Discrete Inequalities of Wirtinger’s Type for Higher Differences

GRADIMIR V. MILOVANOVIĆ and IGOR Ž. MILOVANOVIĆ

Faculty of Electronic Engineering, Department of Mathematics, P.O. Box 73, 18001 Niš, Yugoslavia
e-mail: grade@gauss.elfak.ni.ac.yu

(Received 28 July 1996)

Discrete version of Wirtinger’s type inequality for higher differences,

$$A_{n,m} \sum_{k=1}^{n} x_k^2 \leq \sum_{k=l_m}^{u_m} (\Delta^m x_k)^2 \leq B_{n,m} \sum_{k=1}^{n} x_k^2,$$

where $l_m = 1 - \lfloor m/2 \rfloor$, $u_m = n - \lfloor m/2 \rfloor$ and

$$\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+m-i},$$

is considered. Under some conditions, the best constants $A_{n,m}$ and $B_{n,m}$ are determined.

Keywords: Discrete inequality; difference of higher order; eigenvalue; eigenvector.

AMS 1991 Subject classifications: Primary 26D15; Secondary 41A44.

1 INTRODUCTION AND PRELIMINARIES

In [1] (see also [2]) we presented a general method for finding the best possible constants $A_n$ and $B_n$ in inequalities of the form

$$A_n \sum_{k=1}^{n} p_k x_k^2 \leq \sum_{k=0}^{n} r_k(x_k - x_{k+1})^2 \leq B_n \sum_{k=1}^{n} p_k x_k^2, \quad (1.1)$$

This work was supported in part by the Serbian Scientific Foundation, grant number 04M03.
where \( p = (p_k) \) and \( r = (r_k) \) are given weight sequences and \( x = (x_k) \) is an arbitrary sequence of the real numbers. The basic discrete inequalities of the form (1.1) for \( p_k = r_k = 1 \) were given by K. Fan, O. Taussky, and J. Todd [3]. Here, we mention some references in this direction [4–8].

The first results for the second difference were proved by Fan, Taussky and Todd [3]:

**Theorem 1.1** If \( x_0 (\neq 0), x_1, x_2, \ldots, x_n, x_{n+1} (\neq 0) \) are given real numbers, then

\[
\sum_{k=0}^{n-1} (x_k - 2x_{k+1} + x_{k+2})^2 \geq 16 \sin^4 \frac{\pi}{2(n+1)} \sum_{k=1}^{n} x_k^2, \tag{1.2}
\]

with equality in (1.2) if and only if \( x_k = A \sin \frac{k\pi}{n+1}, \ k = 1, 2, \ldots, n, \) where \( A \) is an arbitrary constant.

**Theorem 1.2** If \( x_0, x_1, \ldots, x_n, x_{n+1} \) are given real numbers such that

\[
\sum_{k=1}^{n} x_k = 0, \tag{1.3}
\]

then

\[
\sum_{k=0}^{n-1} (x_k - 2x_{k+1} + x_{k+2})^2 \geq 16 \sin^4 \frac{\pi}{2n} \sum_{k=1}^{n} x_k^2. \tag{1.4}
\]

The equality in (1.4) is attained if and only if

\[
x_k = A \cos \frac{(2k-1)\pi}{2n}, \quad k = 1, 2, \ldots, n,
\]

where \( A \) is an arbitrary constant.

A converse inequality of (1.2) was proved by Lunter [9], Yin [10] and Chen [11] (see also Agarwal [8]).

**Theorem 1.3** If \( x_0 (\neq 0), x_1, x_2, \ldots, x_n, x_{n+1} (\neq 0) \) are given real numbers, then

\[
\sum_{k=0}^{n-1} (x_k - 2x_{k+1} + x_{k+2})^2 \leq 16 \cos^4 \frac{\pi}{2(n+1)} \sum_{k=1}^{n} x_k^2, \tag{1.5}
\]

with equality in (1.5) if and only if \( x_k = A(-1)^k \sin \frac{k\pi}{n+1}, \ k = 1, 2, \ldots, n, \)

where \( A \) is an arbitrary constant.

