We introduce a new class of normalized norms on $\mathbb{R}^2$ which properly contains all absolute normalized norms. We also give a criterion for deciding whether a given norm in this class is uniformly nonsquare. Moreover, an estimate for the James constant is presented and the exact value of some certain norms is computed. This gives a partial answer to the question raised by Kato et al.

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1. Introduction and preliminaries

A norm $\| \cdot \|$ on $\mathbb{C}^2$ (resp., $\mathbb{R}^2$) is said to be absolute if $\|(z,w)\| = \|(\|z\|, \|w\|)\|$ for all $z,w \in \mathbb{C}$ (resp., $\mathbb{R}$), and normalized if $\|(1,0)\| = \|(0,1)\| = 1$. The $\ell_p$-norms $\| \cdot \|_p$ are such examples:

$$\|(z,w)\|_p = \begin{cases} (|z|^p + |w|^p)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{|z|, |w|\} & \text{if } p = \infty. \end{cases}$$ (1.1)

Let $AN_2$ be the family of all absolute normalized norms on $\mathbb{C}^2$ (resp., $\mathbb{R}^2$), and $\Psi_2$ the family of all continuous convex functions $\psi$ on $[0,1]$ such that $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \leq \psi(t) \leq 1$ ($0 \leq t \leq 1$). According to Bonsall and Duncan [1], $AN_2$ and $\Psi_2$ are in a one-to-one correspondence under the equation

$$\psi(t) = \|(1-t,t)\| \quad (0 \leq t \leq 1).$$ (1.2)

Indeed, for all $\psi \in \Psi_2$, let

$$\|(z,w)\|_\psi = \begin{cases} (|z| + |w|)\psi\left(\frac{|w|}{|z| + |w|}\right) & \text{if } (z,w) \neq (0,0), \\ 0 & \text{if } (z,w) = (0,0). \end{cases}$$ (1.3)
The James constant of normalized norms on $\mathbb{R}^2$

Then $\| \cdot \|_\psi \in AN_2$, and $\| \cdot \|_\psi$ satisfies (1.2). From this result, we can consider many non-$\ell_p$-type norms easily. Now let

$$
\psi_p(t) = \begin{cases} 
((1-t)^p + t^p)^{1/p} & \text{if } 1 \leq p < \infty, \\
\max\{1-t, t\} & \text{if } p = \infty.
\end{cases}
$$

(1.4)

Then $\psi_p(t) \in \Psi_2$ and, as is easily seen, the $\ell_p$-norm $\| \cdot \|_p$ is associated with $\psi_p$.

If $X$ is a Banach space, then $X$ is uniformly nonsquare if there exists $\delta \in (0, 1)$ such that for any $x, y \in X$,

$$
\text{either } \|x + y\| \leq 2(1 - \delta) \text{ or } \|x - y\| \leq 2(1 - \delta),
$$

(1.5)

where $S_X = \{x \in X : \|x\| = 1\}$. The James constant $J(X)$ is defined by

$$
J(X) = \sup \{\min \{\|x + y\|, \|x - y\|\} : x, y \in S_X\}.
$$

(1.6)

The modulus of convexity of $X$, $\delta_X : [0, 2] \to [0, 1]$ is defined by

$$
\delta_X(\varepsilon) = \inf \left\{1 - \frac{1}{2}\|x + y\| : x, y \in S_X, \|x - y\| \geq \varepsilon\right\}.
$$

(1.7)

The preceding parameters have been recently studied by several authors (cf. [4–6, 8, 9]). We collect together some known results.

**Proposition 1.1.** Let $X$ be a nontrivial Banach space, then

(i) $\sqrt{2} \leq J(X) \leq 2$ (Gao and Lau [5]),

(ii) if $X$ is a Hilbert space, then $J(X) = \sqrt{2}$; the converse is not true (Gao and Lau [5]),

(iii) $X$ is uniformly nonsquare if and only if $J(X) < 2$ (Gao and Lau [5]),

(iv) $2J(X) - 2 \leq J(X^*) \leq J(X)/2 + 1$, $J(X^{**}) = J(X)$, and there exists a Banach space $X$ such that $J(X^*) \neq J(X)$ (Kato et al. [8]),

(v) if $2 \leq p \leq \infty$, then $\delta_{\ell_p}(\varepsilon) = 1 - (1 - (\varepsilon/2)^p)^{1/p}$ (Hanner [6]),

(vi) $J(X) = \sup \{\varepsilon \in (0, 2) : \delta_X(\varepsilon) \leq 1 - \varepsilon/2\}$ (Gao and Lau [5]).

