THE METHOD OF LOWER AND UPPER SOLUTIONS FOR
nth-ORDER PERIODIC BOUNDARY VALUE PROBLEMS\textsuperscript{1, 2}

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ABSTRACT

In this paper we develop the monotone method in the presence of lower and upper solutions for the problem

\[ u^{(n)}(t) = f(t, u(t)); u^{(i)}(a) - u^{(i)}(b) = \lambda_i \in \mathbb{R}, \ i = 0, \ldots, n - 1, \]

where \( f \) is a Carathéodory function. We obtain sufficient conditions for \( f \) to guarantee the existence and approximation of solutions between a lower solution \( \alpha \) and an upper solution \( \beta \) for \( n \geq 3 \) with either \( \alpha \leq \beta \) or \( \alpha \geq \beta \).

For this, we study some maximum principles for the operator \( Lu \equiv u^{(n)} + Mu \). Furthermore, we obtain a generalization of the method of mixed monotonicity considering \( f \) and \( u \) as vectorial functions.

\textbf{Key words:} Periodic boundary value problem, lower and upper solutions, monotone method.

\textbf{AMS (MOS) subject classifications:} 34B15, 34C25.

1. INTRODUCTION

In this paper we study the following class of boundary value problems for the ordinary differential equations:

\[ u^{(n)}(t) = f(t, u(t)) \text{ for a.e. } t \in I = [a, b] \]

\[ u^{(i)}(a) - u^{(i)}(b) = \lambda_i \in \mathbb{R}; \ i = 0, 1, \ldots, n - 1, \]

for \( n \geq 3 \) where \( f \) is a Carathéodory function.

\textbf{Definition 1.1:} We say that \( f: I \times \mathbb{R}^l \rightarrow \mathbb{R}^m \) is a Carathéodory function, if \( f \equiv (f_1, \ldots, f_m) \) satisfies the following properties:

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1. \( f_i(\cdot, x) \) is measurable for all \( x \in \mathbb{R}^l \) and \( i \in \{1, \ldots, m\} \).
2. \( f_i(t, \cdot) \) is continuous for a.e. \( t \in I \).
3. For every \( R > 0 \) and \( i \in \{1, \ldots, m\} \), there exists \( h_{i,R} \in L^1(I) \) such that:
   \[
   |f_i(t, x)| \leq h_{i,R}(t) \text{ for a.e. } t \in I,
   \]
   with \( \| x \| \leq R \).

To develop the monotone method we use the concept of lower and upper solutions:

**Definition 1.2:** Let \( \alpha \in W^{n,1}(I) \), we say that \( \alpha \) is a lower solution for the problem (1.1)-(1.2) if \( \alpha \) satisfies

\[
\alpha^{(n)}(t) \geq f(t, \alpha(t)) \text{ for a.e. } t \in I
\]

\[
\alpha^{(i)}(a) - \alpha^{(i)}(b) = \lambda_i, \quad i = 0, 1, \ldots, n - 2
\]

\[
\alpha^{(n-1)}(a) - \alpha^{(n-1)}(b) \geq \lambda_{n-1}.
\]

**Definition 1.3:** Let \( \beta \in W^{n,1}(I) \), we say that \( \beta \) is an upper solution for the problem (1.1)-(1.2) if \( \beta \) satisfies

\[
\beta^{(n)}(t) \leq f(t, \beta(t)) \text{ for a.e. } t \in I
\]

\[
\beta^{(i)}(a) - \beta^{(i)}(b) = \lambda_i, \quad i = 0, 1, \ldots, n - 2
\]

\[
\beta^{(n-1)}(a) - \beta^{(n-1)}(b) \leq \lambda_{n-1}.
\]

We suppose that \( f \) satisfies one of the following conditions, depending on various circumstances:

(H_1) \( f(t, x) - f(t, y) \leq M(x - y) \) for a.e. \( t \in I \) with \( \alpha(t) \leq y \leq x \leq \beta(t) \) and \( M > 0 \).

(H_2) \( f(t, x) - f(t, y) \geq M(x - y) \) for a.e. \( t \in I \) with \( \beta(t) \leq y \leq x \leq \alpha(t) \) and \( M < 0 \).

This problem has been studied by different authors for second order equations when \( \alpha \leq \beta \) ([1]-[4], [6], [8], [10], [11]). If \( \alpha \geq \beta \) the monotone method is not valid if \( f \) satisfies the condition \( (H_2) \) for some \( M < 0 \) ([2], [7], [12], [14]).

