THE PROBABILISTIC APPROACH TO THE ANALYSIS OF THE LIMITING BEHAVIOR OF AN INTEGRO-DIFFERENTIAL EQUATION DEPENDING ON A SMALL PARAMETER, AND ITS APPLICATION TO STOCHASTIC PROCESSES

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ABSTRACT

Using connection between stochastic differential equation with Poisson measure term and its Kolmogorov's equation, we investigate the limiting behavior of the Cauchy problem solution of the integro-differential equation with coefficients depending on a small parameter. We also study the dependence of the limiting equation on the order of the parameter.

Key words: Stochastic process, Kolmogorov's averaging, integro-differential equation, Cauchy problem, limiting behavior, small parameters, white and Poisson noise.

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It is well known that investigation of a nonlinear oscillating systems with a small stochastic white noise at the input, can be accomplished applying the averaging method for Kolmogorov's parabolic equation with coefficients depending on a small parameter [1]. If both white and Poisson types of noise are present, then the corresponding Kolmogorov's equation is integro-differential [2],

and we shall extend here the averaging principle to such equations.

Let us study behavior, as \( \epsilon \to 0 \), of the following equation

\[
\frac{\partial}{\partial t} U(t, x) + \epsilon^{k_1}(f(t, x), \nabla U(t, x)) + \frac{\epsilon^{k_2}}{2} \text{Tr}(g(t, x)g^*(t, x) \nabla^2 U(t, x)) \\
+ \int_{R^d} [U(t, x + \epsilon^{k_3}q(t, x, y)) - U(t, x) - \epsilon^{k_3}(q(t, x, y), \nabla U(t, x))] \Pi(dy) = 0, \\
(t, x) \in [0, T) \times R^d,
\]

where \( \epsilon > 0 \) is a small parameter and \( k_1, k_2, k_3 \), are some positive numbers, and

\[
\nabla U(t, x) = \left\{ \frac{\partial U(t, x)}{\partial x_i}, i = 1, \ldots, d \right\}, \quad \nabla^2 U(t, x) = \left\{ \frac{\partial^2 U(t, x)}{\partial x_i \partial x_j}, i, j = 1, \ldots, d \right\}.
\]

Here \( \Pi \) is a finite measure on Borel sets in \( R^d \), \( f(t, x), q(t, x, y) \) are \( d \)-dimensional vectors, and \( g(t, x) \) is a \( d \times d \) square matrix.

**Lemma:** If

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int A b(t, x) dt = \overline{b}(x)
\]

uniformly with respect to \( A \) for each \( x \), the function \( b(x) \) is continuous, and \( b(t, x) \) is continuous in \( x \) uniformly with respect to \( (t, x) \) in arbitrary compact \( |x| \leq C \), and stochastic process \( \xi(t) \) is continuous, then

\[
\lim_{\epsilon \to 0} \int_0^t b(\frac{\tau}{\epsilon}, \xi(\tau)) d\tau = \int_0^t \overline{b}(\xi(\tau)) d\tau.
\]

The proof is similar to that in [2].

Now, replacing \( t \) with \( t/\epsilon^k \) in (1), where \( k = \min(k_1, k_2, k_3) \), and denoting \( V_\epsilon(t, x) = U(t/\epsilon^k, x) \), we can derive the following equation:

\[
\frac{\partial}{\partial t} V_\epsilon(t, x) + \epsilon^{k_1-k}(f(t/\epsilon^k, x), \nabla V_\epsilon(t, x)) + \frac{\epsilon^{k_2-k}}{2} \text{Tr}(g(t/\epsilon^k, x)g^*(t/\epsilon^k, x) \nabla^2 V_\epsilon(t, x)) \\
+ \frac{1}{\epsilon^k} \int_{R^d} [V_\epsilon(t, x + \epsilon^{k_3}q(t/\epsilon^k, x, y)) - V_\epsilon(t, x) - \epsilon^{k_3}(q(t/\epsilon^k, x, y), \nabla V_\epsilon(t, x))] \Pi(dy) = 0, \\
(t, x) \in [0, T) \times R^d.
\]
The Probabilistic Approach to the Analysis of the Limiting Behavior