The paper is organized as follows. In Section 2 we introduce a new class of normalized norms on $\mathbb{R}^2$. This class properly contains all absolute normalized norms of Bonsall and Duncan [1]. The so-called generalized Day-James space, $\ell_\psi - \ell_\varphi$, where $\psi, \varphi \in \Psi_2$, is introduced and studied. More precisely, we prove that $(\ell_\psi - \ell_\varphi)^* = \ell_{\psi^*} - \ell_{\varphi^*}$ where $\psi^*$ and $\varphi^*$ are the dual functions of $\psi$ and $\varphi$, respectively. In Section 3, the upper bound of the James constant of the generalized Day-James space is given. Furthermore, we compute $J(\ell_\psi - \ell_\infty)$ and deduce that every generalized Day-James space except $\ell_1 - \ell_1$ and $\ell_\infty - \ell_\infty$ is uniformly nonsquare. This result strengthens Corollary 3 of Saito et al. [10].

**2. Generalized Day-James spaces**

In this section, we introduce a new class of normalized norms on $\mathbb{R}^2$ which properly contains all absolute normalized norms of Bonsall and Duncan [1]. Moreover, we introduce a two-dimensional normed space which is a generalization of Day-James $\ell_p - \ell_q$ spaces.
Lemma 2.1. Let $\psi \in \Psi_2$ and let $\| \cdot \|_{\psi,\psi}$ be a function on $\mathbb{R}^2$ defined by, for all $(z,w) \in \mathbb{R}^2$,

$$
\|(z,w)\|_{\psi,\psi} := \max \{ \|(z^+,w^+)\|_{\psi},\|(z^-,w^-)\|_{\psi} \},
$$

where $x^+$ and $x^-$ are positive and negative parts of $x \in \mathbb{R}$, that is, $x^+ = \max\{x,0\}$ and $x^- = \max\{-x,0\}$. Then $\| \cdot \|_{\psi,\psi}$ is a norm on $\mathbb{R}^2$.

For convenience, we put $\mathcal{B}_{\psi_1,\psi_2} := \{(z,w) \in \mathbb{R}^2 : \|(z,w)\|_{\psi_1,\psi_2} \leq 1\}$.

Theorem 2.2. Let $\psi, \varphi \in \Psi_2$ and

$$
\|(z,w)\|_{\psi,\varphi} := \begin{cases} \|(z,w)\|_{\psi} & \text{if } zw \geq 0, \\ \|(z,w)\|_{\varphi} & \text{if } zw \leq 0 \\ \end{cases}
$$

for all $(z,w) \in \mathbb{R}^2$. Then $\| \cdot \|_{\psi,\varphi}$ is a norm on $\mathbb{R}^2$. Denote by $N_2$ the family of all such preceding norms.

Proof. Let $\psi, \varphi \in \Psi_2$, we only show $\| \cdot \|_{\psi,\varphi}$ satisfies the triangle inequality. To this end, it suffices to prove that $\mathcal{B}_{\psi,\varphi}$ is convex. By Lemma 2.1, we have that $\mathcal{B}_{\psi,\psi}$ and $\mathcal{B}_{\varphi,\psi}$ are closed unit balls of $\| \cdot \|_{\psi,\psi}$ and $\| \cdot \|_{\psi,\varphi}$, respectively, and so $\mathcal{B}_{\psi,\psi}$ and $\mathcal{B}_{\varphi,\psi}$ are convex sets. We define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$
T((z,w)) = (-z,w) \quad \forall (z,w) \in \mathbb{R}^2.
$$

Then $T$ is a linear operator and $T(\mathcal{B}_{\psi,\psi}) = \mathcal{B}_{\psi,\varphi}$, which implies that $\mathcal{B}_{\psi,\varphi}$ is convex and so $\mathcal{B}_{\psi,\varphi} = \mathcal{B}_{\psi_1,\varphi} \cap \mathcal{B}_{\psi_2,\varphi}$ is convex. \hfill $\square$

Taking $\psi = \psi_p$ and $\varphi = \psi_q$ ($1 \leq p, q \leq \infty$) in Theorem 2.2, we obtain the following.

Corollary 2.3 (Day-James $\ell_p$-$\ell_q$ spaces). For $1 \leq p, q \leq \infty$, denote by $\ell_p$-$\ell_q$ the Day-James space, that is, $\mathbb{R}^2$ with the norm defined by, for all $(z,w) \in \mathbb{R}^2$,

$$
\|(z,w)\|_{p,q} := \begin{cases} \|(z,w)\|_p & \text{if } zw \geq 0, \\ \|(z,w)\|_q & \text{if } zw \leq 0. \\ \end{cases}
$$

