Research Article

Some Limit Properties of the Harmonic Mean of Transition Probabilities for Markov Chains in Markovian Environments Indexed by Cayley’s Trees

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We prove some limit properties of the harmonic mean of a random transition probability for finite Markov chains indexed by a homogeneous tree in a nonhomogeneous Markovian environment with finite state space. In particular, we extend the method to study the tree-indexed processes in deterministic environments to the case of random environments.

1. Introduction

A tree $T$ is a graph which is connected and doesn’t contain any circuits. Given any two vertices $\alpha \neq \beta \in T$, let $\overline{\alpha \beta}$ be the unique path connecting $\alpha$ and $\beta$. Define the graph distance $d(\alpha, \beta)$ to be the number of edges contained in the path $\overline{\alpha \beta}$.

Let $T$ be an infinite tree with root 0. The set of all vertices with distance $n$ from the root is called the $n$th generation of $T$, which is denoted by $L_n$. We denote by $T^{(n)}$ the union of the first $n$ generations of $T$. For each vertex $t$, there is a unique path from 0 to $t$ and $|t|$ for the number of edges on this path. We denote the first predecessor of $t$ by $1t$. The degree of a vertex is defined to be the number of neighbors of it. If every vertex of the tree has degree $d + 1$, we say it is Cayley’s tree, which is denoted by $T_{C,d}$. Thus, the root vertex has $d + 1$ neighbors in the first generation and every other vertex has $d$ neighbors in the next generation. For any two vertices $s$ and $t$ of tree $T$, write $s \leq t$ if $s$ is on the unique path from the root 0 to $t$. We denote by $s \wedge t$ the farthest vertex from 0 satisfying $s \wedge t \leq s$ and $s \wedge t \leq t$. We use the notation $X^A = \{X_t, t \in A\}$ and denote by $|A|$ the number of vertices of $A$.

In the following, we always let $T$ denote the Cayley tree $T_{C,d}$.

A tree-indexed Markov chain is the particular case of a Markov random field on a tree. Kemeny et al. [1] and Spitzer [2] introduced two special finite tree-indexed Markov chains with finite transition matrix which is assumed to be positive and reversible to its stationary distribution, and these tree-indexed Markov chains ensure that the cylinder probabilities are independent of the direction we travel along a path. In this paper, we omit such assumption and adopt another version of the definition of tree-indexed Markov chains which is put forward by Benjamini and Peres [3]. Yang and Ye [4] extended it to the case of nonhomogeneous Markov chains indexed by infinite Cayley’s tree and we restate it here as follows.

Definition 1 ($T$-indexed nonhomogeneous Markov chains (see [4])). Let $T$ be an infinite Cayley tree, $\mathcal{X}$ a finite state space, and $\{X_t, t \in T\}$ a stochastic process defined on probability space $(\Omega, F, P)$, which takes values in the finite set $\mathcal{X}$. Let

$$p = \{p(i), i \in \mathcal{X}\}$$

be a distribution on $\mathcal{X}$ and

$$P_t = (P_t(j | i)), \quad i, j \in \mathcal{X},$$

be the transition probability at time $t$.
a transition probability matrix on $\mathcal{X}^2$. If, for any vertex $t$, 
\[
P(X_t = j \mid X_{1:t} = i, X_s = x_s, t \wedge s \leq 1t) = P_t(j \mid i), \quad \forall i, j \in \mathcal{X},
\]

then $\{X_s, s \in T\}$ will be called $\mathcal{X}$-valued nonhomogeneous Markov chains indexed by infinite Cayley’s tree with initial distribution $(I)$ and transition probability matrices $[P_t, t \in T]$.

The subject of tree-indexed processes has been deeply studied and made abundant achievements. Benjamin and Peres [3] have given the notion of the tree-indexed Markov chains and studied the recurrence and ray-recurrence for them. Berger and Ye [5] have studied the existence of entropy rate for some stationary random fields on a homogeneous tree. Ye and Berger [6, 7], by using Pemantle’s result [8] and a combinatorial approach, have studied the Shannon-McMillan theorem with convergence in probability for a PPG-invariant and ergodic random field on a homogeneous tree. Yang and Liu [9] and Yang [10] have studied a strong law of large numbers for Markov chains fields on a homogeneous tree (a particular case of tree-indexed Markov chains and PPG-invariant random fields). Yang and Ye [4] have established the Shannon-McMillan theorem for nonhomogeneous Markov chains on a homogeneous tree. Huang and Yang [11] has studied the strong law of large numbers for Markov chains indexed by infinite Cayley’s tree with initial distribution $(1)$ and transition probability matrices $[P_t, t \in T]$.

