ON THE VARIANCE OF THE NUMBER OF REAL ROOTS OF A RANDOM TRIGONOMETRIC POLYNOMIAL*

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ABSTRACT

This paper provides an upper estimate for the variance of the number of real zeros of the random trigonometric polynomial $g_1 \cos \theta + g_2 \cos 2\theta + \ldots + g_n \cos n\theta$. The coefficients $g_i$ ($i = 1, 2, \ldots, n$) are assumed independent and normally distributed with mean zero and variance one.

Key words: random trigonometric polynomial, number of real roots, variance.

AMS subject classification: 60H, 42.

1. INTRODUCTION

Let

$$T(\theta) = T_n(\theta, \omega) = \sum_{i=1}^{n} g_i(\omega) \cos i\theta,$$

where $g_1(\omega), g_2(\omega), \ldots, g_n(\omega)$ is a sequence of independent random variables defined on a probability space $(\Omega, A, P)$ each normally distributed with mathematical expectation zero and variance one. Denote by $N(\alpha, \beta)$ the number of real roots of the

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equation \( T(\theta) = 0 \) in the interval \((\alpha, \beta)\), where multiple roots are counted only once. Dunnage [3] showed that except for a set of functions of \( T(\theta) \) of measure not larger than \( (\log n)^{-1} \)

\[
N(0, 2\pi) = 2n/\sqrt{3} + O\{ n^{11/13} (\log n)^{3/13} \}.
\]

Later Sambandham and Renganathan [9] and Farahmand [4] generalized this result to the case where the coefficients \( g_i \) have a non zero mean. They show that for \( n \) sufficiently large the mathematical expectation of the number of real roots, \( EN \), satisfies

\[
EN(0, 2\pi) \sim (2/\sqrt{3}) n.
\]

The results for the dependent coefficients with constant correlation coefficient or otherwise are due to Renganathan and Sambandham [6] and Sambandham [7] and [8]. A comprehensive treatment of the zeros of random polynomial constitutes the greater part of a book by Bharucha-Reid and Sambandham [1] which gives a rigorous and interesting survey of earlier works in this field.

Qualls [5] resolved the only known variance of the number of real roots of a random trigonometric polynomial. Indeed he considered a different type of random polynomial,

\[
\sum_{i=0}^{n} (a_i \cos i\theta + b_i \sin i\theta)
\]

which has the property of being stationary and for which a special theorem has been developed by Cramer and Leadbetter [2]. Here we shall prove the following theorem:

**Theorem.** Let \( g_1(\omega), g_2(\omega), \ldots, g_n(\omega) \) be the independent random variables
corresponding to a Gaussian distribution with mean zero. Then the variance of the number of real roots of $T(\theta)$ satisfies

$$\text{Var } N(0, 2\pi) = O\left( n^{24/13} (\log n)^{16/13} \right).$$

2. OVERVIEW OF PROOF OF THE THEOREM AND SOME LEMMAS

In general we make use of a delicate analysis suggested by the work of Dunnage in [3] with which we assume the reader is familiar. We divide the interval $(0, 2\pi)$ into intervals $I_1, I_2, \ldots, I_s$, each of equal length $\delta$. Then with each $I_j (j = 1, 2, \ldots, s)$, we associate the following two functions:

$$N_j(\omega) = \text{number of zeros of } T(\theta) \text{ in } I_j, \text{ counted according to their multiplicity}$$

and

$$N^*_j(\omega) = \begin{cases} N_j(\omega) & \text{if } N_j(\omega) \geq 2, \\ 0 & \text{otherwise}. \end{cases}$$

Now if $T(a) T(b) \leq 0$ we shall say, being prompted by a graphical idea, that $T(\theta)$ has a single crossover (s.c.o.) in $(a, b)$, and let

$$\mu_j(\omega) = \begin{cases} 1 & \text{if } T(\theta) \text{ has a (s.c.o.) in } I_j \\ 0 & \text{otherwise} \end{cases}$$
clearly

\[(2.1) \quad 0 \leq N_j(\omega) - \mu_j(\omega) \leq N_j^*(\omega).\]

For the proof of the theorem we need the following lemmas.

