INITIAL AND BOUNDARY VALUE PROBLEMS FOR
FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

S.K. NTOWYAS and P. CH. TSAMATOS
University of Ioannina
Department of Mathematics
451 10 Ioannina, GREECE

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ABSTRACT

In this paper, we study initial and boundary value problems for functional integro-differential equations, by using the Leray-Schauder Alternative.

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1. Introduction

The purpose of this paper is to study the existence of solutions for initial and boundary value problem (IVP and BVP, for short) for functional integro-differential equations. The paper is divided into two parts.

In Section 2 we consider the following IVP for nonlinear Volterra type integro-differential equations

\[ x'(t) = A(t, x_t) + \int_0^t k(t,s)f(s, x_s)ds, \quad t \in [0,T] \]

\[ x_0 = \phi, \quad (1.1) \]

where \( A, f : [0,T] \times C \rightarrow \mathbb{R}^n \) are continuous functions, and for \( t \in [0,T] \), \( A(t, \cdot) \) is a bounded linear operator from \( C \) to \( \mathbb{R}^n \), and \( k \) is a measurable for \( t \geq s \geq 0 \) real valued function. Here \( C = C([-r,0], R^n) \) is the Banach space of all continuous functions \( \phi : [-r,0] \rightarrow R^n \) endowed with the sup-norm

\[ \| \phi \| = \sup \{ |\phi(\theta)| : -r \leq \theta \leq 0 \}. \]

Also, for \( x \in C([-r,T], R^n) \) we have \( x_t \in C \) for \( t \in [0,T] \), \( x_t(\theta) = x(t + \theta) \) for \( \theta \in [-r,0] \) and \( \phi \in C \).

The results of this section generalize recent results of Ntouyas and Tsamatos [5] when the following degenerate case
is studied. In Section 3 we study the following BVP for nonlinear Volterra integro-differential equations

\[ x'(t) = A(t)x(t) + \int_0^t k(t,s)f(s,x_s)ds, \quad t \in [0, T] \]  
\[ x_0 = \phi \]

where \( A, f \) and \( k \) are as above and \( L \) is a bounded linear operator from a Banach space \( C([-r, T], \mathbb{R}^n) \) into \( \mathbb{R}^n \) and \( h \in \text{Im} L \), the image of \( L \). The results of this section extend previous results on BVP for functional differential equations [2], [3], [4], and [7] to functional integro-differential equations.

2. IVP for Volterra Functional Integro-Differential Equations

In this section we consider the following initial value problem

\[ x'(t) = A(t,x_t) + \int_0^t k(t,s)f(t,x_s)ds, \quad 0 \leq t \leq T \]
\[ x_0 = \phi. \]

Before stating our basic existence theorems, we need the following lemma which is an immediate consequence of the Topological Transversality Theorem of Granas [1], known as "Leray-Schauder alternative".

**Lemma 2.1:** Let \( S \) be a convex subset of a normed linear space \( E \) and assume \( 0 \in S \). Let \( F:S \rightarrow S \) be a completely continuous operator, i.e., it is continuous and the image of any bounded set is included in a compact set, and let

\[ E(F) = \{ x \in S : x = \lambda Fx \text{ for some } 0 < \lambda < 1 \}. \]

Then, either \( E(F) \) is unbounded or \( F \) has a fixed point.

For the IVP (2.1)-(2.2) we have the following existence theorem.

**Theorem 2.2:** Let \( f:[0,T] \times C \rightarrow \mathbb{R}^n \) be a completely continuous function (i.e., it is continuous and takes closed bounded sets of \([0,T] \times C \) into bounded sets of \( \mathbb{R}^n \)). Suppose that:

(\( HA \)) There exists a nonnegative integrable function \( p \) on \([0,T] \) such that \( |A(t,\phi)| \leq p(t) \| \phi \|, (t,\phi) \in [0,T] \times C. \)

(\( Hk \)) There exists a constant \( M \) such that \( |k(t,s)| \leq M, \quad t \geq s \geq 0 \).

