APPLICATION OF LAKSHMIKANTHAM'S MONOTONE-ITERATIVE TECHNIQUE TO THE SOLUTION OF THE INITIAL VALUE PROBLEM FOR IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT

In the present paper, a technique of V. Lakshmikantham is applied to approximate finding of extremal quasisolutions of an initial value problem for a system of impulsive integro-differential equations of Volterra type.

Key words: Monotone-iterative technique, impulsive integro-differential equations.

AMS (MOS) subject classifications: 34A37.

1. INTRODUCTION

The monotone-iterative technique of V. Lakshmikantham is one of the most effective methods for finding approximate solutions of initial value and periodic problems for differential equations. This technique is a fruitful combination of the method of upper and lower solutions and a suitably chosen monotone method [1]-[8].

In the present paper, by means of this monotone-iterative technique, minimal and maximal quasisolutions of the initial value problem for a system of impulsive integro-differential equations of Volterra type are obtained.

2. STATEMENT OF THE PROBLEM, PRELIMINARY NOTES

Consider the initial value problem for the system of impulsive integro-differential equations
\[ \dot{x} = f(t, x, Qx(t)) \quad \text{for } t \neq t_i, t \in [0, T] \]
\[ \Delta x |_{t = t_i} = I_i(x(t_i - 0)) \quad \text{for } t \in [-h, 0], \]
where \( x = (x_1, x_2, \ldots, x_n), \quad f: [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad f = (f_1, f_2, \ldots, f_n), \quad Qx = (Q_1x, Q_2x, \ldots, Q_nx), \quad Q_jx(t) = \int_{t-h}^t k_j(t, s) x_j(s) ds, \quad \varphi: [-h, 0] \rightarrow \mathbb{R}^n, \quad \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n), \quad h = \text{const} > 0, \quad 0 < t_1 < t_2 < \ldots < t_p < T, \quad \Delta x |_{t = t_i} = x(t_i + 0) - x(t_i - 0), \quad I_i: \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad I_i = (I_{i1}, I_{i2}, \ldots, I_{in}). \]

With any integer \( j = 1, \ldots, n, \) we associate two nonnegative integers \( p_j \) and \( q_j \) such that \( p_j + q_j = n - 1 \) and introduce the notation

\[ (x_j, [x]_{p_j}, [y]_{q_j}) = \begin{cases} 
(x_1, x_2, \ldots, x_{p_j + 1}, y_{p_j + 2}, \ldots, y_{n}) & \text{for } p_j \geq j \\
(x_1, \ldots, x_{p_j}, y_{p_j + 1}, \ldots, y_{j - 1}, x_j, y_{j + 1}, \ldots, y_{n}) & p_j < j. \end{cases} \]

With the notation introduced, the initial value problem (1) can be written down in the form

\[ \dot{x}_j = f_j(t, x_j, [x]_{p_j}, [x]_{q_j}, Q_jx(t), [Qx(t)]_{p_j}, [Qx(t)]_{q_j}) \quad \text{for } t \neq t_i, t \in [0, T] \]
\[ \Delta x_j |_{t = t_i} = I_{ij}(x_j(t_i), [x_j(t_i)]_{p_j}, [x_j(t_i)]_{q_j}), \]
\[ x_j(t) = \varphi_j(t) \quad \text{for } t \in [-h, 0], j = 1, \ldots, n. \]

Let \( x, y \in \mathbb{R}^n, x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n). \) We shall say that \( x \geq (\leq) y \) if for any \( i = 1, \ldots, n, \) the inequality \( x_i \geq (\leq) y_i \) holds.

Consider the set \( G([a, b], \mathbb{R}^n) \) of all functions \( u: [a, b] \rightarrow \mathbb{R}^n \) which are piecewise continuous with points of discontinuity of the first kind at the points \( t_i \in (a, b), \) \( u(t_i) = u(t_i - 0) \) and the set \( G^1([a, b], \mathbb{R}^n) \) of all functions \( u \in G([a, b], \mathbb{R}^n) \) which are continuously differentiable for \( t \neq t_i, t \in [a, b] \) and have continuous left derivatives at the points \( t_i \in (a, b). \)

