ON THE NONEXISTENCE OF A LAW OF THE ITERATED LOGARITHM FOR WEIGHTED SUMS OF IDENTICALLY DISTRIBUTED RANDOM VARIABLES

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ABSTRACT
For weighted sums of independent and identically distributed random variables, conditions are placed under which a generalized law of the iterated logarithm cannot hold, thereby extending the usual nonweighted situation.

AMS Subject Classification: 60F15.

Key Words: law of the iterated logarithm, strong law of large numbers.

1. INTRODUCTION.
Heyde [1] established the fact that partial sums of independent and identically distributed (i.i.d.) random variables \( \{X_n, n \geq 1\} \) whose common distribution is of the form \( P\{|X| > x\} = L(x)x^{-\alpha} \) \( (0 \leq \alpha < 2, \alpha \neq 1) \), where \( L(x) \) is slowly varying at infinity and where \( EX = 0 \) if \( E|X| < \infty \), cannot be normalized in the sense that there exist constants \( 0 < b_n \uparrow \) with \( \sum_{k=1}^{\infty} X_k/b_n \to 1 \) a.s. The purpose of this paper is to present similar results in the weighted case.

Herein, we define \( S_n = \sum_{k=1}^{n} a_k X_k \) where \( \{a_n, n \geq 1\} \) are constants and the random variables \( \{X, X_n, n \geq 1\} \) are identically distributed with common distribution

\[
P\{|X| > x\} = \begin{cases} 
L(x)x^{-\alpha} & \text{if } x \geq 1, \\
1 & \text{if } x < 1,
\end{cases}
\]

where \( L(cx)/L(x) \to 1 \) as \( x \to \infty \) for all \( c > 0 \), and \( \alpha \geq 0 \).

A remark about notation is needed. Throughout, the symbol \( C \) will denote a generic finite nonzero constant which is not necessarily the same in each appearance. Also, we let \( c_n = b_n/|a_n|, n \geq 1 \), where \( \{b_n, n \geq 1\} \) is our norming sequence.

It should be noted that the techniques involved with the main results (Theorems 2 and 3) follow a similar pattern to those that can be found in Heyde [1]. As usual, via the Borel-Cantelli lemma, one need only consider a truncated version of the random variables \( \{X_n, n \geq 1\} \). Instead of truncating \( X_n \) at \( b_n \) the trick, in the weighted case, is to cut off \( X_n \) at \( c_n \). Then by classical arguments the remaining terms are shown to be almost surely negligible. Also of particular interest is the discussion (Section 3) of the \( \alpha = 1 \) situation.

\[\text{Received: January 1989; Revised: December 1989} \]
2. RESULTS.

Our first theorem examines what happens when \( P\{X_{i} > c_{i}, i.o.(n)\} = 1 \).

**Theorem 1.** Let \( \{X, X_{n}, n \geq 1\} \) be i.i.d. random variables. If \( \{a_{n}, n \geq 1\} \) and \( \{b_{n}, n \geq 1\} \) are constants satisfying \( b_{n} = O(b_{n+1}) \), \( b_{n} \to \infty \), and \( \sum_{n=1}^{\infty} P\{|X| > c_{n}\} = \infty \), then \( \limsup_{n \to \infty} |S_{n}|/b_{n} = \infty \) a.s.

**Proof.** If \( c_{n} \to \infty \), then for all large \( M \)

\[
\sum_{n=1}^{\infty} P\{|a_{n}X_{n}| > Mb_{n}\} = \sum_{n=1}^{\infty} L(Mc_{n})(Mc_{n})^{-\alpha} \\
\geq C \sum_{n=1}^{\infty} L(c_{n})c_{n}^{-\alpha} \\
\geq C \sum_{n=n_{0}}^{\infty} P\{|X_{n}| > c_{n}\} \quad (\text{for a suitably chosen } n_{0}) \\
= \infty.
\]

Otherwise, if \( \liminf_{n \to \infty} c_{n} < \infty \), then there exists a subsequence \( \{n_{k}, k \geq 1\} \) and a finite constant \( B \) such that \( c_{n_{k}} \leq B \). Hence for all \( 0 < M < \infty \)

\[
\sum_{n=1}^{\infty} P\{|X| > Mc_{n}\} \geq \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} P\{|X_{n}| > Mc_{n}\} \\
\geq \sum_{k=1}^{\infty} P\{|X| > MB\} = \infty.
\]