In this paper we assume that $\{\xi_t, t \in T\}$ is a nonhomogeneous $T$-indexed Markov chain on state space $\Theta$. The probability of going from $i$ to $j$ in one step in the $\theta$th environment is denoted by $P_\theta(i, j)$. We also suppose that the one-step transition probability of going from $\alpha$ to $\beta$ for nonhomogeneous $T$-indexed Markov chain $\{\xi_t, t \in T\} = K_\theta(\alpha, \beta)$. In this case, $\{\xi_t, n \in T^0\}$ is a Markov chain indexed by $T$ with initial distribution $q = (q(\theta, i))$ and one-step transition on $\Theta \times \mathcal{X}$ determined by
\[
P_t(\alpha, i; \beta, j) = K_\theta(\alpha, \beta) P_\alpha(i, j),
\]

where $q(\theta, i) = P(\xi_0 = \theta, X_0 = i)$. Then $\{\xi_t, t \in T\}$ will be called the bichain indexed by tree $T$. Obviously, we have
\[
P(\xi_t = \alpha, X_t = x_t) = \sum_{\theta \in \Theta} P(\xi_t = \alpha, X_t = x_t, \theta = \theta).
\]

### 2. Main Results

For every finite $n \in \mathbb{N}$, let $\{X_t, t \in T\}$ be a Markov chain indexed by an infinite Cayley tree $T$ in Markovian environment $\{\xi_t, t \in T\}$, which is defined as in Definition 2. Now we suppose that $g_\theta(\alpha, i, \beta, j)$ are functions defined on $\Theta \times \mathcal{X} \times \Theta \times \mathcal{X}$. Let $\lambda$ be a real number, $L_0 = \{0\}$, $\mathcal{F}_n = \sigma(\xi_t, X_t, \xi_t, X_t)$; now we define a stochastic sequence as follows:
\[
\phi_n(\lambda, \omega) = \frac{e^{\lambda \sum_{t \in \mathcal{F}_n} \log g(\xi_t, X_t, \xi_t, X_t)}}{\prod_{t \in L_0} E[e^{\lambda g(\xi_t, X_t, \xi_t, X_t) \mid X_0, \xi_0]}}.
\]

At first we come to prove the following fact.

**Lemma 5.** $\phi_n(\lambda, \omega), \mathcal{F}_n, n \geq 1$ is a nonnegative martingale.
Proof of Lemma 5. Obviously, we have

\[
\begin{align*}
\mathbb{P} \left( \xi^{(n)} = \alpha^{(n)}, X^{(n)} = X^{(n-1)} \right) \\
= \mathbb{P} \left( \xi^{(n-1)} = \alpha^{(n-1)}, X^{(n-1)} = X^{(n-1)} \right) \\
= \mathbb{P} \left( \xi^{(n-1)} = \alpha^{(n-1)}, X^{(n-1)} = X^{(n-1)} \right) \\
&= \prod_{t \in T_n} \mathbb{P} \left( \xi_t = \alpha_t, X_t = x_t \mid \xi_{t+1}, X_{t+1} = x_{t+1} \right). 
\end{align*}
\]

Here, the second equation holds because of the fact that \( \{\xi, X, t \in T\} \) is a bichain indexed by tree \( T \) and (8) is being used. Furthermore, we have

\[
E \left[ e^\lambda \sum_{n=1}^N g_t(\xi_t, X_t; \alpha, \lambda) \right] = \sum_{\alpha, x} \prod_{t \in T_n(\alpha, x) \in \Theta \times \mathcal{X}} e^{\lambda g_t(\xi_t, X_t; \alpha, \lambda)} \\
\times \mathbb{P} \left( \xi_t = \alpha_t, X_t = x_t \mid \xi_{t+1}, X_{t+1} \right) \\
= \prod_{t \in T_n(\alpha, x) \in \Theta \times \mathcal{X}} \mathbb{P} \left( \xi_t = \alpha_t, X_t = x_t \mid \xi_{t+1}, X_{t+1} \right) \\
\times \mathbb{P} \left( \xi_t = \alpha_t, X_t = x_t \mid \xi_{t+1}, X_{t+1} \right)
\]