**Definition 1:** The couple of functions \( v, w \in G([-h, T], \mathbb{R}^n), v, w \in G^1([0, T], \mathbb{R}^n), \) \( v = (v_1, v_2, \ldots, v_n), w = (w_1, w_2, \ldots, w_n) \) is said to be a couple of lower and upper quasisolutions of the initial value problem (1) if the following inequalities hold.

\[ \dot{v}_j \leq f_j(t, v_j, [v]_{p_j}, [v]_{q_j}, Q_jv, [Qv]_{p_j}, [Qv]_{q_j}) \quad \text{for } t \neq t_i, t \in [0, T] \]
\[ \dot{w}_j \geq f_j(t, w_j, [w]_{p_j}, [w]_{q_j}, Q_jw, [Qw]_{p_j}, [Qw]_{q_j}) \]
\[ \Delta v_j |_{t = t_i} \leq I_{ij}(v_j(t_i), [v(t_i)]_{p_j}, [w(t_i)]_{q_j}) \quad \text{for } t \neq t_i, t \in [0, T] \]
\[ \Delta w_j \big|_{t = t_i} \geq I_{ij}(w_j(t_i), [w(t_i)]_{p_j}, [v(t_i)]_{q_j}) \]

\[ v_j(t) \leq \varphi_j(t) \leq w_j(t) \text{ for } t \in [-h, 0], \quad j = 1, \ldots, n. \] \hfill (4)

**Definition 2:** In the case when (1) is an initial value problem for a scalar impulsive integro-differential equation, i.e. \( n = 1 \) and \( p_1 = q_1 = 0 \), the couple of upper and lower quasisolutions of (1) are said to be upper and lower solutions of the same problem.

**Definition 3:** The couple of functions \( v, w \in G([-h, T], \mathbb{R}^n) \), \( v, w \in G^1([0, T], \mathbb{R}^n) \) is said to be a couple of quasisolutions of the initial value problem (1) if (2), (3) and (4) hold only equalities.

**Definition 4:** The couple of functions \( v, w \in G([-h, T], \mathbb{R}^n) \), \( v, w \in G^1([0, T], \mathbb{R}^n) \) is said to be a couple of minimal and maximal quasisolutions of the initial value problem (1) if they are a couple of quasisolutions of the same problem and for any couple of quasisolutions of (1) \((u, z)\) the inequalities \( v(t) \leq u(t) \leq w(t) \) and \( v(t) \leq z(t) \leq w(t) \) hold for \( t \in [-h, T] \).

**Remark 1:** Note that for the couple of minimal and maximal quasisolutions \((v, w)\) of (1) the inequality \( v(t) \leq w(t) \) holds for \( t \in [-h, T] \), while for an arbitrary couple of quasisolutions \((u, z)\) of (1) an analogous inequality may not be valid.

**Remark 2:** If for any \( j = 1, \ldots, n \), the equalities \( p_j = n - 1 \) and \( q_j = 0 \) hold and the couple of functions \((v, w)\) is a couple of quasisolutions of the initial value problem (1), then the functions \( v(t) \) and \( w(t) \) are two solutions of the same problem. If, in this case, problem (1) has a unique solution \( u(t) \), then the couple of functions \((u, u)\) is a couple of minimal and maximal quasisolutions of (1).

For any couple of functions \( v, w \in G([-h, T], \mathbb{R}^n) \), \( v, w \in G^1([0, T], \mathbb{R}^n) \) such that \( v(t) \leq w(t) \) for \( t \in [-h, T] \) define the set of functions

\[ S(v, w) = \{ u \in G([-h, T], \mathbb{R}^n), \quad u \in G^1([0, T], \mathbb{R}^n) : \quad v(t) \leq u(t) \leq w(t) \text{ for } t \in [-h, T] \}. \]

### 3. MAIN RESULTS

**Lemma 1:** Let the following conditions hold:

1. The function \( k \in C([0, T] \times [-h, T], [0, \infty)) \).
2. The function \( g \in G([-h, T], \mathbb{R}), \quad g \in G^1([0, T], \mathbb{R}^n) \) satisfies the inequalities

\[ \dot{g}(t) \leq -M g(t) - N \int_{t-h}^{t} k(t, s) g(s) ds \text{ for } t \neq t_i, t \in [0, T] \] \hfill (5)
\[ \Delta g \big|_{t=t_i} \leq -L_i g(t_i) \]  

(6)

\[ g(0) \leq g(t) \leq 0 \text{ for } t \in [-h, 0], \]  