So in either case we conclude, via the Borel-Cantelli lemma, that

\[
\limsup_{n \to \infty} \left| \frac{a_{n}X_{n}}{b_{n}} \right| = \infty \quad \text{a.s.}
\]

Since

\[
\left| \frac{a_{n}X_{n}}{b_{n}} \right| \leq \frac{|S_{n}|}{b_{n}} + \frac{b_{n-1}}{b_{n}} \cdot \frac{|S_{n-1}|}{b_{n-1}}
\]

the conclusion follows. \( \square \)

Note that in the next result independence is not necessary.

**Theorem 2.** Let \( \{X, X_{n}, n \geq 1\} \) be identically distributed random variables. Let \( \{a_{n}, n \geq 1\} \) and \( \{b_{n}, n \geq 1\} \) be constants satisfying \( 0 < b_{n} \uparrow \infty \) and \( \sum_{n=1}^{\infty} P\{|X| > c_{n}\} < \infty \). If \( 0 \leq \alpha < 1 \), then \( S_{n}/b_{n} \to 0 \) a.s.

**Proof.** Notice, via the Borel-Cantelli lemma, that

\[
\sum_{k=1}^{n} a_{k}X_{k} I(|X_{k}| > c_{k}) = o(b_{n}) \quad \text{a.s.}
\]

Hence it remains to show that

\[
\sum_{k=1}^{n} a_{k}X_{k} I(|X_{k}| \leq c_{k}) = o(b_{n}) \quad \text{a.s.}
\]
Since, for all large $k$

$$E|X|I(|X| \leq c_k) \leq \int_0^{c_k} P\{|X| > t\} dt$$

$$= \int_0^1 dt + \int_1^{c_k} L(t)t^{-\alpha} dt$$

$$\leq CL(c_k)c_k^{-\alpha+1}$$

(by Theorem 1b of Feller [2, p. 281]), it follows that

$$\sum_{k=1}^{\infty} c_k^{-1}E|X|I(|X| \leq c_k) \leq C \sum_{k=1}^{\infty} L(c_k)c_k^{-\alpha}$$

$$\leq C \sum_{k=1}^{\infty} P\{|X| > c_k\}$$

$$< \infty,$$

whence

$$\sum_{k=1}^{\infty} c_k^{-1}|X_k|I(|X_k| \leq c_k) < \infty \text{ a.s.}$$

This, via Kronecker's lemma, implies (1). □

Next, we examine the mean zero situation.

**Theorem 3.** Let \{X, X_n, n \geq 1\} be i.i.d. mean zero random variables. Let \{a_n, n \geq 1\} and \{b_n, n \geq 1\} be constants satisfying $0 < b_n \to \infty$ and $\sum_{n=1}^{\infty} P\{|X| > c_n\} < \infty$. If $1 < \alpha < 2$, then $S_n/b_n \to 0$ a.s.

**Proof.** Again, note that

$$\sum_{k=1}^{n} a_k X_k I(|X_k| > c_k) = o(b_n) \text{ a.s.}$$

Since

$$\sum_{k=1}^{n} a_k X_k = \sum_{k=1}^{n} a_k [X_k I(|X_k| \leq c_k) - EXI(|X| \leq c_k)]$$

$$+ \sum_{k=1}^{n} a_k EXI(|X| \leq c_k) + \sum_{k=1}^{n} a_k X_k I(|X_k| > c_k)$$

we need only show that the first two terms are $o(b_n)$. In view of the Khintchine-Kolmogorov convergence theorem and Kronecker’s lemma, all that one needs to show, in order to prove that the first term is $o(b_n)$ a.s., is that