Chen [11] also proved the following result:
THEOREM 1.4 If $x_0, x_1, \ldots, x_n, x_{n+1}$ are given real numbers such that $x_0 = x_1$ and $x_{n+1} = x_n$, then

$$\sum_{k=0}^{n-1} (x_k - 2x_{k+1} + x_{k+2})^2 \leq 16 \cos^4 \frac{\pi}{2n} \sum_{k=1}^{n} x_k^2,$$

with equality holding if and only if

$$x_k = A(-1)^k \sin \left( \frac{(2k - 1)\pi}{n} \right), \quad k = 1, 2, \ldots, n,$$

where $A$ is an arbitrary constant.

In this case, the $n \times n$ symmetric matrix corresponding to the quadratic form

$$F_2 = \sum_{k=0}^{n-1} (x_k - 2x_{k+1} + x_{k+2})^2 = (H_{n,2}x, x)$$

is

$$H_{n,2} = \begin{bmatrix}
2 & -3 & 1 \\
-3 & 6 & -4 & 1 \\
1 & -4 & 6 & -4 & 1 \\
& & \ddots & \ddots & \ddots & \ddots \\
& & 1 & -4 & 6 & -4 & 1 \\
& & 1 & -4 & 6 & -3 \\
& & 1 & -3 & 2
\end{bmatrix}.$$

This matrix is the square of the $n \times n$ matrix

$$H_n = H_{n,1} = \begin{bmatrix}
1 & -1 \\
-1 & 2 & -1 \\
-1 & 2 & -1 \\
& & \ddots & \ddots & \ddots \\
& & -1 & 2 & -1 \\
& & 1 & -1 & 1
\end{bmatrix}.$$  \hspace{1cm} (1.6)

The eigenvalues of $H_n$ are

$$\lambda_\nu = \lambda_\nu(H_n) = 4 \cos^2 \left( \frac{(n - \nu + 1)\pi}{2n} \right), \quad \nu = 1, \ldots, n,$$

and therefore, the largest eigenvalue of $H_n$ is

$$\lambda_n(H_n) = 4 \cos^2 \frac{\pi}{2n} > \lambda_{n-1}(H_n).$$
The corresponding eigenvector is \( x^n = [x_1 \ x_2 \ \ldots \ x_n]^T \), where
\[
x_{vn} = (-1)^v \sin \frac{(2v - 1)\pi}{2n}, \quad v = 1, 2, \ldots, n.
\]
Thus, the largest eigenvalue of \( H_{n, 2} \) is
\[
\lambda_n(H_{n, 2}) = 16 \cos^4 \frac{\pi}{2n} > \lambda_{n-1}(H_{n, 2}),
\]
and the associated eigenvector is \( x^n \).

Notice that the minimal eigenvalue of the matrix \( H_n \) (and also \( H_{n, 2} \)) is \( \lambda_1 = 0 \). Therefore, the condition (1.3) must be included in Theorem 1.2 (see Agarwal [8, Ch. 11]) and the best constant is the square of the relevant eigenvalue
\[
\lambda_2 = 4 \cos^2 \frac{(n-1)\pi}{2n} = 4 \sin^2 \frac{\pi}{2n}.
\]

For any \( n \)-dimensional vector \( x = [x_1 \ x_2 \ \ldots \ x_n]^T \), Pfeffer [12] introduced a periodically extended \( n \)-vector by setting \( x_{i+rn} = x_i \) for \( i = 1, 2, \ldots, n \) and \( r \in \mathbb{N} \), and used the \( m \)th difference of \( x \) given by
\[
x^{(m)} = [\Delta^m x_1 \ \Delta^m x_2 \ \ldots \ \Delta^m x_n]^T,
\]
in order to prove the following result:

**Theorem 1.5** If \( x \) is a periodically extended \( n \)-vector and (1.3) holds, then
\[
(x^{(m)}, x^{(m)}) \geq \left(4 \sin^2 \frac{\pi}{n}\right)^m (x, x),
\]
with equality case if and only if \( x \) is the periodic extension of a vector of the form \( C_1 u + C_2 v \), where
\[
u = [v_1 \ v_2 \ \ldots \ v_n]^T
\]
have the following components
\[
u_k = \cos \frac{2k\pi}{n}, \quad \nu_k = \sin \frac{2k\pi}{n}, \quad k = 1, \ldots, n,
\]
and \( C_1 \) and \( C_2 \) are arbitrary real constants.
2 MAIN RESULTS

In this paper we consider inequalities of the form

\[ A_{n,m} \sum_{k=1}^{n} x_k^2 \leq \sum_{k=l_m}^{u_m} (\Delta^m x_k)^2 \leq B_{n,m} \sum_{k=1}^{n} x_k^2, \]  

(2.1)

where \( l_m = 1 - [m/2] \), \( u_m = n - [m/2] \) and

\[ \Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+m-i}. \]

The quadratic form \( F_m = \sum_{k=l_m}^{u_m} (\Delta^m x_k)^2 \) for \( m = 1 \) reduces to

\[ F_1 = x_1^2 + \sum_{k=2}^{n-1} 2x_k^2 + x_n^2 - 2 \sum_{k=1}^{n-1} x_kx_{k+1}, \]

with corresponding tridiagonal symmetric matrix \( H_n = H_{n,1} \) given by (1.6).

We consider inequalities (2.1) under conditions

\[ x_s = x_{1-s}, \quad x_{n+1-s} = x_{n+s} \quad (l_m \leq s \leq 0) \]  

(2.2)

and define

\[ x^{(j)} = \begin{bmatrix} \Delta^j x_1-\lfloor j/2 \rfloor \\ \Delta^j x_2-\lfloor j/2 \rfloor \\ \vdots \\ \Delta^j x_{n-\lfloor j/2 \rfloor} \end{bmatrix}, \]

(2.3)

The quadratic form \( F_m \) can be expressed then in the following form

\[ F_m = F_m(x) = \sum_{k=l_m}^{u_m} (\Delta^m x_k)^2 = (x^{(m)}, x^{(m)}), \]

(2.4)

where

\[ x = x^{(0)} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \]

At the begining we prove three auxiliary results:
**Lemma 2.1** If \( j \) is an even integer, under conditions (2.2), we have that

\[
\Delta^{j+1}x_{-[j/2]} = 0 \quad \text{and} \quad \Delta^{j+1}x_{n-[j/2]} = 0.
\]  

**Proof** Let \( q = 0 \) or \( q = n \). Putting \( j = 2p \) we have

\[
\Delta^{j+1}x_{q-[j/2]} = \Delta^{2p+1}x_{q-p} = \sum_{i=0}^{2p+1} (-1)^i \binom{2p+1}{i} x_{q+p+1-i}
\]

\[
= \sum_{i=0}^{p} (-1)^i \binom{2p+1}{i} x_{q+p+1-i} + \sum_{i=p+1}^{2p+1} (-1)^i \binom{2p+1}{i} x_{q+p+1-i}
\]

\[
= \sum_{i=0}^{p} (-1)^i \binom{2p+1}{i} x_{q+p+1-i} - \sum_{i=0}^{p} (-1)^i \binom{2p+1}{i} x_{q-p+i}
\]

\[
= \sum_{i=0}^{p} (-1)^i \binom{2p+1}{i} (x_{q+p+1-i} - x_{q-p+i}) = 0
\]

because of the conditions (2.2). \( \square \)

**Lemma 2.2** If \( j \) is an even integer, under conditions (2.2), we have that

\[
H_n x^{(j)} = -x^{(j+2)},
\]

where the matrix \( H_n \) is given by (1.6).