For \( n \geq 3 \) the method of lower and upper solutions has been little studied ([2], [9], [13]). In [2] the author obtains the best value on the constant \( M \) for
The Method of Lower and Upper Solutions

$n = 2$, $n = 3$ and $n = 4$ (in this last case, if $M < 0$) for which the conditions $(H_1)$ or $(H_2)$ imply that the monotone method is valid.

To prove the validity of the monotone method to more general cases, we present some maximum principles for the operator

$$L_n: F_{a,b}^n \to L^1(I),$$

defined by $L_n u = u^{(n)} + Mu$. Where $M$ is a real constant different from zero, and

$$F_{a,b}^n = \left\{ u \in W^{n-1}(I); u^{(i)}(a) = u^{(i)}(b), i = 0, \ldots, n-2; u^{(n-1)}(a) \geq u^{(n-1)}(b) \right\}.$$

We say that an operator $L$ is inverse positive in $F_{a,b}^n$ if $L u \geq 0$ implies $u \geq 0$ for all $u \in F_{a,b}^n$ and that $L$ is inverse negative in $F_{a,b}^n$ if $L u \geq 0$ implies $u \leq 0$ for all $u \in F_{a,b}^n$.

In Section 2, we obtain a new maximum principle for the operator $L_n$, using that this operator is given by the composition of the operators of first and second order.

This result is used in Section 3 to extend to more general cases the validity of the monotone method for the problem (1.1)-(1.2) and in Section 4 it is applied to obtain a new generalization of the method of mixed monotony [5] when $f$ and $u$ are vectorial functions.

2. MAXIMUM PRINCIPLES

In this section we improve the following result obtained in [2], which generalizes theorem 4 in [15].

Lemma 2.1: Let $A(n) = \left\lceil \frac{n^2}{2} \right\rceil - \frac{n^n n!}{(b-a)^{n-1} (n-1)^n - 1}$, where $\left\lceil \frac{n}{2} \right\rceil$ is the integer part of $\frac{n}{2}$. Then if $M \in (0, A(n)]$ ($M \in [-A(n), 0]$), the operator $L_n$ is inverse positive (inverse negative) on $F_{a,b}^n$.

Furthermore, if $M \in [-A(n)^2, 0)$ the operator $L_{2n}$ is inverse negative on $F_{a,b}^{2n}$.

For this, we use the following known result.
Lemma 2.2:
1. $L_1$ is inverse positive (inverse negative) on $F^{1}_{a,b}$ for all $M > 0$ ($M < 0$).
2. $L_2$ is inverse negative on $F^{2}_{a,b}$ for all $M < 0$.
3. ([13], Lemma 2.1) The operator $N_{A,B}u = u'' - 2Au' + (A^2 + B^2)u$ is 
   inverse positive on $F^2_{0,2\pi}$ if and only if $(0 < ) B \leq \frac{1}{2}$.

Now, we prove the following preliminary lemma.

Lemma 2.3: Let $L_3 = u^{(n)} + \sum_{i=0}^{n-1} a_i u^{(i)}$ and $N_3 = u^{(m)} + \sum_{i=0}^{m-1} b_i u^{(i)}$. Then 
   if $L$ is inverse positive on $F^{n}_{a,b}$ and $N$ is inverse positive (inverse negative) on $F^{m}_{a,b}$ then $L \circ N$ is inverse positive (inverse negative) on $F^{n+m}_{a,b}$. 

Proof: Since $u \in F^{n+m}_{a,b}$ it is clear that 
   $$(Nu)^{(i)}(a) = (Nu)^{(i)}(b), \quad i = 0, \ldots, n-2$$
   and 
   $$\quad (Nu)^{(n-1)}(a) \geq (Nu)^{(n-1)}(b).$$

In consequence, since $L$ is inverse positive on $F^{n}_{a,b}$, we have that $Nu \geq 0$. 
Now, using that $N$ is inverse positive (inverse negative) on $F^{m}_{a,b}$, we obtain that $u \geq 0$ ($u \leq 0$).

Thus, we are in position to prove the following lemma.

Lemma 2.4: Let $M > 0$. The following properties hold:
1. Let $n = 4k$, $k \in \{1, 2, \ldots\}$. 
   If $M \leq \left[ \frac{\pi}{(b-a)\sin\left(\frac{n+2\pi}{2n}\right)} \right]^n$, then $L_n$ is inverse positive on $F^n_{a,b}$.
2. Let $n = 2 + 4k$, $k \in \{1, 2, \ldots\}$. 
   If $M \leq \left[ \frac{\pi}{b-a} \right]^n$, then $L_n$ is inverse positive on $F^n_{a,b}$.
3. Let $n$ be odd. 
   If $M \leq \left[ \frac{\pi}{(b-a)\sin\left(\frac{n+1\pi}{2n}\right)} \right]^n$, then $L_n$ is inverse positive on $F^n_{a,b}$.