James [7] considered the $\ell_p$-$\ell_p$ space as an example of a Banach space which is isometric to its dual but which is not given by a Hilbert norm when $p \neq 2$. Day [2] considered even more general spaces, namely, if $(X,\| \cdot \|)$ is a two-dimensional Banach space and $(X^*,\| \cdot \|^*)$ its dual, then the $X$-$X^*$ space is the space $X$ with the norm defined by, for all $(z,w) \in \mathbb{R}^2$,

$$
\|(z,w)\|_{x,x^*} := \begin{cases} \|(z,w)\| & \text{if } zw \geq 0, \\ \|(z,w)\|^* & \text{if } zw \leq 0. \\ \end{cases}
$$
For $\psi, \varphi \in \Psi_2$, denote by $\ell_{\psi} - \ell_{\varphi}$ the generalized Day-James space, that is, $\mathbb{R}^2$ with the norm $\| \cdot \|_{\psi, \varphi}$ defined by (2.2). For $\psi_p$ defined by (1.4), we write $\ell_{\psi} - \ell_p$ for $\ell_{\psi} - \ell_{\psi_p}$. For example, if $1 \leq p, q \leq \infty$, $\ell_p - \ell_q$ means $\ell_{\psi_p} - \ell_{\psi_q}$.

It is worthwhile to mention that there is a normalized norm which is not absolute.

**Proposition 2.4.** There is $\psi \in \Psi_2$ such that $\ell_{\psi} - \ell_{\infty}$ is not isometrically isomorphic to $\ell_{\varphi} - \ell_{\varphi}$ for all $\varphi \in \Psi_2$.

**Proof.** Let

\[
\psi(t) := \begin{cases}
1 - t & \text{if } 0 \leq t \leq \frac{1}{8}, \\
\frac{11 - 4t}{12} & \text{if } \frac{1}{8} \leq t \leq \frac{1}{2}, \\
\frac{1 + t}{2} & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]  

(2.6)

We observe that the sphere of $\ell_{\psi} - \ell_{\infty}$ is the octagon whose right half consists of 4 segments of different lengths. Suppose that there are $\varphi \in \Psi_2$ and an isometric isomorphism from $\ell_{\psi} - \ell_{\infty}$ onto $\ell_{\varphi} - \ell_{\varphi}$. Since the image of each segment in $\ell_{\psi} - \ell_{\infty}$ is again a segment of the same length in $\ell_{\varphi} - \ell_{\varphi}$, the sphere of $\ell_{\varphi} - \ell_{\varphi}$ must be the octagon whose each corresponding side has the same length (measured by $\| \cdot \|_{\varphi}$). We show that this cannot happen. Consider $(1, 0) \in S_{\ell_{\varphi} - \ell_{\varphi}}$. If $(1, 0)$ is an extreme point of $B_{\ell_{\varphi} - \ell_{\varphi}}$, then $S_{\ell_{\psi} - \ell_{\psi}}$ contains 4 segments of same lengths since $\| \cdot \|_{\varphi}$ is absolute. On the other hand, if $(1, 0)$ is an not extreme point of $B_{\ell_{\varphi} - \ell_{\varphi}}$, again $S_{\ell_{\psi} - \ell_{\psi}}$ contains 4 segments of same lengths. □

Next, we prove that the dual of a generalized Day-James space is again a generalized Day-James space. Recall that, for $\psi \in \Psi_2$, the dual function $\psi^*$ of $\psi$ is defined by

\[
\psi^*(s) = \max_{0 \leq t \leq 1} \frac{(1 - s)(1 - t) + st}{\psi(t)}
\]  

for all $s \in [0, 1]$. It was proved that $\psi^* \in \Psi_2$ and $(\ell_{\psi} - \ell_{\psi})^* = \ell_{\psi^*} - \ell_{\psi^*}$ (see [3, Proposition 1 and Theorem 2]). We generalize this result to our spaces as follows.

**Theorem 2.5.** For $\psi, \varphi \in \Psi_2$, there is an isometric isomorphism that identifies $(\ell_{\psi} - \ell_{\varphi})^*$ with $\ell_{\psi^*} - \ell_{\varphi^*}$ such that if $f \in (\ell_{\psi} - \ell_{\varphi})^*$ is identified with the element $(z, w) \in \ell_{\psi^*} - \ell_{\varphi^*}$, then

\[
f(u, v) = zu + wv
\]  

for all $(u, v) \in \mathbb{R}^2$.

**Proof.** We can prove analogous to [3, Theorem 2]. □

### 3. The James constant and uniform nonsquareness

The next lemmas are crucial for proving the main theorems.