(7)

where \( M, N, L_i (i = 1, \ldots, p) \) are constants such that \( M, N > 0, \ 0 \leq L_i < 1. \)

3. The inequality

\[ (M + N \kappa_0 h) p \tau < (1 - L)^p \]  

(8)

holds, where

\[ \kappa_0 = \max \{ \kappa(t, s) : t \in [0, T], s \in [-h, T] \}, \]

\[ \tau = \max \{ t_1, T - t_p, \max [t_{i+1} - t_i : i = 1, 2, \ldots, p - 1] \}, \]

\[ L = \max \{ L_i : i = 1, 2, \ldots, p \}. \]

Then \( g(t) \leq 0 \text{ for } t \in [-h, T]. \)

Proof: Suppose that this is not true, i.e. that there exists a point \( \xi \in [0, T] \) such that \( g(\xi) > 0. \) The following three cases are possible:

Case 1: Let \( g(0) = 0 \) and \( g(t) \geq 0, \ g(t) \neq 0 \text{ for } t \in [0, b) \) where \( b > 0 \) is a sufficiently small number. From inequality (7), it follows that \( g(t) \equiv 0 \text{ for } t \in [-h, 0] \). Then by assumption there exist points \( \xi_1, \xi_2 \in [0, T], \xi_1 < \xi_2, \) such that \( g(t) = 0 \text{ for } t \in [-h, \xi_1] \) and \( g(t) > 0 \text{ for } t \in (\xi_1, \xi_2] \). From inequality (5), it follows that \( \dot{g}(t) \leq 0 \text{ for } t \in [\xi_1, \xi_2] \cap [\xi_1, \xi_1 + h], \ t \neq t_i \), which together with inequality (6) shows that the function \( g(t) \) is monotone nonincreasing in the interval \( [\xi_1, \xi_2] \cap [\xi_1, \xi_1 + h], \) i.e. \( g(t) \leq g(\xi_1) = 0 \text{ for } t \in [\xi_1, \xi_2] \cap [\xi_1, \xi_1 + h]. \) The last inequality contradicts the choice of points \( \xi_1 \) and \( \xi_2. \)

Case 2: Let \( g(0) < 0. \) By assumption and inequality (7) there exists a point \( \eta \in (0, T], \eta \neq t_i \ (i = 1, \ldots, p), \) such that \( g(t) \leq 0 \text{ for } t \in [-h, \eta], \ g(\eta) = 0 \) and \( g(t) > 0 \text{ for } t \in (\eta, \eta + \epsilon) \) where \( \epsilon > 0 \) is a sufficiently small number. Introduce the notation \( \inf \{ g(t) : t \in [-h, \eta] \} = -\lambda, \ \lambda = \text{const} > 0. \) Then there are two possibilities:

Case 2.1: Let a point \( \rho \in [0, \eta] \) and \( \rho \neq t_i \ (i = 1, \ldots, p) \) such that \( g(\rho) = -\lambda. \) For the sake of definiteness, let \( \rho \in (t_k, t_{k+1}], \) and \( \eta \in (t_{k+m}, t_{k+m+1}], \) \( m \geq 0. \) Choose a point \( \eta_1 \in (t_{k+m}, t_{k+m+1}], \) \( \eta_1 < \eta \) such that \( g(\eta_1) > 0. \) By the mean value theorem, the following equations are valid.

\[ g(\eta_1) - g(t_{k+m} + 0) = \dot{g}(\xi_m)(\eta_1 - t_{k+m}) \]
\begin{align*}
g(t_k + m - 0) - g(t_k + m - 1 + 0) &= \dot{g}(\xi_{m - 1})(t_k + m - t_k + m - 1) \\
&\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)
\end{align*}

where \( \xi_0 \in (\rho, t_k + 1), \xi_m \in (t_k + m, \eta_1), \xi_i \in (t_k + i, t_k + i + 1), i = 1, \ldots, m - 1. \)

From (6) and (9) we obtain the inequalities

\begin{align*}
g(t_k + m) - (1 - L_{k + m})g(t_k + m) &\leq \dot{g}(\xi_m)r, \\
g(t_k + m) - (1 - L_{k + m - 1})g(t_k + m - 1) &\leq \dot{g}(\xi_{m - 1})r, \\
&\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)
\end{align*}