Also we assume that there exists a constant \( K \) such that

\[ \| x \|_1 \leq K, \]

for each solution \( x \) of
\[
x'(t) = \lambda A(t, x_t) + \int_0^t k(t, s)f(t, x_s)ds, \quad 0 \leq t \leq T
\]
\[
x_0 = \phi
\]

for any \( \lambda \in (0, 1) \).

Then the initial value problem (2.1)-(2.2) has at least one solution on \([-r, T]\).

**Proof:** We will rewrite (2.1) as follows. For \( \phi \in C \) define \( \tilde{\phi} \in B, B = C([-r, T], \mathbb{R}^n) \) by

\[
\tilde{\phi}(t) = \begin{cases} 
\phi(t), & -r \leq t \leq 0 \\
\phi(0), & 0 \leq t \leq T.
\end{cases}
\]

If \( x(t) = y(t) + \tilde{\phi}(t), t \in [-r, T] \) it is easy to verify that \( y \) satisfies

\[
y_0 = 0,
\]
\[
y(t) = \int_0^t A(s, y_s + \tilde{\phi}_s)ds + \int_0^r k(t, s)f(s, y_s + \tilde{\phi}_s)dsd\tau, \quad 0 \leq t \leq T
\]

if and only if \( x \) satisfies

\[
x(t) = \phi(0) + \int_0^t A(s, x_s)ds + \int_0^r k(t, s)f(s, x_s)dsd\tau, \quad 0 \leq t \leq T
\]

and \( x_0 = \phi \).

Define \( N: B_0 \to B_0, B_0 = \{y \in B: y_0 = 0\} \) by

\[
N y(t) = \begin{cases} 
0, & -h \leq t \leq 0 \\
\int_0^t A(s, y_s + \tilde{\phi}_s)ds + \int_0^r k(t, s)f(s, y_s + \tilde{\phi}_s)dsd\tau, & 0 \leq t \leq T.
\end{cases}
\]

\( N \) is clearly continuous. We shall prove that \( N \) is completely continuous.

Let \( \{h_\nu\} \) be a bounded sequence in \( B_0 \), i.e.,

\[
\| h_\nu \| \leq b, \text{ for all } \nu,
\]

where \( b \) is a positive constant. We obviously have \( \| h_\nu \| \leq b, \) \( t \in [0, T] \), for all \( \nu \). Hence we obtain

\[
\| N h_\nu \| \leq p_0(b + \| \phi \|) + MM_0M_0,
\]

where

\[
p_0 = \int_0^T p(t)dt,
\]
\[ M_0 = \sup \{ |f(t,u)| : t \in [0,T], \|u\| \leq b + \|\phi\| \} \]

and
\[ m_0 = \int_0^T \int_0^\tau p(t)dtd\tau. \]

This means that \( \{Nh_\nu\} \) is uniformly bounded.

Moreover, the sequence \( \{Nh_\nu\} \) is equicontinuous, since for \( t_1, t_2 \in [-r,T] \) we have
\[ |Nh_\nu(t_1) - Nh_\nu(t_2)| \leq [p_0(b + \|\phi\|) + MM_0m_0] |t_1 - t_2|. \]

Thus, by the Arzela-Ascoli theorem, the operator \( N \) is completely continuous.

Finally, the set \( E(N) = \{y \in B_0 : y = \lambda Ny, \lambda \in (0,1)\} \) is bounded by assumption, since
\[ \|x\|_1 \leq K \]
implies
\[ \|y\|_1 \leq K + \|\phi\|. \]

Consequently, by Lemma 2.1, the operator \( N \) has a fixed point \( y^* \) in \( B_0 \). Then \( x^* = y^* + \tilde{\phi} \) is a solution of the IVP (2.1)-(2.2). This proves the theorem.

The applicability of Theorem 2.1 depends upon the existence of a priori bounds for the solutions of the initial value problem (2.1)-(2.2), which are independent of \( \lambda \). Conditions on \( f \) which imply the desired a priori bounds are given in the following:

**Theorem 2.3:** Assume that \((HA)\) and \((Hk)\) hold. Also assume that \( (Hf) \) There exists a continuous function \( m \) such that \( |f(t,\phi)| \leq m(t)f_2(\|\phi\|), \)
\[ 0 \leq t \leq T, \phi \in C \] where \( f_2 \) is a continuous nondecreasing function defined on \([0,\infty)\) and positive on \((0,\infty)\).