Proof: Since, if $u \in W^{n,1}(I)$ satisfies 
$$L_n u(t) = \sigma(t), u^{(i)}(a) = u^{(i)}(b), i = 0, \ldots, n-2 \text{ and } u^{(n-1)}(a) - u^{(n-1)}(b) = \lambda,$$
then $v(t) = \left( \frac{2\pi}{b-a} \right)^{n-1} u \left( \frac{b-a}{2\pi} t + a \right)$ satisfies
It is sufficient to study the operator $L_n$ on $F^n_{0,2\pi}$ because to obtain the estimate on the interval $[a,b]$ we multiply by $\left(\frac{2\pi}{b-a}\right)^n$ the estimate obtained on $[0,2\pi]$.

Let $m > 0$ such that $m^n = M$.

First, we suppose that $n$ is even.

In this case the polynomial function $p(\lambda) = \lambda^n + m^n = 0$ if and only if

\[ \lambda = \lambda_l = m \left[ \cos \left( \frac{2l + 1}{n} \pi \right) \pm i \sin \left( \frac{2l + 1}{n} \pi \right) \right] \equiv a_l \pm i \beta_l, \]

$l = 0, 1, \ldots, \frac{n-2}{2}$.

As consequence we have that

\[ \lambda^n + m^n = \prod_{l=0}^{\frac{n-2}{2}} \left( \lambda^2 - 2a_l\lambda + m^2 \right), \]

and

\[ L_n \equiv T_0 \circ T_1 \circ \ldots \circ T_{\frac{n-2}{2}}. \tag{2.3} \]

Where $T_l u = u'' - 2a_l u' + m^2 u$.

If $n = 4k$ for some $k \in \{1,2,\ldots\}$, then $\beta_l \leq \beta_{\frac{n}{4}} = m \sin \left( \frac{n+2}{2n} \pi \right)$ for all $l \in \{0,1,\ldots,\frac{n-2}{2}\}$. Thus, using lemma 2.2, if $m \leq \left[ 2 \sin \left( \frac{n+2}{2n} \pi \right) \right]^{-1}$ the operator $T_l$ is inverse positive on $F^2_{0,2\pi}$ for all $l \in \{0,1,\ldots,\frac{n-2}{2}\}$. Therefore lemma 2.3 implies that $L_n$ is inverse positive on $F^n_{0,2\pi}$.

If $n = 2 + 4k$ for some $k \in \{1,2,\ldots\}$, then $\beta_l \leq \beta_{\frac{n-2}{4}} = m$ for all $l \in \{0,1,\ldots,\frac{n-2}{2}\}$ and as a consequence, $T_l$ is inverse positive on $F^2_{0,2\pi}$ when $m \leq \frac{1}{2}$. By (2.3) and the two previous lemmas, we obtain that $L_n$ is inverse positive on $F^n_{0,2\pi}$.

Now, we suppose that $n$ is odd.

In this case, $p(\lambda) = 0$ if and only if $\lambda = -m$ or $\lambda = \lambda_l = a_l \pm i \beta_l$. 

\[ v^{(n)}(t) + \left( \frac{b-a}{2\pi} \right)^n M v(t) = \left( \frac{b-a}{2\pi} \right) \sigma \left( \frac{b-a}{2\pi} t + a \right), \]

with

\[ v^{(i)}(0) = v^{(i)}(2\pi), \quad i = 0, \ldots, n-2 \quad \text{and} \quad v^{(n-1)}(0) - v^{(n-1)}(2\pi) = \lambda. \]
Thus
\[\lambda^n + m^n = (\lambda + m) \prod_{l=0}^{n-1} (\lambda^2 - 2\alpha_l \lambda + m^2),\]
and
\[L_n \equiv T_0 \circ T_1 \circ \ldots \circ T_{n-3} \circ S_1.\]

Where \(S_1 u = u' + mu.\)