**Lemma 3.1.** Let $\psi, \varphi \in \Psi_2$. Then

(i) $\| \cdot \|_{\infty} \leq \| \cdot \|_{\psi, \varphi} \leq \| \cdot \|_1$,
(ii) \((1/M_{\psi,\varphi}) \| \| \psi \leq \| \| \psi, \varphi \leq M_{\psi,\varphi} \| \| \psi, \varphi\)

(iii) \((1/M_{\psi,\varphi}) \| \| \varphi \leq \| \| \varphi, \psi \leq M_{\psi,\varphi} \| \| \varphi, \psi\)

where \(M_{\psi,\varphi} = \max_{0 \leq t \leq 1} \varphi(t)/\psi(t)\) and \(M_{\varphi,\psi} = \max_{0 \leq t \leq 1} \psi(t)/\varphi(t)\).

**Lemma 3.2.** Let \(\psi, \varphi \in \Psi_2\) and let \(Q_i\) \((i = 1, \ldots, 4)\) denote the \(i\)th quadrant in \(\mathbb{R}^2\). Suppose that \(x, y \in S_{\psi, \epsilon}\), then the following statements are true.

(i) If \(x, y \in Q_1\), then \(x + y \in Q_1\) and \(x - y \in Q_2 \cup Q_4\).

(ii) If \(x, y \in Q_2\), then \(x + y \in Q_2\) and \(x - y \in Q_1 \cup Q_3\).

(iii) If \(\psi(t) \leq \varphi(t)\) for all \(t \in [0, 1]\) and \(x - y \in Q_2^c \cup Q_4^c\), where \(Q_2^c\) and \(Q_4^c\) are the interiors of \(Q_2\) and \(Q_4\), respectively, then \(x + y \in Q_1 \cup Q_3\).

We will estimate the James constant of \(\ell_{\psi, \varphi}\).

**Theorem 3.3.** Let \(\psi, \varphi \in \Psi_2\) with \(\psi(t) \leq \varphi(t)\) for all \(t \in [0, 1]\), let \(M_{\psi,\varphi} = \max_{0 \leq t \leq 1} \varphi(t)/\psi(t)\), and let \(\delta\) be the modulus of convexity of \(\ell_{\psi, \varphi}\). Then for \(\epsilon \in [0, 2]\),

\[
\delta_{\psi,\varphi}(\epsilon) = \min \left\{1 - M_{\psi,\varphi} (1 - \delta_{\psi,\varphi}(\epsilon)), \delta, \left(\frac{\epsilon}{M_{\psi,\varphi}}\right)\right\},
\]

where \(\delta_{\psi,\varphi}(\cdot)\) is the modulus of convexity of \(\ell_{\psi, \varphi}\). Consequently,

\[
J(\ell_{\psi, \varphi}) \leq \sup \left\{\epsilon \in (0, 2) : \epsilon \leq 2M_{\psi,\varphi} (1 - \delta_{\psi,\varphi}(\epsilon)) \right\}.
\]

**Proof.** By Lemma 3.1(ii), we have

\[
\| \| \psi \| \psi, \varphi \leq M_{\psi,\varphi} \| \| \psi, \varphi\.
\]

We now evaluate the modulus of convexity \(\delta_{\psi,\varphi}\) for \(\ell_{\psi, \varphi}\). We consider two cases.

**Case 1.** Take \(\| x \|_{\psi, \varphi} = \| y \|_{\psi, \varphi} = 1\) with \(\| x - y \|_{\psi, \varphi} \geq \epsilon\), where \(x - y \in Q_1 \cup Q_3\). Thus \(\| x \|_{\psi} \leq 1\), \(\| y \|_{\psi} \leq 1\), and \(\| x - y \|_{\psi} \geq \epsilon\), which implies that

\[
\frac{1}{2} \| x + y \|_{\psi} \leq 1 - \epsilon.
\]

This in turn implies

\[
\frac{1}{2} \| x + y \|_{\psi, \varphi} \leq \frac{1}{2} M_{\psi,\varphi} \| x + y \|_{\psi} \leq M_{\psi,\varphi} (1 - \delta_{\psi,\varphi})\)

thus

\[
1 - \frac{1}{2} \| x + y \|_{\psi, \varphi} \geq 1 - M_{\psi,\varphi} (1 - \delta_{\psi,\varphi})\).
\]

**Case 2.** Now take \(x, y\) as above, but with \(x - y \in Q_2^c \cup Q_4^c\). By Lemma 3.2(iii), \(x + y \in Q_1 \cup Q_3\). Since \(\| x - y \|_{\psi, \varphi} \geq \epsilon\),

\[
\| x - y \|_{\psi} \geq \frac{\| x - y \|_{\psi, \varphi}}{M_{\psi,\varphi}} \geq \frac{\epsilon}{M_{\psi,\varphi}}.
\]
The James constant of normalized norms on $\mathbb{R}^2$

Then

$$\frac{1}{2} \| x + y \|_{\psi,\varphi} = \frac{1}{2} \| x + y \|_{\psi} \leq 1 - \delta_{\psi} \left( \frac{\varepsilon}{M_{\psi,\varphi}} \right), \quad (3.8)$$

and so

$$1 - \frac{1}{2} \| x + y \|_{\psi,\varphi} \geq \delta_{\psi} \left( \frac{\varepsilon}{M_{\psi,\varphi}} \right). \quad (3.9)$$

Hence we obtain (3.1). By Proposition 1.1(vi), (3.2) follows.