Then, the initial value problem (2.1)-(2.2) has a solution on \([-r,T]\) provided
\[ \int_0^T m_1(s)ds < \int_0^\infty \frac{ds}{s + \Omega(s)}, \quad m_1(t) = \sup \{1, p(t), Mm(t)\}. \]

**Proof:** To prove the existence of a solution of the IVP (2.1)-(2.2), we apply Theorem 2.1. In order to apply this theorem, we must establish the a priori bounds for the solutions of the IVP (2.1)-(2.2). Let \( x \) be a solution of (2.1)_\lambda. From
\[ x(t) = \phi(0) + \lambda \int_0^t A(s,x_s)ds + \lambda \int_0^\tau k(t,s)f(s,x_s)dsd\tau, \quad 0 \leq t \leq T \]
we have
\[ |x(t)| \leq |\phi(0)| + \lambda \int_0^t |A(s,x_s)|ds + \lambda \int_0^\tau |k(t,s)||f(s,x_s)|dsd\tau, \quad 0 \leq t \leq T, \]
from which, by \((HA)\), \((Hf)\), and \((Hk)\), we get
\[ |x(t)| \leq \|\phi\| + \int_0^t p(s)||x_s||ds + M \int_0^\tau m(s)\Omega(||x_s||)dsd\tau. \]
We consider the function $\mu$ given by

$$\mu(t) = \sup\{|x(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq T.$$  

Let $t^* \in [-r, t]$ be such that $\mu(t) = |x(t^*)|$. If $t^* \in [0, t]$, by the previous inequality we have

$$\mu(t) \leq \|\phi\| + \int_0^t p(s)\mu(s)ds + M \int_0^t \int_0^\tau m(s)\Omega(\mu(s))d\sigma d\tau, \quad 0 \leq t \leq T \quad (2.3)$$

If $t^* \in [-r, 0]$ then $\mu(t) = \|\phi\|$ and (2.3) obviously holds.

Denoting by $u(t)$ the right-hand side of (2.3) we have

$$\mu(t) \leq u(t), \quad 0 \leq t \leq T,$$

$$u(0) = \|\phi\|,$$

and

$$u'(t) = p(t)\mu(t) + M \int_0^t m(s)\Omega(\mu(s))ds \leq p(t)u(t) + M \int_0^t m(s)\Omega(u(s))ds \leq m_1(t)[u(t) + \int_0^t \Omega(u(s))ds], \quad 0 \leq t \leq T.$$

Let

$$v(t) = u(t) + \int_0^t \Omega(u(s))ds, \quad 0 \leq t \leq T.$$  

Then

$$v(0) = u(0), \quad u(t) \leq v(t), \quad u'(t) \leq m_1(t)v(t), \quad 0 \leq t \leq T$$

and

$$v'(t) = u'(t) + \Omega(u(t)) \leq m_1(t)v(t) + \Omega(v(t)) \leq m_1(t)[v(t) + \Omega(v(t))], \quad 0 \leq t \leq T$$

or

$$\frac{v'(t)}{v(t) + \Omega(v(t))} \leq m_1(t), \quad 0 \leq t \leq T.$$  

This implies

$$\int \frac{ds}{v(t)} \leq \int_0^T m_1(t)dt < \int_{v(0)}^\infty \frac{ds}{s + \Omega(s)}, \quad 0 \leq t \leq T.$$
This inequality implies that there is a constant $K$ such that $u(t) \leq K$, $t \in [0, T]$, and hence $\mu(t) \leq K$, $t \in [0, T]$. Therefore,

$$\| x \|_1 \leq K,$$

and the proof of the theorem is complete.

By applying Theorem 2.3, we have the following result which concerns the global existence of solutions for the IVP (1.1)-(1.2). The proof is omitted since it is similar to that of Theorem 2.3 of [5].