In this case \(\beta_l \leq \beta_{n-1} = m \sin\left(\frac{n+1}{2n} \pi\right)\) for all \(l \in \{0, 1, \ldots, \frac{n-3}{2}\}.\) Thus, if \(m \geq [2 \sin\left(\frac{n+1}{2n} \pi\right)]^{-1}\) lemmas 2.2 and 2.3 imply that the operator \(L_n\) is inverse positive on \(F^n_{0,2\pi}.\) \(\square\)

Analogously we can prove the following result for \(M < 0.\)

**Lemma 2.5:** Let \(M < 0.\) The following properties hold:

1. Let \(n = 4k, k \in \{1, 2, \ldots\}.\)
   If \(M \geq -\left[\frac{\pi}{b-a}\right]^n\) then \(L_n\) is inverse negative on \(F^n_{a,b}.\)

2. Let \(n = 2 + 4k, k \in \{0, 1, \ldots\}.\)
   If \(M \geq -\left[\frac{\pi}{(b-a) \sin\left(\frac{n+2}{2n} \pi\right)}\right]^n\) then \(L_n\) is inverse negative on \(F^n_{a,b}.\)

3. Let \(n\) be odd.
   If \(M \geq -\left[\frac{\pi}{(b-a) \sin\left(\frac{n+1}{2n} \pi\right)}\right]^n\) then \(L_n\) is inverse negative on \(F^n_{a,b}.\)

**Remark 2.1:** Note that these estimates are not the best possible for all \(n \in \mathbb{N}.\)

In [2] it is proved that \(L_3\) is inverse positive (inverse negative) on \(F^3_{0,2\pi}\) if and only if \(M \in (0, M_3^3)(M \in [-M_3^3, 0]).\) Where \(M_3\) is the unique solution of the equation

\[\arctan\left(\frac{\sin\sqrt{3} m\pi}{\cos\sqrt{3} m\pi - e^{m\pi}}\right) + \pi = \frac{\sqrt{3}}{3} \log\left(\frac{e^{3m\pi} - e^{m\pi}}{\sqrt{1 + e^{2m\pi} - 2e^{m\pi} \cos\sqrt{3} m\pi}}\right).\]

Furthermore, \(L_4\) is inverse negative on \(F^4_{0,2\pi}\) if and only if \(M \in [-M_4^4, 0),\)
with \(M_4\) given as the unique solution in \((\frac{1}{2}, 1)\) of the equation

\[-\tanh m\pi = \tan m\pi.\]
Note that the estimates obtained in lemmas 2.4 and 2.5 are the best possible for \( n = 1 \) and \( n = 2 \).

3. THE MONOTONE METHOD

In this section we study the existence of solutions of the problem (1.1)-(1.2) in the sector \([\alpha, \beta]\) or \([\beta, \alpha]\), where \([v, w] = \{ u \in L^1(I) : v \leq u \leq w \text{ on } I \}\). We improve the following result given in [2], which generalizes theorem 5 in [15].

**Theorem 3.1:** The following properties hold.
1. If there exists \( \alpha \leq \beta \) (\( \alpha \geq \beta \)) lower and upper solutions respectively of the problem (1.1)-(1.2), and \( f \) satisfies the condition \((H_1)\) ((\(H_2)\)) for some \( M \in (0, A(n)] \) (\( M \in [-A(n), 0) \)) then there exists a solution of the problem (1.1)-(1.2) in \([\alpha, \beta]\) ([\(\beta, \alpha]\)). Furthermore, there exist two monotone sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) with \( \alpha_0 = \alpha \) and \( \beta_0 = \beta \) which converge uniformly to the extremal solutions in \([\alpha, \beta]\) ([\(\beta, \alpha]\)) of the problem (1.1)-(1.2).
2. The previous property is true when \( n \) is even and \( f \) satisfies the condition \((H_2)\) for some \( M \in ( -A(n/2)^2, 0] \).

Using lemma 2.4 we prove the following result.

**Theorem 3.2:** If there exists \( \alpha \geq \beta \) lower and upper solutions respectively of the problem (1.1)-(1.2) and if any of the following properties are true:
1. Let \( n = 4k, \ k \in \{1, 2, \ldots\} \). Suppose that \( f \) satisfies the property \((H_2)\)
   for some \( M \in [-\left( \frac{\pi}{(b-a)\sin(n+2\pi/2n)} \right)^n, 0) \).
2. Let \( n = 2 + 4k, \ k \in \{1, 2, \ldots\} \). Suppose that \( f \) satisfies the property \((H_2)\)
   for some \( M \in [-\left( \frac{\pi}{(b-a)\sin(n+\pi/2n)} \right)^n, 0) \).
3. Let \( n \) be odd. Suppose that \( f \) satisfies the property \((H_2)\)
   for some \( M \in [-\left( \frac{\pi}{(b-a)\sin(n+1\pi/2n)} \right)^n, 0) \).