From inequalities (10), by means of elementary transformations, we obtain the inequalities

\begin{align*}
g(\eta_1) - (1 - L_{k + 1})(1 - L_{k + 2}) \ldots (1 - L_{k + m})g(\rho) &\leq \dot{g}(\xi_m) - (1 - L_{k + m})\dot{g}(\xi_{m - 1}) + \ldots + \\
&\quad (1 - L_{k + m})(1 - L_{k + m - 1}) \ldots (1 - L_{k + 1})\dot{g}(\xi_0)r. \quad (11)
\end{align*}

Inequalities (6) and (11) and the choice of the points \( \rho \) and \( \eta_1 \) imply the inequality

\((1 - L)^m \lambda < [1 + (1 - L_{k + m}) + \ldots + (1 - L_{k + m})(1 - L_{k + m - 1}) + \ldots (1 - L_{k + 1})](M + N\xi_0 h)r\lambda \)

or

\(1 < \frac{(M + N\xi_0 h)}{(1 - L)^p} r. \quad (12)\)

Inequality (12) contradicts inequality (8).

Case 2.2: Let a point \( t_\kappa \in [0, \eta] \) exist such that \( g(t_\kappa + 0) < g(t) \) for \( t \in [0, \eta] \), i.e. \( g(t_\kappa + 0) = -\lambda \). By arguments analogous to those in Case 2.1, where \( \rho = t_\kappa + 0 \), we again obtain a contradiction.

Case 3: Let \( g(0) = 0 \) and \( g(t) \leq 0, g(t) \neq 0 \) for \( t \in (0, b) \) where \( b > 0 \) is a sufficiently small number. By arguments analogous to those in Case 2 we obtain a contradiction.

This completes the proof of Lemma 1.
Theorem 1: Let the following conditions hold:

1. The couple of functions \( v,w \in G([\alpha, T], \mathbb{R}^n) \), \( v,w \in G^1([\alpha, T], \mathbb{R}^n) \) is a couple of lower and upper quasisolutions of the initial value problem (1) and satisfies the inequalities \( v(t) \leq w(t) \) for \( t \in [\alpha, T] \) and \( v(0) - \varphi(0) \leq v(t) - \varphi(t), w(0) - \varphi(0) \geq w(t) - \varphi(t) \) for \( t \in [\alpha, T] \).

2. The functions \( \kappa_j \in C([0, T] \times [-\alpha, T], [0, \infty)), j = 1, \ldots, n. \)

3. The function \( f \in C([0, T] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n), f = (f_1, f_2, \ldots, f_n), f_j(t, x, y) = f_j(t, x, [x]_{p_j}, [x]_{q_j}, y, [y]_{p_j}, [y]_{q_j}) \) is monotone nondecreasing with respect to \( [x]_{p_j} \) and \( [y]_{q_j} \) and monotone nonincreasing with respect to \( x, y \in S(v, w), y(t) \leq x(t) \) satisfies the inequalities

\[
\begin{align*}
&f_j(t, x, [x]_{p_j}, [x]_{q_j}, Q_j, [Q_j]_{p_j}, [Q_j]_{q_j}) \\
&- f_j(t, y, [x]_{p_j}, [x]_{q_j}, Q_j, [Q_j]_{p_j}, [Q_j]_{q_j}) \\
&\geq -M_j(x - y) - N_j(Q - Q_j), j = 1, \ldots, n,
\end{align*}
\]

where \( M_j, N_j \) are positive constants.

4. The functions \( I_i \in C(\mathbb{R}^n, \mathbb{R}^n), I_i = (I_{i1}, I_{i2}, \ldots, I_{in}), (i = 1, \ldots, p), I_{ij}(x) = I_{ij}(x, [x]_{p_j}, [x]_{q_j}) \) are monotone nondecreasing with respect to \( x, y \in S(v, w), y(t) \leq x(t) \) satisfy the inequalities

\[
\begin{align*}
&I_{ij}(x(t), [x(t)]_{p_j}, [x(t)]_{q_j}) - I_{ij}(y(t), [x(t)]_{p_j}, [x(t)]_{q_j}) \\
&\geq -L_{ij}(x(t) - y(t)), j = 1, \ldots, n, i = 1, \ldots, p,
\end{align*}
\]

where \( L_{ij} \) are nonnegative constants, \( L_{ij} < 1. \)