Lemma 1. Provided that the interval of \( I \), of length \( \delta = o(1/n) \) does not overlap the \( \varepsilon \) - neighborhood of \( 0, \pi \) and \( 2\pi \), where \( \varepsilon \sim n^{-6/13} (\log n)^{-4/13} \), the probability that \( T(\theta) \) has at least two zeros (counted according to their multiplicity) in \( I \) is \( O(n^3, \delta^3) \).

Proof. This is lemma 11 of [3].

We denote by \( N(\omega) \) the number of real zeros that \( T(\theta) \) has in \( I \) and we define

\[
N^*(\omega) = \begin{cases} 
N(\omega) & \text{if } N(\omega) \geq 2 \\
0 & \text{otherwise}. 
\end{cases}
\]

Lemma 2. For a constant \( A \)

\[
E[N^*(\omega)]^2 < A n^3 \delta^3 \log n.
\]

Proof. Suppose \( T(\theta) \) has at least \( k (\geq 2) \) zeros in \( I \). Then if \( I \) is divided into \( 2p \) equal parts where \( p \) is chosen as an integer satisfying \( 2p < k < 2p+1 \) at least one part must contain two or more zeros, and by lemma 1, the probability of this occurring does not exceed
\[ 2^p n^{3} (\delta / 2^{p})^{3} = A n^{3} \delta^{3} 2^{-2p} < A n^{3} \delta^{2} / k^{2}. \]

Hence if \( q_{k} \) is the probability that \( T(\theta) \) has at least \( k \) zeros in \( I \), we have

\[ q_{k} < A n^{3} \delta^{3} / k^{2}. \]

Now we find the mathematical expectation of \( N^*^{2} \) as

\[
E [N^*^{2}] = \sum_{k=2}^{n} k^2 \text{Prob} (n = k) \sum_{k=2}^{n} k^2 (q_{k} - q_{k+1})
\]

\[
= \sum_{k=2}^{n} k^2 q_{k} - \sum_{k=3}^{n+1} (k - 1)^2 q_{k}
\]

\[
\leq 4q_{2} + \sum_{k=3}^{n+1} (2k - 1) q_{k} < A n^{3} \delta^{3} \log n
\]

which completes the proof of lemma 2.

Now we define

\[ \alpha_{j} = E (N_{j}) \quad \text{and} \quad m_{j} = E (\mu_{j}). \]

Lemma 3.

\[
\sum m_{j} = (N / \sqrt{3}) + O \{N^{11/13} (\log n^{3/13}) \}.
\]

Proof. This is lemma 16 of [3].
3. PROOF OF THE THEOREM.

First we consider the interval \((\varepsilon, \pi - \varepsilon)\). We have

\[
(3.1) \quad \text{Var} \ N(\varepsilon, \pi - \varepsilon) \leq 4E \left\{ \sum_{j} (N_j - \mu_j)^2 \right\}
\]

\[
+ 4E \left\{ \sum_{j} (\mu_j - m_j)^2 \right\} + 4E \left\{ \sum_{j} (m_j - \alpha_j)^2 \right\}.
\]

From (2.1) and lemma 2 we have

\[
(3.2) \quad \mathbb{E} \left[ \sum_{j} (N_j - \mu_j)^2 \right] \leq \mathbb{E} \left[ \sum_{j} N_j^2 \right] < \delta \mathbb{E} \left[ \sum_{j=1}^{s} (N_j^*)^2 \right]
\]

\[
\leq \frac{\pi}{\delta} \sum_{j=1}^{s} \mathbb{E}(N_j^*)^2 < A \pi \delta n^3 \delta^2 \log n.
\]