**Theorem 2.4:** Assume that (HA) and (Hk) hold. Also assume that

(Hf) There exists a continuous function $m$ such that $|f(t,\phi)| \leq m(t)\Omega(\| \phi \|)$, $0 \leq t < \infty$, $\phi \in C$, where $\Omega$ is a continuous nondecreasing function defined on $[0, \infty)$ and positive on $(0, \infty)$,

and

$$\int_{s}^{\infty} \frac{ds}{s+\Omega(s)} = +\infty.$$  

Then the initial value problem

$$x'(t) = A(t)x_t + \int_{0}^{t} k(t,s)f(t, x_t)ds, \quad t \geq 0$$ (2.1)'

$$x_0 = \phi$$ (2.2)'

has a solution defined on $[0, \infty)$.

Consider now the following special case of initial value problem (2.1)-(2.2), i.e.,

$$x'(t) = A(t)x(t) + \int_{0}^{t} k(t,s)f(t, x_t)ds, \quad 0 \leq t \leq T$$ (2.5)

$$x_0 = \phi,$$ (2.6)

where $A(t)$ is an $n \times n$ continuous matrix for $t \in [0, T]$ and $f$ is a continuous mapping from $[0, T] \times C$ to $R^n$.

Any solution of this problem may be represented as follows:

$$x(t) = \Phi(t)\Phi^{-1}(t)\phi(0) + \int_{0}^{t} \Phi(t)\Phi^{-1}(t) \int_{0}^{s} k(t,s)f(s, x_s)dsdr, \quad 0 \leq t \leq T,$$

where $\Phi(t)$ is the fundamental matrix of solutions of the homogeneous system $x'(t) = A(t)x(t)$, $0 \leq t \leq T$. $\Phi(t)$ is extended to $[-r, 0]$ by $I$, the identity matrix.

Let $M_1 = \max\{\sup |\Phi(t)\Phi^{-1}(t)| : t, s \in [0, T], 1\}$. Using this formula, we obtain the following theorem proved earlier in [5].

**Theorem 2.5:** If (Hf) and (Hk) hold, then the initial value problem (2.5)-(2.6) has at least one solution on $[-r, T]$, provided that
3. BVP For Volterra Functional Integro-Differential Equations

Consider in this section the following BVP for nonlinear Volterra type integro-differential equations

\[ x'(t) = A(t, x_t) + \int_0^t k(t, s)f(s, x_s)ds, \quad t \in [0, T] \]  \hspace{1cm} (3.1)

\[ Lx = h, \]  \hspace{1cm} (3.2)

where \( A, f \) and \( k \) are as in the previous section and \( L \) is a bounded linear operator from a Banach space \( C([-r, T], \mathbb{R}^n) \) into \( \mathbb{R}^n \) and \( h \in \text{Im} L \), is the image of \( L \).

We will now introduce some necessary preliminaries. Consider a linear nonhomogeneous system of differential equations

\[ x'(t) = A(t, x_t) + g(t) \]  \hspace{1cm} (3.3)

\[ x_0 = \phi \]  \hspace{1cm} (3.4)

for which we assume that \((HA)\) holds.

For any initial function \( \phi \in C \) we denote by \( x(\phi, g)(t) \), the solution of (3.3) satisfying \( x(\phi, g) = \phi \). For each \( \phi \in C \) and \( g \) as above, the initial value problem (3.3)-(3.4) has a unique solution \( x(\phi, g) \) defined on \([-r, T]\) such that

\[ x(\phi, g)(t) = x(\phi, 0)(t) + \int_0^t U(t, s)g(s)ds, \quad t \in [0, T], \]  \hspace{1cm} (3.5)

where \( U(t, s) \) is the fundamental matrix of \( x'(t) = A(t, x_t) \). Denote by \( |U(t, s)| \), the operator norm of the matrix \( U(t, s) \) and set

\[ P = \sup\{ |U(t, s)| : 0 \leq s, t \leq T \}. \]

Set \( S: C \rightarrow C([-r, T], \mathbb{R}^n) \) be the solution mapping defined by

\[ S\phi = x(\phi, 0). \]

Then \( S \) is a bounded linear operator and hence the composite mapping \( L_S = LS \) is a bounded linear operator from \( C \) into \( \mathbb{R}^n \). We assume that

\((HL)\) There exists a bounded linear operator \( L^*_S: \mathbb{R}^n \rightarrow C \) such that \( L_S L^*_S L_S = L_S \).