(2)

$$\sum_{k=1}^{\infty} c_k^{-2}EX^2 I(|X| \leq c_k) < \infty.$$
Hence (2) holds. Finally, we need to show that

$$\sum_{k=1}^{n} a_k E X I(|X| \leq c_k) = o(b_n).$$

Due to the fact that $E X I(|X| \leq c_k) \leq E |X| I(|X| > c_k)$ it is sufficient to show that

$$\sum_{k=1}^{n} |a_k| E |X| I(|X| > c_k) = o(b_n).$$

However, since

$$\sum_{k=1}^{\infty} \frac{1}{c_k} E |X| I(|X| > c_k) = \sum_{k=1}^{\infty} P(|X| > c_k) + \sum_{k=1}^{\infty} \frac{1}{c_k} \int_{c_k}^{\infty} P(|X| > t) dt$$

$$\leq O(1) + C \sum_{k=1}^{\infty} \frac{1}{c_k} \int_{c_k}^{\infty} L(t) t^{-\alpha} dt$$

$$\leq O(1) + C \sum_{k=1}^{\infty} L(c_k) c_k^{-\alpha} \quad \text{(see Feller, [2, p.281])}$$

$$\leq O(1) + C \sum_{k=1}^{\infty} P(|X| > c_k)$$

$$= O(1),$$

it is clear that (3) obtains. □

3. DISCUSSION.

In this section we combine the previous theorems. The conclusion is that for all $\alpha \in [0, 1) \cup (1, 2)$ a law of the iterated logarithm cannot hold.

Theorem 4. Let $\{X, X_n, n \geq 1\}$ be i.i.d. random variables with

$$P(|X| > x) = \begin{cases} L(x)x^{-\alpha} & x \geq 1, \\ 1 & x < 1, \end{cases}$$

with $EX = 0$ if $\alpha > 1$. If $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ are constants with $0 < b_n \uparrow \infty$, then for all $\alpha \in [0, 1) \cup (1, 2)$

$$\lim \sup_{n \to \infty} \left| \frac{\sum_{k=1}^{n} a_k X_k}{b_n} \right| = 0 \ \text{or} \ \infty \ \text{a.s.}$$
depending on whether \( \sum_{n=1}^{\infty} P(|X| > c_n) \) converges or diverges.

**Proof.** In view of Theorems 1, 2, and 3 the conclusion is immediate. \( \square \)

Now, clearly if a law of the iterated logarithm does not exist, then a strong law of large numbers (with limit one) is also not feasible.

**Corollary.** If the hypotheses of Theorem 4 hold, then

\[
P\left( \lim_{n \to \infty} \sum_{k=1}^{n} a_k X_k / b_n = 1 \right) = 0.
\]

It is well known that if \( \alpha > 2 \), then a classical law of the iterated logarithm can be obtained provided suitable conditions are imposed on the constants \( \{a_n, n \geq 1\} \). An interesting question is what happens when \( \alpha = 1 \). If we allow \( \alpha = 1 \), then not only can a law of the iterated logarithm obtain, but a strong law of large numbers can also occur where the limit is one. The following example is of the flavor of those that can be found in Adler [3].

**Example.** If \( \{X_n, n \geq 1\} \) are i.i.d. random variables with common density \( f(z) = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}z^2} \), \( -\infty < z < \infty \), then

\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} \frac{1}{k} X_k}{(\log n)^2} = 1 \text{ a.s.}
\]

**Proof.** Since

\[
\sum_{n=1}^{\infty} P(|X| > \frac{n(\log n)^2}{2}) = 2 + \sum_{n=3}^{\infty} \frac{2}{n(\log n)^2} < \infty
\]

and

\[
\left( \frac{n(\log n)^2}{2} \right)^2 \sum_{j=n}^{\infty} \frac{2}{j(\log j)^2} = O(n)
\]

we have, by Theorem 1 of Adler and Rosalsky [4],

\[
\sum_{k=1}^{n} \frac{1}{k} (X_k - \mu_k) \to 0 \text{ a.s.}
\]

where

\[
\mu_n = \mathbb{E}X I(|X| \leq \frac{n(\log n)^2}{2})
\]

\[
= \int_{1}^{n(\log n)^{1/2}} x^{-1} dx
\]

\[
\sim \log n.
\]

Noting that

\[
\sum_{k=1}^{n} \frac{1}{k} \log k \to 1
\]

the proof is complete. \( \square \)

Here we exhibited a strong law in the nonintegrable case. One can obtain similar strong laws for mean zero random variables when \( P(|X| > x) = L(x)/x \) (see, e.g., Adler and Rosalsky [5]).
Acknowledgement: The author wishes to thank the Referee for his/her valuable comments and helpful suggestions which led to an improved presentation of this paper.

REFERENCES.