**Proof** We have

\[
H_n x^{(j)} = \begin{bmatrix}
\Delta^j x_{1-[j/2]} - \Delta^j x_{2-[j/2]} \\
-\Delta^{j+2} x_{1-[j/2]} \\
\vdots \\
-\Delta^{j+2} x_{n-2-[j/2]} \\
-\Delta^{j+2} x_{n-1-[j/2]} + \Delta^j x_{n-[j/2]}
\end{bmatrix}.
\]  

(2.6)

Since

\[
\Delta^{j+2} x_{-[j/2]} = \Delta^{j+1} x_{1-[j/2]} - \Delta^{j+1} x_{-[j/2]}
\]

and

\[
\Delta^{j+2} x_{n-1-[j/2]} = \Delta^{j+1} x_{n-[j/2]} - \Delta^{j+1} x_{n-1-[j/2]},
\]
because of Lemma 2.1, we conclude that
\[ \Delta^{j+2} x_{[j/2]} = \Delta^{j+1} x_{1-[j/2]} \quad \text{and} \quad \Delta^{j+2} x_{n-1-[j/2]} = -\Delta^{j+1} x_{n-1-[j/2]}, \]
respectively. Therefore,
\[ \Delta^{j} x_{1-[j/2]} - \Delta^{j} x_{2-[j/2]} = -\Delta^{j+1} x_{1-[j/2]} = -\Delta^{j+2} x_{[j/2]} \]
and
\[ -\Delta^{j} x_{n-1-[j/2]} + \Delta^{j} x_{n-[j/2]} = \Delta^{j+1} x_{n-1-[j/2]} = -\Delta^{j+2} x_{n-1-[j/2]}, \]
Then (2.6) becomes
\[ H_n x^{(j)} = - \begin{bmatrix} \Delta^{j+2} x_{[j/2]} \\ \Delta^{j+2} x_{1-[j/2]} \\ \vdots \\ \Delta^{j+2} x_{n-2-[j/2]} \\ \Delta^{j+2} x_{n-1-[j/2]} \end{bmatrix} = - \begin{bmatrix} \Delta^{j+2} x_{1-[(j+2)/2]} \\ \Delta^{j+2} x_{2-[(j+2)/2]} \\ \vdots \\ \Delta^{j+2} x_{n-1-[(j+2)/2]} \\ \Delta^{j+2} x_{n-[(j+2)/2]} \end{bmatrix} = -x^{(j+2)}. \]

**Lemma 2.3** If \( j \) is an even integer, under conditions (2.2), we have that
\[ (x^{(j)}, x^{(j+2)}) = -(x^{(j+1)}, x^{(j+1)}). \]

**Proof** Let \( j \) is an even integer. Using (2.3) we have
\[
(x^{(j)}, x^{(j+2)}) = \sum_{k=1}^{n} \Delta^{j} x_{k-[j/2]} \Delta^{j+2} x_{k-1-[j/2]}
\]
\[
= \sum_{k=1}^{n} \Delta^{j} x_{k-[j/2]} \left( \Delta^{j} x_{k-1-[j/2]} - 2\Delta^{j} x_{k-[j/2]} + \Delta^{j} x_{k+1-[j/2]} \right)
\]
\[
= \sum_{k=1}^{n} \Delta^{j} x_{k-[j/2]} \left( \Delta^{j} x_{k+1-[j/2]} - \Delta^{j} x_{k-[j/2]} \right)
- \sum_{k=1}^{n} \Delta^{j} x_{k-[j/2]} \left( \Delta^{j} x_{k-[j/2]} - \Delta^{j} x_{k-1-[j/2]} \right)
\]
\[
= \sum_{k=1}^{n} \Delta^{j} x_{k-[j/2]} \Delta^{j+1} x_{k-[j/2]} - \sum_{k=1}^{n} \Delta^{j} x_{k-[j/2]} \Delta^{j+1} x_{k-1-[j/2]}
\]
\[
= \sum_{k=1}^{n} \Delta^{j} x_{k-[j/2]} \Delta^{j+1} x_{k-[j/2]} - \sum_{k=0}^{n-1} \Delta^{j} x_{k+1-[j/2]} \Delta^{j+1} x_{k-[j/2]}. \]
Because of (2.5) we can write
\[
(x^{(j)}, x^{(j+2)}) = \sum_{k=1}^{n} \Delta^j x_{k-[j/2]} \Delta^{j+1} x_{k-[j/2]} - \sum_{k=1}^{n} \Delta^j x_{k+1-[j/2]} \Delta^{j+1} x_{k-[j/2]}
\]
\[
= - \sum_{k=1}^{n} \left( \Delta^{j+1} x_{k-[j/2]} \right)^2.
\]