5. The inequalities

\[
(M_j + N_j \kappa_0 h) \tau p \leq (1 - L_i)^p, j = 1, \ldots, n
\]

hold, where

\[
\kappa_0 = \max(\kappa_j(t, s): t \in [0, T], s \in [-h, T]),
\]

\[
\tau = \max\{t_1, T - t_p, \max[t_{i+1} - t_i: i = 1, 2, \ldots, p - 1]\},
\]

\[
L_i = \max\{L_{ij}: i = 1, 2, \ldots, p\}.
\]

Then there exist two monotone sequences of functions \( \{v(\kappa)(t)\}_0^{\infty} \) and \( \{w(\kappa)(t)\}_0^{\infty} \), \( v(0)(t) \equiv v(t), w(0)(t) \equiv w(t) \) which are uniformly convergent in the interval \([\alpha, T]\) and their limits \( \bar{v}(t) = \lim_{\kappa \to \infty} v(\kappa)(t) \) and \( \bar{w}(t) = \lim_{\kappa \to \infty} w(\kappa)(t) \) are a couple of minimal and maximal
quasisolutions of the initial value problem (1). Moreover, if \( u(t) \) is any solution of the initial value problem (1) such that \( u \in S(v, w) \), then the inequalities \( \bar{v}(t) \leq u(t) \leq \bar{w}(t) \) hold for \( t \in [-h,T] \).

**Proof:** Fix two functions \( \eta, \mu \in S(v, w), \eta(\eta_1, \eta_2, \ldots, \eta_n), \mu = (\mu_1, \mu_2, \ldots, \mu_n) \). Consider the initial value problems for the linear impulsive integro-differential equations

\[
\begin{align*}
\dot{x}_j(t) + M_j x_j(t) + N_j \int_{t-h}^t \kappa_j(t,s)x_j(s)ds = \sigma_j(t,\eta,\mu) & \quad \text{for } t \neq t_i, t \in [0,T] \\
\Delta x_j |_{t=t_i} = -L_{ij} x_j(t_i) + \gamma_{ij}(\eta,\mu) & \quad \text{for } t \in [-h,0]
\end{align*}
\]

where

\[
\begin{align*}
\sigma_j(t,\eta,\mu) &= f_j(t,\eta_j,\eta(t),\eta(t),\eta(t),\eta(t),\eta(t),\eta(t),\eta(t),\eta(t)) \\
& \quad + M_j \eta_j(t) + N_j \eta(t), \\
\gamma_{ij}(\eta,\mu) &= I_{ij}(\eta_j(t),\eta(t),\eta(t),\eta(t),\eta(t),\eta(t),\eta(t),\eta(t),\eta(t),\eta(t)) + L_{ij} \eta_j(t).
\end{align*}
\]

The initial value problem (13)-(15) has a unique solution for any fixed couple of functions \( \eta, \mu \in S(v, w) \).

Define the map \( A: S(v, w) \times S(v, w) \rightarrow S(v, w) \) by the equality \( A(\eta, \mu) = x \), where \( x = (x_1, x_2, \ldots, x_n) \) and \( x_j(t) \) is the unique solution of the initial value problem (13)-(15) for the couple of functions \( \eta, \mu \in S(v, w) \).

We shall prove that \( v \leq A(v, w) \). Introduce the notations \( x^{(1)} = A(v, w), g = v - x^{(1)}, g = (g_1, g_2, \ldots, g_n) \). Then the following inequalities hold:

\[
\begin{align*}
\dot{g}_j(t) &= v - x^{(1)} \leq f_j(t, v_j, \eta_j, \eta(t), \eta(t), \eta(t), \eta(t), \eta(t), \eta(t), \eta(t)) \\
& \quad + M_j x^{(1)} + N_j \eta(t) - \sigma_j(t, \eta, \mu) \\
& = -M_j g_j(t) - N_j \int_{t-h}^t \kappa_j(t,s)g_j(s)ds \quad \text{for } t \neq t_i, t \in [0,T], \\
\Delta g_j |_{t=t_i} & \leq I_{ij}(v_j(t_i), \eta(t_i), \eta(t_i), \eta(t_i), \eta(t_i), \eta(t_i), \eta(t_i), \eta(t_i), \eta(t_i), \eta(t_i)) + L_{ij} x^{(1)}(t_i) - \gamma_{ij}(v, \mu) \\
& = -L_{ij} g_j(t_i), \\
g_j(0) & \leq g_j(t) \leq 0 \quad \text{for } t \in [-h,0], j = 1, \ldots, n.
\end{align*}
\]
By Lemma 1, the functions $g_j(t)$, $j = 1, \ldots, n$ are nonpositive, i.e. $v \leq A(v, w)$. In an analogous way it is proved that $w \geq A(v, w)$.