**Theorem:** Let the following conditions hold:

1) the functions $f(t, x), g(t, x), q(t, x, y)$ are continuous in $(t, x)$, bounded and twice continuously differentiable with respect to $x$, with derivatives also bounded;

2) uniformly with respect to $A$ for each $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ there exists the following three limits

$$
\lim_{s \to A} \frac{1}{s-A} \int_{s-A}^{s+A} f(t, x) dt = \bar{f}(x), \quad \lim_{s \to A} \frac{1}{s-A} \int_{s-A}^{s+A} g(t, x) g^*(t, x) dt = \bar{G}(x),
$$

and

$$
\lim_{s \to A} \frac{1}{s-A} \int_{s-A}^{s+A} q(t, x, y) q^*(t, x, y) dt = \bar{Q}(x, y).
$$

3) The functions $\bar{f}(x), \bar{G}(x), \bar{Q}(x, y)$ satisfy the Lipschitz condition in $x$, and the matrix

$$
\bar{B}(x) = \bar{G}(x) + \int_{\mathbb{R}^d} Q(x, y) \Pi(dy)
$$

is uniformly parabolic.

Then,

a) if $k_1 = k_2 = 2k_3$ and $V_\epsilon(t, x)$ satisfies (2) and the “Cauchy” condition

$$
\lim_{t \to T} V_\epsilon(t, x) = F(x), \quad F(x) \in C^2(\mathbb{R}^d),
$$

then $\lim_{\epsilon \to 0} V_\epsilon(t, x) = \bar{V}(t, x)$, where $\bar{V}(t, x)$ is a solution of the problem:

$$
\frac{\partial}{\partial t} \bar{V}(t, x) + (\bar{f}(x), \nabla \bar{V}(t, x)) + \frac{1}{2} \text{Tr}(\bar{B}(x) \nabla^2 \bar{V}(t, x)) = 0,
$$

$$
\lim_{t \to T} \bar{V}(t, x) = F(x).
$$

b) If $k < k_1$, then $V$ satisfies (4)-(5) but in this case there is no term containing $\bar{f}(x)$ in (4); Similarly, if $k < k_2$, then $\bar{B}(x)$ does not depend on $\bar{G}(x)$; and if $k < 2k_3$, then $\bar{B}(x)$ does not contain the term

$$
\int_{\mathbb{R}^d} \bar{Q}(x, y) \Pi(dy).
$$

**Proof:** Applying the results of [2-3] to the conditions of the theorem, it follows that the solution of the problem (2)-(3) exists for each $\epsilon$, is unique and can be represented in the form
\[ V_\varepsilon(t, x) = \mathbb{E}[F(\xi_\varepsilon(t, x, T))], \]

where \( \xi_\varepsilon(t, x, T) \) is the solution of the stochastic equation

\[
\xi_\varepsilon(t, x, s) = x + \varepsilon^{k_1-k} \int_t^s f(\tau/\varepsilon^k, \xi_\varepsilon(t, x, \tau)) d\tau \\
+ \varepsilon^{k_2-k} \int_t^s g(\tau/\varepsilon^k, \xi_\varepsilon(t, x, \tau)) d\omega(\tau) \\
+ \varepsilon^{k_3-k} \int_t^s \int_{R^d} q(\tau/\varepsilon^k, \xi_\varepsilon(t, x, \tau), y) \nu(\tau, dy),
\]

where \( \omega(t) \) is a \( d \)-dimensional Wiener process, \( \nu(t, A) \) is a Poisson measure independent of \( \omega \), \( A \) is a Borel set of \( R^d \), and

\[
\nu(t, A) = \nu(t/\varepsilon^k, A) - t \Pi(A)/\varepsilon^k; \quad \mathbb{E} \nu(t, A) = t \Pi(A).
\]