Then there exists \( u \) a solution of the problem (1.1)-(1.2) in \([\beta, \alpha]\).

Furthermore, there exist two monotone sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) with \( \alpha_0 = \alpha \) and \( \beta_0 = \beta \), which converge uniformly to the extremal solutions in \([\beta, \alpha]\) of
the problem (1.1)-(1.2).

Proof: We consider the problem:

\[ u^{(n)}(t) - Mu(t) = f(t, \eta(t)) - M\eta(t) \text{ for a.e. } t \in I \]  \hspace{1cm} (3.1)

\[ u^{(i)}(a) - u^{(i)}(b) = \lambda_i, \quad i = 0, 1, \ldots, n - 1 \]  \hspace{1cm} (3.2)

with \( \eta \in L^1(I) \), \( \beta(t) \leq \eta(t) \leq \alpha(t) \).

We have:

\[
(\alpha - u)^{(n)}(t) - M(\alpha - u)(t) \geq -f(t, \eta(t)) + M\eta(t) + f(t, \alpha(t)) - M\alpha(t) \geq 0
\]

\[
(\alpha - u)^{(i)}(a) - (\alpha - u)^{(i)}(b) = 0; \quad i = 0, \ldots, n - 2
\]

\[
(\alpha - u)^{(n-1)}(a) - (\alpha - u)^{(n-1)}(b) \geq 0.
\]

Lemma 2.4 implies that \( u \leq \alpha \).

Analogously we can prove that \( u \geq \beta \).

Let \( u_i = Q\eta_i \) the unique solution of the problem (3.1)-(3.2) for \( \eta = \eta_i \in L^1(I) \). Since for \( \beta \leq \eta_1 \leq \eta_2 \leq \alpha \),

\[
(u_2 - u_1)^{(n)}(t) - M(u_2 - u_1)(t) = f(t, \eta_2(t)) - M\eta_2(t) - f(t, \eta_1(t)) + M\eta_1(t) \geq 0
\]

\[
(u_2 - u_1)^{(i)}(a) - (u_2 - u_1)^{(i)}(b) = 0; \quad i = 0, \ldots, n - 1,
\]

the following property holds:

If \( \beta \leq \eta_1 \leq \eta_2 \leq \alpha \) then \( u_1 = Q\eta_1 \leq Q\eta_2 = u_2 \).

The sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) are obtained by recurrence: \( \alpha_0 = \alpha, \beta_0 = \beta, \alpha_n = Q\alpha_{n-1} \) and \( \beta_n = Q\beta_{n-1}; \quad n \geq 1 \).

By standard arguments we prove that \( \{\alpha_n\} \) and \( \{\beta_n\} \) converge to the extremal solutions on \([\beta, \alpha]\) of the problem (1.1)-(1.2). \(\Box\)

Analogously, using lemma 2.5 we can prove the following theorem.

**Theorem 3.3:** If there exists \( \alpha \leq \beta \) lower and upper solutions respectively of the problem (1.1)-(1.2) and any of the following properties are
verified:

1. Let $n = 4k$, $k \in \{1, 2, \ldots\}$. Suppose that $f$ satisfies the property $(H_1)$ for some $M \in (0, \left[\frac{\pi}{b-a}\right]^n]$. 

2. Let $n = 2 + 4k$, $k \in \{1, 2, \ldots\}$. Suppose that $f$ satisfies the property $(H_1)$ for some $M \in (0, \left[\frac{\pi}{(b-a)\sin\left(\frac{n+2}{2n}\pi\right)}\right]^n]$. 

3. Let $n$ be odd. Suppose that $f$ satisfies the property $(H_1)$ for some $M \in (0, \left[\frac{\pi}{(b-a)\sin\left(\frac{n+1}{2n}\pi\right)}\right]^n]$. 

Then there exists $u$ a solution of the problem (1.1)-(1.2) in $[\alpha, \beta]$. 

Furthermore there exist two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_0 = \alpha$ and $\beta_0 = \beta$ which converge uniformly to the extremal solutions in $[\alpha, \beta]$ of the problem (1.1)-(1.2).

Remark 3.1: Similarly to the remark 2.1, note that the estimates obtained for the function $f$ in theorems 3.2 and 3.3 are not the best possible for all $n \in \mathbb{N}$.