Let $\eta, \mu \in S(v, w)$ be such that $\eta(t) \leq \mu(t)$ for $t \in [-h, T]$. Set $x^{(1)} = A(\eta, \mu)$, $x^{(2)} = A(\mu, \eta)$. For $g = (g_1, t_2, \ldots, g_n)$, $g = (g_1, t_2, \ldots, g_n)$. By Lemma 1 the functions $g_j(t)$, $j = 1, \ldots, n$, are nonpositive, i.e. $A(\eta, \mu) \leq A(\mu, \eta)$.

Define the sequences of functions $\{v^{(\kappa)}(t)\}_{0}^{\infty}$ and $\{w^{(\kappa)}(t)\}_{0}^{\infty}$ by the equations

$$v^{(0)}(t) \equiv v(t), \quad w^{(0)}(t) \equiv w(t),$$
$$v^{(\kappa+1)}(t) = A(v^{(\kappa)}, w^{(\kappa)}), \quad w^{(\kappa+1)}(t) = A(w^{(\kappa)}, v^{(\kappa)}).$$

The functions $v^{(\kappa)}(t)$ and $w^{(\kappa)}(t)$ for $t \in [-h, T]$ and $\kappa \geq 0$ satisfy the inequalities

$$v^{(0)}(t) \leq v^{(1)}(t) \leq \ldots \leq v^{(\kappa)}(t) \leq \ldots \leq w^{(\kappa)}(t) \leq \ldots \leq w^{(0)}(t). \quad (17)$$

Hence the sequences of functions $\{v^{(\kappa)}(t)\}_{0}^{\infty}$ and $\{w^{(\kappa)}(t)\}_{0}^{\infty}$ are uniformly convergent for $t \in [-h, T]$. Introduce the notation $\overline{v}(t) = \lim_{k \to \infty} v^{(k)}(t)$ and $\overline{w}(t) = \lim_{k \to \infty} w^{(k)}(t)$. We shall show that the couple of functions $(\overline{v}, \overline{w})$ is a couple of minimal and maximal quasisolutions of the initial value problem (1). From the definitions of the functions $v^{(\kappa)}(t)$ and $w^{(\kappa)}(t)$, it follows that these functions satisfy the initial value problem

$$\dot{v}^{(\kappa+1)}_j + M_j v^{(\kappa+1)}_j + N_j Q_j v^{(\kappa+1)} = \sigma_j(t, v^{(\kappa)}, w^{(\kappa)}) \quad \text{for } t \neq t_i, \ t \in [0, T]$$
$$\dot{w}^{(\kappa+1)}_j + M_j w^{(\kappa+1)}_j + N_j Q_j w^{(\kappa+1)} = \sigma_j(t, w^{(\kappa)}, v^{(\kappa)}), \quad (18)$$
$$\Delta v_j^{(\kappa+1)} \big|_{t = t_i} = - L_{ij} v_j^{(\kappa+1)}(t_i) + \gamma_{ij}(v^{(\kappa)}, w^{(\kappa)})$$
$$\Delta w_j^{(\kappa+1)} \big|_{t = t_i} = - L_{ij} w_j^{(\kappa+1)}(t_i) + \gamma_{ij}(w^{(\kappa)}, v^{(\kappa)}), \quad (19)$$
$$v_j^{(\kappa+1)}(t) = w_j^{(\kappa+1)}(t) = \varphi_j(t) \quad \text{for } t \in [-h, 0], \ j = 1, \ldots, n. \quad (20)$$

We pass to the limit in equations (18)-(20) and obtain that the functions $\overline{v}(t)$ and $\overline{w}(t)$ are a couple of quasisolutions of the initial value problem (1). From inequalities (17) it follows that the inequality $\overline{v}(t) \leq \overline{w}(t)$ holds for $t \in [-h, T]$.

