over $S$ and $y^T g$ is $\rho_1 - (\eta, \theta)$-invex at $u$ over the primal feasible region $S = \{x \mid x \in X : g(x) \leq 0\}$ with $\rho + \rho_1 \geq 0$, then $(\bar{x}, \bar{\lambda}, \bar{y})$ is also properly efficient of the dual multiobjective programming problem ($DVP$).

**Proof.** By Theorem 3.1 we have
\[
\lambda^T f(x) \geq \lambda^T f(\bar{u}) + \bar{y}^T g(\bar{u}) \quad \forall x \in S.
\] (4.1)

Now since there exists $\bar{x} \in S$ such that
\[
\lambda^T f(\bar{x}) = \lambda^T f(\bar{u}) + \bar{y}^T g(\bar{u}),
\] (4.2)

so
\[
\lambda^T f(x) \geq \lambda^T f(\bar{x}) \quad \forall x \in S.
\] (4.3)

Since $x \in S$, (4.3) implies that for $\lambda$, $\bar{x}$ is an optimal solution of the parametric programming ($PC_\lambda$). As $\lambda > 0$, from Geoffrion's sufficient conditions, $\bar{x}$ is properly efficient solution of the primal multiobjective programming problem ($PVP$).

Again because $\lambda^T f$ is $\rho - (\eta, \theta)$-invex and $\bar{y}^T g$ is $\rho_1 - (\eta, \theta)$-invex at $u$ over $S$ with $\rho + \rho_1 \geq 0$, we have, for each feasible $(u, \bar{\lambda}, y)$ in the dual programming problem ($DVP$),
\[
\lambda^T f(x) \geq \lambda^T f(u) + y^T g(u), \quad \forall x \in S,
\] (4.4)

and because $\bar{x} \in S$ and $\lambda^T f(\bar{x}) = \lambda^T f(\bar{u}) + \bar{y}^T g(\bar{u})$, it follows that
\[
\lambda^T f(u) + y^T f(u) \leq \lambda^T f(\bar{x}) = \lambda^T f(\bar{u}) + \bar{y}^T g(\bar{u}),
\] (4.5)
and (4.5) holds for all feasible \((u, \bar{\lambda}, y)\) in the dual programming problem \((DVP)\). This implies that for \(\bar{\lambda}, (\bar{u}, \bar{y})\) is an optimal solution of the parametric programming problem \((DC_\lambda)\). Since \(\bar{\lambda} > 0\), it now follows from Geoffrion’s [5] sufficient condition that \((\bar{x}, \bar{\lambda}, \bar{y})\) is a properly efficient solution of the dual programming problem \((DVP)\). □

Concluding remark. As \(\rho - (\eta, \theta)\)-invexity/pseudoinvexity is a generalization of invexity, multiobjective variational problem and multiobjective control problem under \(\rho - (\eta, \theta)\)-invexity will orient future research of the authors.

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References


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