4. THE METHOD OF MIXED MONOTONY

In this section we study the method of mixed monotony, studied by Khovanin and Lakshmikantham in [5], in which they consider the initial and periodic first order problems. In this case, under stronger conditions on the function $f$ it is possible to guarantee the unicity of the solution when we have an $n$th-order system.

In [5] the following results are obtained.

Theorem 4.1: Consider the following system

$$u'(t) = f(t, u(t)); t \in [0, T]$$

with $f \in C([0, T] \times \mathbb{R}^N, \mathbb{R}^N)$.

If there exists $F \in C([0, T] \times \mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$, $\alpha, \beta \in C^1([0, T], \mathbb{R}^N)$ which satisfy the following conditions:

(i) $\alpha'(t) \geq F(t, \alpha(t), \beta(t)), \beta'(t) \leq F(t, \beta(t), \alpha(t))$. With $\beta \leq \alpha$ on $[0, T]$.

(ii) $F(t, u, v)$ is nondecreasing on $u$ and nonincreasing on $v$.

(iii) $F(t, u, u) = f(t, u)$ and
\[-B(z_1 - z_2) \leq F(t, y_1, z_1) - F(t, y_2, z_2) \leq B(y_1 - y_2),\]

with \( \beta(t) \leq y_2 \leq y_1 \leq \alpha(t) \), \( \beta(t) \leq z_2 \leq z_1 \leq \alpha(t) \) and \( B \) an \( N \times N \) matrix with nonnegative elements.

Then:

If \( \beta(0) \leq u_0 \leq \alpha(0) \), then there exist two sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) nonincreasing and nondecreasing respectively which converge uniformly to the unique solution of the problem

\[ u'(t) = f(t, u(t)); u(0) = u_0. \]

Furthermore, if \( (T = 2\pi) \beta(0) \leq \beta(2\pi) \) and \( \alpha h(0) \geq \alpha(2\pi) \) with \( I \neq e^{2B\pi} \) the same result is valid for the problem

\[ u'(t) = f(t, u(t)); u(0) = u(2\pi). \]

**Theorem 4.2:** If there exists \( \alpha, \beta \in C^1([0, T], \mathbb{R}^N) \), with \( \beta \leq \alpha \) on \([0, T]\) verifying:

\[ \alpha'(t) \geq f(t, \alpha(t)) + B(\alpha(t) - \beta(t)) \quad \text{and} \quad \beta'(t) \leq f(t, \beta(t)) - B(\alpha(t) - \beta(t)), \]

and \( f \) satisfies

\[-B(x - y) \leq f(t, x) - f(t, y) \leq B(x - y),\]

with \( \beta(t) \leq y \leq x \leq \alpha(t) \), where \( B \) is an \( N \times N \) matrix with nonnegative elements, then the conclusions of theorem 4.1 are valid.

Using lemma 2.5 we prove the following result.

**Theorem 4.3:** Let

\[ u^{(n)}(t) = f(t, u(t)) \text{ for a.e. } t \in [a, b] \]  \hspace{1cm} (4.1)

\[ u_j^{(i)}(a) - u_j^{(i)}(b) = \lambda_i, j \in \mathbb{R}, \text{ } i = 0, \ldots, n - 1; \text{ } j = 1, \ldots, N, \]  \hspace{1cm} (4.2)

with \( f: I \times \mathbb{R}^N \rightarrow \mathbb{R} \) a Carathéodory function and \( n \geq 2 \).

If there exists a Carathéodory function \( F: I \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) and \( \alpha, \beta \in W^{n,1}(I, \mathbb{R}^N) \), \( \alpha \leq \beta \) on \( I \), verifying the following properties:

(i) \[ \alpha^{(n)}(t) \geq F(t, \alpha(t), \beta(t)) \text{ for a.e. } t \in I \]
\( \alpha_j^{(i)}(a) - \alpha_j^{(i)}(b) = \lambda_i, j = 1, \ldots, N \)
\( \alpha_j^{(n-1)}(a) - \alpha_j^{(n-1)}(b) \geq \lambda_{n-1}, j = 1, \ldots, N. \)

(ii) 
\( \beta^{(n)}(t) \leq F(t, \beta(t), \alpha(t)) \) for a.e. \( t \in I \)
\( \beta_j^{(i)}(a) - \beta_j^{(i)}(b) = \lambda_i, j = 1, \ldots, N \)
\( \beta_j^{(n-1)}(a) = \beta_j^{(n-1)}(b) \leq \lambda_{n-1}, j = 1, \ldots, N. \)