Since \( j \) is an even integer we have that
\[
(x^{(j)}, x^{(j+2)}) = - \sum_{k=1}^{n} \left( \Delta^{j+1} x_{k-[j+(j+1)/2]} \right)^2 = -(x^{(j+1)}, x^{(j+1)}).
\]

Now, we give the main result:

**Theorem 2.4** If \( x_1, x_2, \ldots, x_n \) are given real numbers and conditions (2.2) are satisfied, then
\[
\sum_{k=l_m}^{u_m} (\Delta^m x_k)^2 \leq 4^m \cos^{2m} \frac{\pi}{2n} \sum_{k=1}^{n} x_k^2,
\]
(2.7)

where \( l_m = 1 - \lfloor m/2 \rfloor \) and \( u_m = n - \lfloor m/2 \rfloor \). The equality in (2.7) is attained if and only if
\[
x_k = A(-1)^k \sin \frac{(2k-1)\pi}{n}, \quad k = 1, 2, \ldots, n,
\]
where \( A \) is an arbitrary constant.

**Proof** We will prove that the corresponding matrix of the quadratic form (2.4) is exactly the \( m \)th power of the matrix \( H_n = H_{n,1} \) so that the best constant in the right inequality (2.1), i.e., (2.7), is given by
\[
B_{n,m} = 4^m \cos^{2m} \frac{\pi}{2n}.
\]

Evidently, \( A_{n,m} = 0 \).

Let \( m \) be an even integer. Then, using Lemma 2.2, we find
\[
F_m = (x^{(m)}, x^{(m)}) = (H_n x^{(m-2)}, H_n x^{(m-2)}),
\]
Similarly, for an odd \( m \), using Lemmas 2.3 and 2.4, we obtain
\[
F_m = (x^{(m)}, x^{(m)}) = -(x^{(m-1)}, x^{(m+1)}) = (x^{(m-1)}, H_n x^{(m-1)}).
\]
Now, using Lemma 2.2 again, we find
\[
F_m = (H_n^{(m-1)/2} x^{(0)}, H_n^{(m+1)/2} x^{(0)}) = (H_n^m x, x).
\]

By restriction (1.3), we can obtain the following result:

**Theorem 2.5** If \( x_1, x_2, \ldots, x_n \) are given real numbers and conditions (2.2) and (1.3) are satisfied, then
\[
4^m \sin^2 \frac{\pi m}{2n} \sum_{k=1}^{n} x_k^2 \leq \sum_{k=l_m}^{u_m} \left( \Delta^m x_k \right)^2, \tag{2.8}
\]
where \( l_m = 1 - \lfloor m/2 \rfloor \) and \( u_m = n - \lfloor m/2 \rfloor \). The equality in (2.8) is attained if and only if
\[
x_k = A \cos \frac{(2k - 1)\pi}{2n}, \quad k = 1, 2, \ldots, n,
\]
where \( A \) is an arbitrary constant.

For other generalizations of discrete Wirtinger’s inequalities see [13–15]. There are also generalizations for multidimensional sequences and partial differences (see [16] and [17]).

**Acknowledgements**

This work was supported in part by the Serbian Scientific Foundation, grant number 04M03.

**References**