So far \(\delta = o(1/n)\) has been an arbitrary constant; now since the total number of \(\delta\)-intervals is \((\pi - 2\varepsilon) / \delta\), we choose \(\delta\) such that

\[
(\pi - 2\varepsilon) / \delta = n^{15/13} (\log n)^{-3/13}.
\]

So from (3.2) we have

\[
(3.3) \quad \mathbb{E} \sum_{j} \left[ (N_j - \mu_j)^2 \right] < A n^{24/13} (\log n)^{16/13}.
\]

Also from lemma 3 and the fact that
\( \sum_j x_j = n / \sqrt{3} + O \{ n^{11/13} (\log n)^{3/13} \} \)

we have

\[
\begin{align*}
(3.4) \quad E \left[ \sum_j (m_j - x_j) \right]^2 & = E \left[ n / \sqrt{3} + O \{ n^{11/13} (\log n)^{3/13} \} - n / \sqrt{3} 

+ O \{ n^{11/13} (\log n)^{3/13} \} \right]^2 = O \{ n^{22/13} (\log n)^6 \! / \! 13 \}.
\end{align*}
\]

Hence from (3.1), (3.2), (3.3) and since from [3, page 81]

\[
E \left[ \sum_j (\mu_j - m_j) \right]^2 = O \{ n^{22/13} (\log n)^{6/13} \}
\]

we have

\[
(3.5) \quad \text{Var} N (\varepsilon, \pi - \varepsilon) = O \{ n^{24/13} (\log n)^{16/13} \}.
\]

To find the variance in the interval \((- \varepsilon, \varepsilon)\) let \( \eta (r) = \eta (r, \omega) \) be the number of zeros of \( T (\theta) \) in the circle \(| z | \leq r \). From [3, page 83] we know that outside an exceptional set of measure at most \( \exp (- n^2 / 2) + (2 \pi)^{1/2} / n \)

\[
\eta (\varepsilon) \leq 1 + (2 \log n + 2 n \varepsilon) / \log 2.
\]

Since the number of real roots in the segment of the real axis joining points \( \pm \varepsilon \) does not exceed the number in the circle \(| z | \leq \varepsilon \), we can obtain

\[
(3.6) \quad N (- \varepsilon, \varepsilon) = O \{ n^{7/13} (\log n)^{-4/13} \}
\]

except for sample functions in an \( \omega \)-set of measure not exceeding \( \exp (- n^2 / 2) + \).
(2\pi)^{1/2} / n$. Now let $d$ be any integer of $O\{n^{7/13} (\log n)^{-4/13}\}$, then since the trigonometric polynomial has at most $2n$ zeros in $(0, 2\pi)$ from (3.6) we have

\[
(3.7) \quad \text{Var } N (-\varepsilon, \varepsilon) \leq \sum_{i=0}^{2n} i^2 \text{Prob } (N = i)
\]

\[
= \sum_{i \leq d} i^2 \text{Prob } (N = i) + \sum_{i > d} i^2 \text{Prob } (N = i)
\]

\[
< B n^{23/13} \text{Prob } \{N < C n^{7/13} (\log n)^{-4/13}\}
\]

\[
+ 4 n^2 \text{Prob } \{N > C' n^{7/13} (\log n)^{-4/13}\}
\]

\[
< D n^{23/13} + 4 n^2 \{\exp (-n^2 / 2) + (\sqrt{2/\pi}) / n\}
\]

\[
= O(n^{23/13}),
\]

where $B, C, C'$ and $D$ are constants. Finally from (3.5) and (3.7) we have proof of the theorem.

Remark. Although in this paper we assumed that the coefficients $g_i(\omega), i = 1, 2, ..., n$ are independent with means zero and variance one, we can show that our theorem for the case of dependent coefficients with mean zero or non-zero (finite or infinite) and any finite variance would remain valid. However a subsequent study could be directed to reduce the upper bound obtained in our theorem, or further, to establish an asymptotic formula for the variance.
REFERENCES