(iii) \( F(t, u, v) \) is nonincreasing on \( u \) and nondecreasing on \( v. \)
(iv) \( F(t, u, u) = f(t, u) \) and
\( F(t, y, z) - F(t, z, y) = -B(y - z), \)
\( B \) being an \( N \times N \) matrix with nonnegative elements such that
\( \exp(C(b-a)) \neq I. \) Where \( C \) is given by the expression
\[
C \equiv \begin{pmatrix}
0 & I_{(n-1)N} \\
-B & 0
\end{pmatrix}.
\]

Here \( I_{(n-1)N} \) is the \( (n-1)N \times (n-1)N \) identity matrix.

Then there exist two monotone sequences \( \{\alpha_n\} \) and \( \{\beta_n\}, \) with \( \alpha_0 = \alpha \) and \( \beta_0 = \beta, \) which converge uniformly to the unique solution of the problem (4.1)-(4.2).

Proof: Let \( M_1 = -\left[\frac{\pi}{\pi - a}\right]^n \) and \( \eta, \nu \in L^1(I, \mathbb{R}^N), \eta, \nu \in [\alpha, \beta]. \)

Consider the following linear problem for each \( j = 1, \ldots, N: \)
\[
u_j^{(n)}(t) + M_1 \nu_j(t) = F_j(t, \eta(t), \nu(t)) + M_1 \eta_j(t) \text{ for a.e. } t \in [a, b] \quad (4.3)
\]
\[
u_j^{(i)}(a) - \nu_j^{(i)}(b) = \lambda_{i, j} \in \mathbb{R}, i = 0, \ldots, n-1; j = 1, \ldots, N. \quad (4.4)
\]

Let \( \nu = A[\eta, \nu] \) be the unique solution of the problem (4.3)-(4.4) for each \( \eta, \nu. \)

First, we prove that \( \alpha \leq A[\alpha, \beta] = \alpha_1, \)
\[
(\alpha_j^{(n)} - \alpha_j^{(n)}(t) + M_1(\alpha_j - \alpha_{1, j})(t) \geq 0
\]
\[
(\alpha_j^{(i)} - \alpha_j^{(i)}(a) - (\alpha_j^{(i)} - \alpha_{1, j})(b) = 0; i = 0, \ldots, n-2
\]
\[
(\alpha_j^{(n-1)} - \alpha_j^{(n-1)}(a) - (\alpha_j^{(n-1)} - \alpha_{1, j}^{(n-1)})(b) \geq 0.
\]
Thus, lemma 2.5 implies that $\alpha \leq \alpha_1$ on $I$.

Similarly, we obtain that $\beta \geq \beta_1 = A[\beta, \alpha]$.

Let $\eta_1, \eta_2, \nu \in [\alpha, \beta]$, with $\eta_1 \leq \eta_2$. Let $u_1 = A[\eta_1, \nu]$ and $u_2 = A[\eta_2, \nu]$. We have that

$$(u_{1,j} - u_{2,j})^{(n)}(t) + M_1(u_{1,j} - u_{2,j})(t) = F_j(t, \eta_1, \nu) + M_1\eta_1,j$$
$$- F_j(t, \eta_2, \nu) - M_1\eta_2,j \geq 0$$

which implies that $u_1 \leq u_2$.

Analogously, one can prove that $A[\eta, \nu_1] \leq A[\eta, \nu_2]$ if $\nu_1 \geq \nu_2$.

It is now easy to define the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$, $\alpha_{n+1} = A[\alpha_n, \beta_n]$ and $\beta_{n+1} = A[\beta_n, \alpha_n]$.

Clearly, $\alpha \leq \alpha_1 \leq \ldots \leq \alpha_n \leq \beta_n \leq \ldots \leq \beta_1 \leq \beta$ on $I$.

By standard arguments we can show that $\lim_{n \to \infty} \alpha_n = \phi$ and $\lim_{n \to \infty} \beta_n = \psi$ exist uniformly on $I$ and $\phi$ and $\psi$ satisfy

$$\phi^{(n)}(t) = F(t, \phi, \psi), \quad \psi^{(n)}(t) = F(t, \psi, \phi)$$
$$\phi^{(i)}(a) - \phi^{(i)}(b) = \psi^{(i)}(a) - \psi^{(i)}(b) = \lambda_i,j;$$

$i = 0, \ldots, n - 1; \quad j = 1, \ldots, N$.