Therefore \( L^*_S \) is the generalized inverse of \( L_S \). Then any solution to the BVP (3.1)-(3.2) is a fixed point of the operator \( F \) with

\[ Fx = F_1x + F_2x, \]
where
\[(F_1x)(t) = SL^*_S(h - LF_2x)(t), \quad -r \leq t \leq T,\] (3.6)
and
\[F_2x(t) = \begin{cases} 0, & -r \leq t \leq 0 \\ \int_0^T U(t,s)k(t,s)f(s,x_s)dsdt, & 0 \leq t \leq T. \end{cases}\] (3.7)

For a proof of this fact, the reader is referred to Kaminogo [4].

Now, we present our main result on the existence of solutions of the BVP (3.1)-(3.2).

**Theorem 3.1:** Assume that (HA), (Hk), (Hf) and (HL) hold, then, if
\[T \geq \gamma \left( \int_0^T \Phi(s)ds \right), \quad c = \max \{ |SL^*_S|, |h|, \| \phi \| \},\]
the BVP (3.1)-(3.2) has at least one solution on \([-r, T]\).

**Proof:** To prove the existence of a solution of the BVP (3.1)-(3.2), we apply Lemma 2.1. In order to apply this lemma, we must establish the a priori bounds for the BVP (3.1)(3.2), Let \(x\) be a solution of the BVP (3.1). Then,
\[x(t) = \lambda SL^*_S(h - LF_2x)(t) + (F_2x)(t), \quad t \in [0, T]\]
where \(F_2(t)\) is given by (3.6). From this, we get
\[|x(t)| \leq |SL^*_S||h| + |L|P \int_0^t \int_0^\tau m(s)\Omega(\|x_s\|)dsdt\]
\[+ P \int_0^t \int_0^\tau m(s)\Omega(\|x_s\|)dsdt\]
\[\leq |SL^*_S||h| + P(|SL^*_S||L| + 1) \int_0^t \int_0^\tau m(s)\Omega(\|x_s\|)dsdt, \quad 0 \leq t \leq T.\]

As in Theorem 2.3, we consider the function \(\mu\) given by
\[\mu(t) = \sup \{|x(s)|: -r \leq s \leq t\}, \quad 0 \leq t \leq T.\]
Let \(t^* \in [-r, t]\) be such that \(\mu(t) = |x(t^*)|\). If \(t^* \in [0, t]\), by the previous inequality we have
\[\mu(t) = |x(t^*)| \leq |SL^*_S||h| + P(|SL^*_S||L| + 1) \int_0^t \int_0^\tau m(s)\Omega(\mu(s))dsdt\]
\[\leq c + P(|SL^*_S||L| + 1) \int_0^t \int_0^\tau m(s)\Omega(\mu(s))dsdt,\]
where \( c = \max \{ |SL^*_S|, |h|, \| \phi \| \} \).

If \( t^* \in [-r, 0] \), then \( \mu(t) = \| \phi \| \) and the previous inequality obviously holds true.

Denoting by \( u(t) \) the right-hand side of the above inequality, we have

\[
\mu(t) \leq u(t), \quad 0 \leq t \leq T,
\]

\[
u(0) = c,
\]

and

\[
u'(t) = P(|SL^*_S| |L| + 1) \int_0^t m(s)\Omega(\mu(s))ds
\]

\[
\leq P(|SL^*_S| |L| + 1) \int_0^t m(s)\Omega(\mu(s))ds
\]

\[
\leq P(|SL^*_S| |L| + 1)\Omega(u(t)) \int_0^t m(s)ds, \quad 0 \leq t \leq T.
\]

or

\[
u'(t) \leq P(|SL^*_S| |L| + 1) \int_0^t m(s)ds, \quad 0 \leq t \leq T.
\]