so that

\[
\lim_{n \to \infty} \frac{\ln \varphi_n(\lambda, \omega)}{|T^{(n)}|} = 0 \quad \text{a.s.} 
\]

which implies that

\[
\lim_{n \to \infty} \frac{\ln \varphi_n(\lambda, \omega)}{|T^{(n)}|} \leq 0 \quad \text{a.s.} 
\]

On the other hand, we also have

\[
\varphi_n(\lambda, \omega) = \varphi_{n-1}(\lambda, \omega) \\
\times \frac{e^{\lambda \sum g_{t-1}(\xi_t, X_t; \alpha, \lambda)} \prod_{t \in T_{n-1}} E \left[ e^{\lambda g_{t-1}(\xi_t, X_t; \alpha, \lambda)} \mid \xi_{t+1}, X_{t+1} \right]}. 
\]

Combining (11) and (12), we get

\[
E \left[ \varphi_n(\lambda, \omega) \mid \mathcal{F}_{n-1} \right] = \varphi_{n-1}(\lambda, \omega) \quad \text{a.s.} 
\]

Thus, we complete the proof of Lemma 5. \( \square \)

Theorem 6. Let \( \{X, t \in T\} \) be a Markov chain indexed by an infinite Cayley tree \( T \) in a nonhomogeneous Markovian environment \( \{\xi_t, t \in T\} \). Suppose that the initial distribution and the transition probability functions satisfy

\[
q(\alpha_0, x_0) > 0, \quad P_t(\alpha, i; \theta, j) > 0, \\
\text{for all } \alpha, \theta \in \Theta; \ x_0, i, j \in \mathcal{X}, \\
a_t = \min \{P_t(\alpha, i; \theta, j) \mid \alpha, \theta \in \Theta; \ i, j \in \mathcal{X}, t \in T \setminus \{0\}, \\
\text{if there exist two positive constants } c \text{ and } m \text{ such that}
\]

\[
\lim_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}(0)} e^{c |t|} = m < \infty. 
\]

Denote \(|\Theta| = M, |\mathcal{X}| = N, \) and

\[
P_t(\xi_{t+1}, X_{t+1}; \xi_t, X_t) = P_t(\xi_{t+1}, X_{t+1} \mid \xi_t, X_t) 
\]

then we have

\[
\lim_{n \to \infty} \frac{|T^{(n)}|}{\sum_{t \in T^{(n)}(0)} P_t(\xi_{t+1}, X_{t+1} \mid \xi_t, X_t)^{-1}} = \frac{1}{MN} \quad \text{a.s.} 
\]

Proof. By Lemma 5, we have known that \( \{\varphi_n(\lambda, \omega), \mathcal{F}_n, n \geq 1\} \) is a nonnegative martingale. According to Doob martingale convergence theorem, we have

\[
\lim_{n \to \infty} \frac{\ln \varphi_n(\lambda, \omega)}{|T^{(n)}|} = 0 \quad \text{a.s.} 
\]

We arrive at

\[
\lim_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}(0)} P_t(\xi_{t+1}, X_{t+1} \mid \xi_t, X_t)^{-1} \leq 0 \quad \text{a.s.} 
\]

Combining (20) with the inequalities \( \ln x \leq x - 1 \) \((x > 0)\) and \( 0 \leq e^x - 1 - x \leq 2^{-1}x^2 e^x \) and taking \( g_t(\xi_{t-1}, X_{t-1}, \xi_t, X_t) = P_t(\xi_{t-1}, X_{t-1} \mid \xi_t, X_t)^{-1} \), it follows that

\[
\lim_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}(0)} \left[ \ln \left[ E \left[ e^{\lambda g_t(\xi_{t-1}, X_{t-1}, \xi_t, X_t)} \mid \xi_{t-1}, X_{t-1}\right] \right] \right] 
\]