That is

$$(\phi - \psi)^{(n)}(t) = F(t, \phi, \psi) - F(t, \psi, \phi) = -B(\phi - \psi) \quad (4.5)$$
$$(\phi - \psi)^{(i)}(a) = (\phi - \psi)^{(i)}(b); \quad i = 0, \ldots, n - 1. \quad (4.6)$$

Now, we define $p(t) = ((\phi - \psi)(t), (\phi - \psi)'(t), \ldots, (\phi - \psi)^{(n-1)}(t)) \in \mathbb{R}^{nN}$. Therefore $p' = Cp$, $p(a) = p(b)$. Since $p(b) = \exp(C(b - a))p(a)$, we obtain that $p \equiv 0$ and, in consequence, $\phi = \psi$. That is, $\phi^{(n)}(t) = F(t, \phi, \phi) = f(t, \phi)$, which concludes the proof.

Similarly, using lemma 2.4 we prove the following result.

**Theorem 4.4:** The conclusions obtained in theorem 4.3 are valid if $\alpha \geq \beta$ and the properties (iii) and (iv) are changed by

(iii)' $F(t, u, v)$ is nondecreasing on $u$ and nonincreasing on $v$.  

(iv) \( F(t, u, u) = f(t, u) \) and 

\[ F(t, y, z) - F(t, z, y) = B(y - z), \]

\( B \) being an \( N \times N \) matrix with nonnegative elements as such that 
\( \exp(D(b - a)) \neq I \), where \( D \) is defined as follows:

\[
D \equiv \left( \begin{array}{c|c}
0 & I_{(n-1)N} \\
\hline
B & 0
\end{array} \right).
\]

Here \( I_{(n-1)N} \) is the \( (n-1)N \times (n-1)N \) identity matrix.

As consequence of the two previous lemmas we prove the following result.

**Theorem 4.5:** Let \( n \geq 2 \). Suppose that there exist \( \alpha, \beta \in W^{n,1}(I, \mathbb{R}^N) \), \( \alpha \leq \beta \) (\( \alpha \geq \beta \)) and \( f \) a Carathéodory function, satisfying

\[-B(x - y) \leq f(t, x) - f(t, y) \leq B(x - y),\]

with \( y \leq x \) between \( \alpha(t) \) and \( \beta(t) \), where \( B \) is an \( N \times N \) matrix with nonnegative elements.

If \( \alpha \) and \( \beta \) satisfies

\[
\alpha^{(n)}(t) \geq f(t, \alpha(t)) + B \mid \beta(t) - \alpha(t) \mid \quad \text{for a.e. } t \in I
\]

\[
\alpha_j^{(i)}(a) - \alpha_j^{(i)}(b) = \lambda_i, j; i = 0, 1, ..., n - 2, j = 1, ..., N
\]

\[
\alpha_j^{(n-1)}(a) - \alpha_j^{(n-1)}(b) \geq \lambda_{n-1}, j; j = 1, ..., N
\]

and

\[
\beta^{(n)}(t) \leq f(t, \beta(t)) - B \mid \beta(t) - \alpha(t) \mid \quad \text{for a.e. } t \in I
\]

\[
\beta_j^{(i)}(a) - \beta_j^{(i)}(b) = \lambda_i, j; i = 0, 1, ..., n - 2, j = 1, ..., N
\]

\[
\beta_j^{(n-1)}(a) - \beta_j^{(n-1)}(b) \leq \lambda_{n-1}, j; j = 1, ..., N.
\]

And \( \exp(C(b - a)) \neq I \) (\( \exp(D(b - a)) \neq I \)) (\( C \) and \( D \) given in theorems 4.3 and 4.4).

Then there exists a unique solution \( u \) between \( \alpha \) and \( \beta \) of the problem (4.1)-(4.2). Furthermore, there exist two monotone sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \), with \( \alpha_0 = \alpha \) and \( \beta_0 = \beta \), which converge uniformly to the solution \( u \).

**Proof:** If \( \alpha \leq \beta \) we define \( F \) as follows:
\[ F(t, u, v) = \frac{1}{2}[f(t, u) + f(t, v) - B(u - v)]. \]

It is easy to prove that the function \( F \) satisfies the conditions of theorem 4.3. If \( \alpha \geq \beta \) the function \( F \) is defined as follows:

\[ F(t, u, v) = \frac{1}{2}[f(t, u) + f(t, v) + B(u - v)]. \]

Clearly, the function \( F \) satisfies the conditions of theorem 4.4. \( \Box \)

REFERENCES