Then,

\[
\int_0^t \frac{ds}{\Omega(u(s))} \leq P(|SL^*_S| |L| + 1) \int_0^T m(s)ds + \int_0^T m(s)ds \int_0^\infty \frac{ds}{\Omega(s)} \leq T.
\]

This inequality implies that there is a constant \( K \) such that \( u(t) \leq K, t \in [0, T] \), and hence \( \mu(t) \leq K, t \in [0, T] \). Since for every \( t \in [0, T] \), \( \| x_t \| \leq \mu(t) \), we have

\[
\| x \|_1 \leq K,
\]

where \( K \) depends only on \( T \) and the functions \( m \) and \( \Omega \).

In the second step, we notice that any solution of the BVP (3.1)-(3.2) is a fixed point of the operator \( F \) with

\[
Fx = SL^*_S(h - LF_2x) + F_2x
\]

which is a completely continuous operator ([4]).

Finally, the set \( E(F) = \{ x \in B: x = \lambda Fx \text{ for some } 0 < \lambda < 1 \} \) is bounded, since in the first step we have proved that \( \| x \|_1 \leq K \).

Consequently, by Lemma 2.1, the BVP (3.1)-(3.2) has at least one solution, completing the proof of the theorem.

We shall now consider equation (3.1) when the linear part \( A(t, x_t) \) is not a functional on \( C \). More precisely, we shall consider the functional differential equation of the form
\[ x'(t) = A(t)x(t) + \int_0^t k(t, s)f(s, x_s)ds, \quad t \in [0, T], \tag{3.8} \]

where \( A(t) \) is a continuous \( n \times n \) matrix for \( t \in [0, T] \).

Let us assume that \( \Phi(t) \) is the fundamental matrix of solutions of the homogeneous system
\[ x'(t) = A(t)x(t), \quad 0 \leq t \leq T \tag{3.9} \]

with \( \Phi(0) = I \), the identity matrix. \( \Phi(t) \) is extended to \([-r, 0]\) by \( I \). We denote by \( L_0 \), the \( n \times n \) matrix whose elements are the values of \( L \) on the corresponding columns of \( \Phi(t) \). Assume that \( L_0 \) is nonsingular with inverse \( L_0^{-1} \). Then it is well known (Opial [6]) that:

(I) The BVP (3.8)-(3.2) has a solution for any \( h \in \mathbb{R}^n \), if and only if, the corresponding homogeneous BVP
\[ x'(t) = A(t)x(t) \\
Lx = 0 \]

has only the trivial solution \( x(t) = 0 \).

(II) The solution of the BVP (3.8)-(3.2) is unique and is given by the explicit formula
\[ x(t) = \Phi(t)L_0^{-1}(h - LF_2(t)) + F_2(t), \]

where
\[ F_2(t) = \begin{cases} 
0, & -r \leq t \leq 0 \\
\int_0^t \Phi(t) \int_0^s \Phi^{-1}(\tau)k(t, \tau)f(\tau, x_\tau)d\tau ds, & 0 \leq t \leq T.
\end{cases} \]

Let
\[ \alpha = \sup\{ |\Phi(t)| : 0 \leq t \leq T \}, \]
\[ \beta = \sup\{ |\Phi^{-1}(t)| : 0 \leq t \leq T \}. \]

Then we have:

**Theorem 3.2:** Assume that (Hf) and (Hk) hold. Assume also that the linear operator \( L \) is such that the operator \( L_0 \) has a bounded inverse \( L_0^{-1} \).

Then if
\[ \alpha \beta M(\alpha |L_0^{-1}| |L| + 1) \int_0^T \int_0^t m(s)ds < \frac{\int_0^\infty ds}{\Omega(s)}, \tag{3.10} \]

where \( c = \max\{\alpha |L_0^{-1}| |h|, \|\phi\|\} \), the BVP (3.8)-(3.2) has at least one solution.

**Proof:** The proof is similar to that of the previous theorem and it is omitted.
References