\[
\leq \lim_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)}(0)} \left[ \ln \left[ E \left[ e^{\lambda g_t(\xi_{t-1}, X_{t-1}, \xi_t, X_t)} \mid \xi_{t-1}, X_{t-1}\right] \right] 
\]

\[
- \lambda MN \right] 
\]
If $-c < \lambda < 0$, similar to the analysis of inequality (24), by using (15), (22), and (23) again, we can arrive at

\[
\liminf_{n \to \infty} \frac{1}{|T|^{(n)}} \sum_{t \in T^{(n)} \setminus \{0\}} \left[ P_t(\xi_{1i}, X_{1i}; \xi_i, X_i)^{-1} - MN \right] 
\geq \frac{\lambda MN}{2} \limsup_{n \to \infty} \frac{1}{|T|^{(n)}} \sum_{t \in T^{(n)} \setminus \{0\}} \frac{1}{a_t} e^{\lambda/|a_t|} \geq \frac{\lambda MN}{2e(\lambda + c)} m \quad \text{a.s.}
\]

Letting $\lambda \to 0^-$ in inequality (26), we get

\[
\liminf_{n \to \infty} \frac{1}{|T|^{(n)}} \sum_{t \in T^{(n)} \setminus \{0\}} \left[ P_t(\xi_{1i}, X_{1i}; \xi_i, X_i)^{-1} - MN \right] \geq 0 \quad \text{a.s.}
\]

(27)

Combining (25) and (27), we obtain that our assertion (17) is true. $\Box$

**Corollary 7.** Let $\{X_n, t \in T\}$ be a nonhomogeneous Markov chain indexed by an infinite Cayley tree $T$. Suppose that the initial distribution and the transition probability functions satisfy

\[
p(x_0) > 0, \quad P_t(i, j) > 0, \quad a_t = \min \{P_t(i, j), i, j \in \mathcal{X}\}, \quad t \in T \setminus \{0\},
\]

if there exist two positive constants $c$ and $m$ such that

\[
\limsup_{n \to \infty} \frac{1}{|T|^{(n)}} \sum_{t \in T^{(n)} \setminus \{0\}} e^{c/|a_t|} = m < \infty.
\]

(29)

Denote $|\mathcal{X}| = N$; then we have

\[
\lim_{n \to \infty} \frac{|T|^{(n)}}{\sum_{t \in T^{(n)} \setminus \{0\}} P_t(X_t | X_{1i})^{-1}} = \frac{1}{N} \quad \text{a.s.}
\]

(30)

**Proof.** If we take $\Theta = \{|\theta| = 1\}$, that is, $|\Theta| = 1$, then the model of Markov chain indexed by tree $T$ in Markovian environment reduces to the formulation of a nonhomogeneous Markov chain indexed by tree $T$. Then we arrive at our conclusion (30) directly from Theorem 6. $\Box$

**Corollary 8 (see [15]).** Let $\{X_n, n \geq 0\}$ be a Markov chain in a nonhomogeneous Markovian environment $\{\xi_n, n \geq 0\}$. Suppose that the initial distribution and the transition probability functions satisfy

\[
q(x_0, x_0) > 0, \quad P_n(\alpha, i; \theta, j) > 0, \quad a_n = \min \{P_n(\alpha, i; \theta, j), \alpha, \theta \in \Theta; i, j \in \mathcal{X}\}, \quad n \geq 1,
\]

(31)
if there exist two positive constants $c$ and $m$ such that
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} e^{c/k} = m < \infty.
\] (32)

Denote $|\Theta| = M$, $|\mathcal{X}| = N$, and
\[
P_k(\xi_{k-1}, X_{k-1}; \xi_k, X_k) = P_k(\xi_k, X_k | \xi_{k-1}, X_{k-1});
\] (33)

then we have
\[
\lim_{n \to \infty} \frac{1}{\sum_{k=1}^{n} P_k(\xi_{k-1}, X_{k-1}; \xi_k, X_k)} = \frac{1}{MN} \text{ a.s.} \] (34)

Proof. If every vertex of the tree $T$ has degree 2, then the nonhomogeneous Markov chain indexed by tree $T$ degenerates into the nonhomogeneous Markov chain on line; thus, this corollary can be obtained from Theorem 6 directly.

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References


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