Let

\[
\zeta_\varepsilon(t, x, s) = \varepsilon^{k_2-k} \int_t^s g(\tau/\varepsilon^k, \xi_\varepsilon(t, x, \tau)) d\omega(\tau) \\
+ \varepsilon^{k_3-k} \int_t^s \int_{R^d} q(\tau/\varepsilon^k, \xi_\varepsilon(t, x, \tau), y) \nu(\tau, dy).
\]

Then we can obtain the following estimates:

\[
\mathbb{E} |\xi_\varepsilon(t, x, s)|^2 \leq C[|x^2 + (\varepsilon^{2(k_1-k)} + \varepsilon^{k_2-k} + \varepsilon^{k_3-k})| s - t |],
\]

\[
\mathbb{E} |\xi_\varepsilon(t, x, s)|^2 \leq C(\varepsilon^{k_2-k} + \varepsilon^{k_3-k}) | s - t |,
\]

\[
\mathbb{E} |\xi_\varepsilon(t, x, s_2) - \xi_\varepsilon(t, x, s_1)|^2 \leq C[\varepsilon^{2(k_1-k)} | s_2 - s_1 |^2 + (\varepsilon^{k_2-k} + \varepsilon^{k_3-k}) | s_2 - s_1 |],
\]

\[
\mathbb{E} |\zeta_\varepsilon(t, x, s_2) - \zeta_\varepsilon(t, x, s_1)|^2 \leq C(\varepsilon^{k_2-k} + \varepsilon^{k_3-k}) | s_2 - s_1 |.
\]

From these estimates we infer that the family of processes \((\xi_\varepsilon(t, x, s), \zeta_\varepsilon(t, x, s))\) satisfies the Skorokhod’s compactness conditions [4]. Therefore, for any sequence \( \varepsilon \to 0 \) there exists a subsequence \( \varepsilon_m \to 0, m = 1, 2, \ldots \), and processes \( \tilde{\xi}(t, x, s), \tilde{\zeta}(t, x, s) \) such that \( \xi_{\varepsilon_m}(t, x, s) \to \xi(t, x, s), \ \zeta_{\varepsilon_m}(t, x, s) \to \zeta(t, x, s) \) in probability as \( \varepsilon_m \to 0 \). From (6) we can also find that

\[
\xi_{\varepsilon_m}(t, x, s) = x + \varepsilon_m^{k_1-k} \int_t^s f(\tau/\varepsilon_m^k, \xi_{\varepsilon_m}(t, x, \tau)) d\tau + \zeta_{\varepsilon_m}(t, x, s). \tag{7}
\]

Then, for any fixed \((t, x) \in [0, T]\) we have:
Therefore,

\[ E | \xi(t, x, s_2) - \xi(t, x, s_1) |^4 \leq C[\epsilon^{4(k_1 - k)} | s_2 - s_1 |^4 + E | \zeta(t, x, s_2) - \zeta(t, x, s_1) |^4], \]

\[ E | \zeta(t, x, s_2) - \zeta(t, x, s_1) |^4 \leq C[(\epsilon^{2(k_2 - k)} + \epsilon^{2(2k_3 - k)}) | s_2 - s_1 |^2 + \epsilon^{4k_3 - 3k/2} | s_2 - s_1 |]. \]

Therefore,

\[ E | \xi(t, x, s_2) - \xi(t, x, s_1) |^4 \leq C[| s_2 - s_1 |^4 + | s_2 - s_1 |^2], \]

\[ E | \zeta(t, x, s_2) - \zeta(t, x, s_1) |^4 \leq C | s_2 - s_1 |^2, \]

and the processes \( \xi(t, x, s), \zeta(t, x, s) \) satisfy the Kolmogorov’s continuity condition on \( s \) [5].