The following corollary shows that we can have equality in (3.2).

**Corollary 3.4** [4, 8]. If $1 \leq q \leq p < \infty$ and $p \geq 2$, then

$$J(\ell_p - \ell_q) \leq 2 \left( \frac{2^{p/q}}{2^{p/q} + 2} \right)^{1/p}. \quad (3.10)$$

In particular, if $p = 2$ and $q = 1$, then $J(\ell_2 - \ell_1) = \sqrt{8/3}$.

**Proof.** It follows that since

$$M_{\psi,\varphi} = 2^{1/q - 1/p}, \quad \delta_{\ell_p - \ell_q}(\varepsilon) = 1 - \left( 1 - \left( \frac{\varepsilon}{2} \right)^p \right)^{1/p}. \quad (3.11)$$

Moreover, if $p = 2$ and $q = 1$, then $J(\ell_2 - \ell_1) \leq \sqrt{8/3}$. Now we put

$$x_0 = \left( \frac{2 + \sqrt{2}}{2\sqrt{3}}, \frac{2 - \sqrt{2}}{2\sqrt{3}} \right), \quad y_0 = \left( \frac{2 - \sqrt{2}}{2\sqrt{3}}, \frac{2 + \sqrt{2}}{2\sqrt{3}} \right). \quad (3.12)$$

Then

$$\| x_0 \|_{2,1} = \| y_0 \|_{2,1} = 1, \quad \| x_0 \pm y_0 \|_{2,1} = \sqrt{\frac{8}{3}}. \quad (3.13) \quad \square$$

**Theorem 3.5.** Let $\psi, \varphi \in \Psi_2$ with $\psi(t) \leq \varphi(t)$ for all $t \in [0,1]$, let $M_{\psi,\varphi} = \max_{0 \leq t \leq 1} \varphi(t)/\psi(t)$, and let $\delta_{\psi}(\cdot)$ be the modulus of convexity of $\ell_{\psi} - \ell_{\psi}$. Then for $\varepsilon \in [0,2]$,

$$\delta_{\psi,\varphi}(\varepsilon) \geq 1 - M_{\psi,\varphi} \left( 1 - \delta_{\psi} \left( \frac{\varepsilon}{M_{\psi,\varphi}} \right) \right), \quad (3.14)$$

where $\delta_{\psi,\varphi}(\cdot)$ is the modulus of convexity of $\ell_{\psi} - \ell_{\varphi}$. Consequently,

$$J(\ell_{\psi} - \ell_{\varphi}) \leq \sup \left\{ \varepsilon \in (0,2) : \varepsilon \leq 2M_{\psi,\varphi} \left( 1 - \delta_{\psi} \left( \frac{\varepsilon}{M_{\psi,\varphi}} \right) \right) \right\}. \quad (3.15)$$

**Proof.** By Lemma 3.1(iii), we have

$$\frac{1}{M_{\psi,\varphi}} \| \cdot \|_{\psi} \leq \| \cdot \|_{\psi,\varphi} \leq \| \cdot \|_{\varphi}. \quad (3.16)$$
Let \( \| x \|_{\psi,\varphi}, \| y \|_{\psi,\varphi} = \| y \|_{\psi,\varphi} = 1 \) with \( \| x - y \|_{\psi,\varphi} \geq \varepsilon \).

Then
\[
\frac{1}{M_{\psi,\varphi}} \| x \|_{\varphi} \leq 1, \quad \frac{1}{M_{\psi,\varphi}} \| y \|_{\varphi} \leq 1,
\]
\[
\frac{1}{M_{\psi,\varphi}} \| x - y \|_{\varphi} \geq \frac{1}{M_{\psi,\varphi}} \| x - y \|_{\psi,\varphi} \geq \frac{\varepsilon}{M_{\psi,\varphi}},
\]
which implies that
\[
\frac{1}{2M_{\psi,\varphi}} \| x + y \|_{\varphi} \leq 1 - \delta_{\varphi} \left( \frac{\varepsilon}{M_{\psi,\varphi}} \right).
\]
This in turn implies that
\[
\frac{1}{2M_{\psi,\varphi}} \| x + y \|_{\psi,\varphi} \leq 1 - \delta_{\psi} \left( \frac{\varepsilon}{M_{\psi,\varphi}} \right),
\]
thus
\[
1 - \frac{1}{2} \| x + y \|_{\psi,\varphi} \geq 1 - M_{\psi,\varphi} \left( 1 - \delta_{\psi} \left( \frac{\varepsilon}{M_{\psi,\varphi}} \right) \right).
\]
Hence we obtain (3.14). By Proposition 1.1(vi), (3.15) follows. \( \square \)

**Corollary 3.6.** If \( 2 \leq q \leq p < \infty \), then
\[
J(\ell_p - \ell_q) \leq 2^{1 - 1/p}.
\]

It is easy to see that the estimate (3.22) is better than one obtained in [4, Example 2.4(3)].