Let $\zeta, \xi \in S(v, w)$ be a couple of quasisolutions of problem (1). From inequalities (17) it follows that there exists an integer $\kappa \geq 1$ such that $v^{(\kappa-1)}(t) \leq \zeta(t) \leq w^{(\kappa-1)}(t)$ and $v^{(\kappa-1)}(t) \leq \xi(t) \leq w^{(\kappa-1)}(t)$ for $t \in [-h, T]$. Introduce the notation $g(t) = v^{(\kappa)}(t) - \zeta(t)$, $g = (g_1, g_2, \ldots, g_n)$. By Lemma 1, the inequality $g_j(t) \leq 0$ holds for $t \in [-h, T], \ j = 1, \ldots, n$, i.e. $v^{(\kappa)}(t) \leq \zeta(t)$. 
In an analogous way, it is proved that the inequalities $\zeta(t) \leq w^{(\kappa)}(t)$ and $v^{(\kappa)}(t) \leq z(t) \leq w^{(\kappa)}(t)$ hold for $t \in [-h, T]$, which shows that the couple of functions $(\overline{v}, \overline{w})$ is a couple of minimal and maximal quasisolutions of the initial value problem (1).

Let $u(t)$ be a solution of (1) such that $u \in S(v, w)$. Consider the couple of functions $(u, u)$ which is a couple of quasisolutions of problem (1). By what was proved above, the inequalities $\overline{v}(t) \leq u(t) \leq \overline{w}(t)$ hold for $t \in [-h, T]$.

This completes the proof of Theorem 1.

In the case when (1) is an initial value problem for a scalar impulsive integro-differential equation, the following theorem is valid.

**Theorem 2:** Let the following conditions hold:

1. The functions $v, w \in G([-h, T], \mathbb{R})$, $v, w \in G^{1}([0, T], \mathbb{R})$ are a couple of lower and upper solutions of the initial value problem (1) and satisfy the inequalities $v(t) \leq w(t)$ for $t \in [-h, T]$ and $v(0) - \varphi(0) \leq v(t) - \varphi(t), w(0) - \varphi(0) \geq w(t) - \varphi(t)$ for $t \in [-h, 0]$.
2. The function $\kappa(t, s) \in C([0, T] \times [-h, T], [0, \infty))$.
3. The function $f \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ satisfies for $x, y \in S(v, w)$, $y(t) \leq z(t)$ the inequality
   \[
   f(t, x(t), \int_{t-h}^{t} \kappa(t, s)x(s)ds) - f(t, y(t), \int_{t-h}^{t} \kappa(t, s)y(s)ds) \geq -M(x(t) - y(t)) - N \int_{t-h}^{t} \kappa(t, s)(x(s) - y(s))ds,
   \]
   where $M$ and $N$ are positive constants.
4. The function $I_{i} \in C(\mathbb{R}, \mathbb{R})$ ($i = 1, \ldots, p$) satisfies for $x, y \in S(v, w), y(t_{i}) \leq x(t_{i})$ the inequality $I_{i}(x(t_{i})) - I_{i}(y(t_{i})) \geq -L_{i}(x(t_{i}) - y(t_{i})), i = 1, \ldots, p$ where $L_{i}$ ($i = 1, \ldots, p$) are nonnegative constants such that $L_{i} < 1$.
5. The inequality
   \[
   (M + N\kappa_{0}h)\tau < (1 - L)^{p}
   \]
   holds, where
   \[
   \kappa_{0} = \max\{\kappa(t, s); t \in [0, T], s \in [-h, T]\},
   \]
   \[
   \tau = \max\{t_{1}, T - t_{p}, \max[t_{i+1} - t_{i}; i = 1, 2, \ldots, p - 1]\},
   \]
   \[
   L = \max\{L_{i}; i = 1, 2, \ldots, n\}.
   \]

Then there exist two sequences of functions $\{v^{(\kappa)}(t)\}_{0}^{\infty}$ and $\{w^{(\kappa)}(t)\}_{0}^{\infty}$ which are uniformly
convergent in the interval \([-h,T]\) and their limits \(v(t) = \lim_{\kappa \to \infty} v^{(\kappa)}(t)\) and \(\bar{w}(t) = \lim_{\kappa \to \infty} w^{(\kappa)}(t)\)

are a couple of minimal and maximal solutions of the initial value problem (1).

The proof of Theorem 2 is analogous to the proof of Theorem 1.

REFERENCES