\( a) \) Let us consider the case \( k_1 = k_2 = 2k_3 \). Then from (7) we obtain:

\[ \xi(t, x, s) = x + \int_t^s f(\tau/e^k, \xi(t, x, \tau))d\tau + \zeta(t, x, s). \]

From this point we shall omit the subindex \( m \) in \( \epsilon_m \) for simplicity. Then for each fixed \( (t, x) \in [0, T] \) the process

\[ \zeta(t, x, s) = \int_t^s g(\tau/e^k, \xi(t, x, \tau))d\omega(\tau) + e^{k/2} \int_t^s \int_{R^d} q(\tau/e^k, \xi(t, x, \tau), y)\tilde{\nu}(d\tau, dy) \]

is a vector-valued martingale with matrix characteristic

\[ \langle \zeta(t, x, s), \zeta(t, x, s) \rangle = \int_t^s g(\tau/e^k, \xi(t, x, \tau))g^*(\tau/e^k, \xi(t, x, \tau))d\tau \]

\[ + \int_t^s \int_{R^d} q(\tau/e^k, \xi(t, x, \tau), y)q^*(\tau/e^k, \xi(t, x, \tau), y)\Pi(dy)\,d\tau. \]

Using the above lemma, it is easy to show that

\[ P - \lim_{\epsilon \to 0} \int_t^s f(\tau/e^k, \xi(t, x, \tau))d\tau = \int_t^s f(\zeta(t, x, \tau))d\tau, \]

and

\[ P - \lim_{\epsilon \to 0} \langle \zeta(t, x, s), \zeta(t, x, s) \rangle = \int_t^s \tilde{B}(\zeta(t, x, \tau))d\tau. \]

Hence, from (8), (9), and (10) we obtain a continuous square integrable vector-valued martingale

\[ \zeta(t, x, s) = x + \int_t^s f(\zeta(t, x, \tau))d\tau + \tilde{\zeta}(t, x, s), \]

with matrix characteristic
It follows from [6] that there exists a d-dimensional Wiener process \( w(t) \) such that
\[
\bar{\zeta}(t, x, s) = \int_t^s \bar{\sigma}(\zeta(t, x, \tau))d\bar{w}(\tau),
\]
where
\[
\bar{\sigma}(x) = \bar{B}(x).
\]
Consequently, the process \( \bar{\xi}(t, x, s) \) satisfies the equation which, according [2], has a unique solution:
\[
\bar{\xi}(t, x, s) = x + \int_t^s \bar{f}(\bar{\xi}(t, x, \tau))d\tau + \int_t^s \bar{\sigma}(\bar{\xi}(t, x, \tau))d\bar{w}(\tau). \tag{11}
\]

The matrix \( \bar{B}(x) \) is positive definite for all \( x \in \mathbb{R}^d \), satisfies Lipschitz conditions, and therefore matrix \( \bar{\sigma}(x) \) satisfies Lipschitz condition as well. Then, using the Lebesgue dominated convergence theorem, we obtain
\[
\lim_{\epsilon_m \to 0} V_{\epsilon_m}(t, x) = \bar{V}(t, x) = E[F(\bar{\xi}(t, x, T))]
\]
for any sequence \( \epsilon_m \to 0 \). But as it follows from [7] the function \( \bar{V}(t, x) \) is a unique solution of the problem (4)-(5), which completes the proof of the part a) of the theorem.

b) When \( k < k_1 \), the boundedness of \( f(t, x) \) implies that
\[
E | \int_t^s f(\tau/e^k, \xi_e(t, x, \tau))d\tau | \leq C
\]
and therefore the second term in the right side of (6) converges to 0 with \( \epsilon \to 0 \) in probability. The matrix characteristic of the martingale \( \zeta_e(t, x, s) \) in (7) has the form
\[
\langle \zeta_e, \zeta_e \rangle_e = e^{k_2 - k} \int_t^s g(\tau/e^k, \xi_e(t, x, \tau))g^*(\tau/e^k, \xi_e(t, x, \tau))d\tau
\]
\[
+ e^{k_3 - k} \int_t^s \int_{\mathbb{R}^d} g(\tau/e^k, \xi_e(t, x, \tau), y)g^*(\tau/e^k, \xi_e(t, x, \tau), y)\Pi(dy)d\tau.
\]
that either first or second term in the right side of (12) converges to 0 (respectively to the $k < k_2$ or $k < 2k_3$ case) as $\epsilon \to 0$, which allows to complete the proof of the theorem as in part a).

REFERENCES