For some generalized Day-James spaces, [8, Corollary 4] of Kato et al. gives only rough result for the estimate of the James constant, that is, for \( \psi \in \Psi_2 \),
\[
\frac{2}{M} \leq J(\ell_\psi - \ell_\infty) \leq 2M,
\]
where \( M = \max_{0 \leq t \leq 1} \psi_\infty(t)/\psi(t) \).

However, the following theorem gives the exact value of the James constant of these spaces.

**Theorem 3.7.** Let \( \psi \in \Psi_2 \). Then
\[
J(\ell_\psi - \ell_\infty) = 1 + \frac{1/2}{\psi(1/2)}.
\]
Proof. For our convenience, we write $\| \cdot \|$ instead of $\| \cdot \|_{\psi,\psi}$. Let $x,y \in S_{\ell_\psi-\ell_\infty}$. We prove that

$$\text{either } \| x + y \| \leq 1 + \frac{1/2}{\psi(1/2)} \quad \text{or} \quad \| x - y \| \leq 1 + \frac{1/2}{\psi(1/2)}. \quad (3.25)$$

Let us consider the following cases.

Case 1. $x,y \in Q_1$. Let $x = (a,b)$ and $y = (c,d)$ where $a,b,c,d \in [0,1]$. By Lemma 3.2(i), we have $x - y \in Q_2 \cup Q_4$. Then

$$\| x - y \| = \max \{|a-c|, |b-d|\} \leq 1 \leq 1 + \frac{1/2}{\psi(1/2)}. \quad (3.26)$$

Case 2. $x,y \in Q_2$. If $x,y$ lies in the same segment, then $\| x - y \| \leq 1$. We now suppose that $x = (-1,a)$ and $y = (-c,1)$ where $a,c \in [0,1]$.

Subcase 2.1. $a \leq (1/2)/\psi(1/2)$ and $c \leq (1/2)/\psi(1/2)$. Then

$$\| x + y \| = \|(1-c,1+a)\|_\infty = \max\{1+c, 1+a\} \leq 1 + \frac{1/2}{\psi(1/2)}. \quad (3.27)$$

Subcase 2.2. $a \geq (1/2)/\psi(1/2)$ or $c \geq (1/2)/\psi(1/2)$. Put $z = (-1,1)$, then

$$\| x - y \| \leq \| x - z \| + \| z - y \| = 1 - a + 1 - c \leq 1 + 1 - \frac{1/2}{\psi(1/2)} \leq 1 + \frac{1/2}{\psi(1/2)}. \quad (3.28)$$

From now on, we may assume without loss of generality that there is $\beta \in [1/2,1]$ such that $\psi(\beta) \leq \psi(t)$ for all $t \in [0,1]$. Indeed, $J(\ell_\psi-\ell_\infty) = J(\ell_\psi-\ell_\infty)$ where $\tilde{\psi}(t) = \psi(1-t)$ for all $t \in [0,1]$.

Case 3. $x \in Q_1$ and $y \in Q_2$. Let $x = (a,b)$, $y = (c,1)$ where $a,b,c \in [0,1]$. We consider three subcases.

Subcase 3.1. $a \leq (1/2)/\psi(1/2)$ or $c \leq (1/2)/\psi(1/2)$. Then

$$\| x - y \| = \|(a+c,b-1)\|_\infty = \max\{a+c, 1-b\} \leq 1 + \frac{1/2}{\psi(1/2)}. \quad (3.29)$$

Subcase 3.2. $(1/2)/\psi(1/2) \leq a \leq c$. Then $b \leq (1/2)/\psi(1/2)$ and

$$\| x + y \| = \|(a-c,b+1)\|_\infty = \max\{c-a, 1+b\} \leq 1 + \frac{1/2}{\psi(1/2)}. \quad (3.30)$$

Subcase 3.3. $(1/2)/\psi(1/2) < c \leq a$. We write $a = (1-t_0)/\psi(t_0)$, $b = t_0/\psi(t_0)$ where $t_0 = b/(a+b)$ and $0 \leq t_0 \leq 1/2$. By the convexity of $\psi$ and $\psi(t) \geq \psi(\beta)$ for all $0 \leq t \leq 1$, we
have $\psi(t_0) \geq \psi(1/2)$ and so $1/\psi(t_0) \leq 1/\psi(1/2)$. By Lemma 3.1(i),

$$
\|x + y\| = \|(a, b) + (-c, 1)\| \leq \|(a - c, b + 1)\|_1
= a - c + b + 1 = \frac{1}{\psi(t_0)} + 1 - c
\leq \frac{1}{\psi(1/2)} + 1 - \frac{1/2}{\psi(1/2)} = 1 + \frac{1/2}{\psi(1/2)}.
$$

(3.31)

Case 4. Let $x = (a, b), y = (-1, c)$ where $a, b, c \in [0, 1]$. We consider three subcases.

Subcase 4.1. $b \leq (1/2)/\psi(1/2)$ or $c \leq (1/2)/\psi(1/2)$. Then

$$
\|x + y\| = \|(a - 1, b + c)\|_{\infty} = \max\{1 - a, b + c\} \leq 1 + \frac{1/2}{\psi(1/2)}.
$$

(3.32)

Subcase 4.2. $(1/2)/\psi(1/2) < b \leq c$. Then $a \leq (1/2)/\psi(1/2)$ and

$$
\|x - y\| = \|(1 + a, b - c)\|_{\infty} = \max\{1 + a, c - b\} \leq 1 + \frac{1/2}{\psi(1/2)}.
$$

(3.33)

Subcase 4.3. $(1/2)/\psi(1/2) < c \leq b$. We write $a = (1 - t_0)/\psi(t_0)$, $b = t_0/\psi(t_0)$, where $t_0 = b/(a + b)$ and $1/2 \leq t_0 \leq 1$. We choose $\alpha = b/(a + 2b - 1)$, then

$$
\frac{1}{2} \leq \alpha \leq 1, \quad a = \frac{1 - 2\alpha}{\alpha}b + 1.
$$

(3.34)

Since $b - c \leq 1 + a$ and $b \leq 1,$

$$
\frac{b - c}{1 + a + b - c} \leq \frac{1}{2} \leq t_0 \leq \alpha.
$$

(3.35)

Let

$$
\psi_{\alpha}(t) = \begin{cases} \alpha - 1 \frac{t}{t + 1} & \text{if } 0 \leq t \leq \alpha, \\ \frac{t}{1} & \text{if } \alpha \leq t \leq 1. \end{cases}
$$

(3.36)

We see that $\psi_{\alpha}(t_0) = \psi(t_0).$ By the convexity of $\psi$, we have

$$
\psi(t) \leq \psi_{\alpha}(t) \quad \forall t \leq t_0.
$$

(3.37)
Therefore, \[
\|x - y\| = \|(a + 1, b - c)\|_\psi = (1 + a + b - c)\psi\left(\frac{b - c}{1 + a + b - c}\right)
\]
\[\leq (1 + a + b - c)\psi(\alpha)\left(\frac{\alpha - 1}{\alpha}b - \frac{\alpha - 1}{\alpha}c\right) = 1 + a + \frac{\alpha - 1}{\alpha}b - \frac{\alpha - 1}{\alpha}c = 1 + 1 - \frac{2\alpha - 1}{\alpha}c
\]
\[< 1 + 1 - \frac{2\alpha - 1}{\alpha}c \frac{1}{\psi(1/2)} = 1 + \frac{1}{\psi(1/2)} + 1 - \frac{3\alpha - 1}{2\alpha} \frac{1}{\psi(1/2)}
\]
\[= 1 + \frac{1}{\psi(1/2)} + 1 - \frac{\psi(1/2)(1/2)}{\psi(1/2)} \leq 1 + \frac{1}{\psi(1/2)}.
\] (3.38)

Finally, we conclude that
\[
J(\ell_\psi - \ell_\infty) \leq 1 + \frac{1/2}{\psi(1/2)}.
\] (3.39)

Now, we put \(x_0 = ((1/2)/\psi(1/2), (1/2)/\psi(1/2))\) and \(y_0 = (-1, 1)\), then
\[
\|x_0\| = \|y_0\| = 1, \quad \|x_0 \pm y_0\| = 1 + \frac{1/2}{\psi(1/2)}.
\] (3.40)

Thus,
\[
J(\ell_\psi - \ell_\infty) \geq \min\{\|x_0 - y_0\|, \|x_0 + y_0\|\} = 1 + \frac{1/2}{\psi(1/2)}.\] (3.41)

This together with (3.39) completes the proof. \(\Box\)

**Corollary 3.8** [4, Example 2.4(2)]. Let \(1 \leq p \leq \infty\), then
\[
J(\ell_p - \ell_\infty) = 1 + \left(\frac{1}{2}\right)^{1/p}.
\] (3.42)

Indeed, \(\psi_p(1/2) = 2^{1/p-1}\).

We now obtain the bounds for \(J(\ell_\psi - \ell_1)\).

**Corollary 3.9.** Let \(\psi \in \Psi_2\). Then
\[
2 \min_{0 \leq t \leq 1} \psi(t) \leq J(\ell_\psi - \ell_1) \leq \frac{3}{2} + \frac{1}{2} \min_{0 \leq t \leq 1} \psi(t).
\] (3.43)

**Proof.** Note that \(\psi^*(1/2) = \max_{0 \leq t \leq 1}(1/2)/\psi(t) = 1/2 \min_{0 \leq t \leq 1} \psi(t)\). By Theorem 3.7, we have \(J(\ell_\psi - \ell_\infty) = 1 + \min_{0 \leq t \leq 1} \psi(t)\). Applying Proposition 1.1(iv), the assertion is obtained. \(\Box\)

We now improve the upper bound for \(J(\ell_p - \ell_1)\) (see also Corollary 3.4).
Corollary 3.10. Let \( 1 \leq p < \infty \). Then
\[
J(\ell_p - \ell_1) \leq \frac{3}{2} + \left( \frac{1}{2} \right)^{2-1/p}.
\] (3.44)

In particular, if \( p \geq 2 \), then
\[
J(\ell_p - \ell_1) \leq \min \left\{ \frac{4}{(2p + 2)^{1/p}}, \frac{3}{2} + \left( \frac{1}{2} \right)^{2-1/p} \right\}.
\] (3.45)

The following corollary follows by Theorem 3.7 and Corollary 3.9.

Corollary 3.11. Let \( \psi \in \Psi_2 \). Then

(i) \( \ell_\psi - \ell_\infty \) is uniformly nonsquare if and only if \( \psi \neq \psi_\infty \),

(ii) \( \ell_\psi - \ell_1 \) is uniformly nonsquare if and only if \( \psi \neq \psi_1 \).

We can say more about the uniform nonsquareness of \( \ell_\psi - \ell_\varphi \).

Theorem 3.12. Let \( \psi, \varphi \in \Psi_2 \). Then all \( \ell_\psi - \ell_\varphi \) except \( \ell_1 - \ell_1 \) and \( \ell_\infty - \ell_\infty \) are uniformly nonsquare.

Proof. If \( \psi = \varphi \), we are done by [10, Corollary 3]. Assume that \( \psi \neq \varphi \). We prove that \( \ell_\psi - \ell_\varphi \) is uniformly nonsquare. Suppose not, that is, there are \( x, y \in S_{\ell_\psi - \ell_\varphi} \) such that \( \|x + y\|_{\psi, \varphi} = 2 \). We consider three cases.

Case 1. \( x, y \in Q_1 \). Then
\[
\|x\|_{\psi, 1} = \|x\|_{\psi} = \|x\|_{\psi, \varphi} = 1,
\]
\[
\|y\|_{\psi, 1} = \|y\|_{\psi} = \|y\|_{\psi, \varphi} = 1.
\] (3.46)

It follows by Lemma 3.2(i) that \( x + y \in Q_1 \) and \( x - y \in Q_2 \cup Q_4 \). Therefore
\[
\|x + y\|_{\psi, 1} = \|x + y\|_{\psi, \varphi} = 2,
\]
\[
2 = \|x - y\|_{\psi, \varphi} \leq \|x - y\|_1 = \|x - y\|_{\psi, 1} \leq 2.
\] (3.47)

Hence \( \|x + y\|_{\psi, 1} = 2 \) and this implies that \( \ell_\psi - \ell_1 \) is not uniformly nonsquare. By Corollary 3.11(ii), we have \( \psi = \psi_1 \). Again, since \( \ell_\psi - \ell_\varphi = \ell_1 - \ell_\varphi \) is not uniformly nonsquare, \( \varphi = \psi_1 = \psi \); a contradiction.

Case 2. \( x, y \in Q_2 \). It is similar to Case 1, so we omit the proof.

Case 3. \( x := (a, b) \in Q_1 \) and \( y := (-c, d) \in Q_2 \) where \( a, b, c, d \in [0, 1] \). Since \( \|x + y\|_{\psi, \varphi} = 2 \), the line segment joining \( x \) and \( y \) must lie in the sphere. In particular, there is \( \alpha \in [0, 1] \) such that
\[
(0, 1) = \alpha x + (1 - \alpha)y.
\] (3.48)

It follows that \( b = 1 \) since \( b, d \leq 1 \). Similarly consider \( x \) and \( -y \) instead of \( x \) and \( y \), we can also conclude that \( a = 1 \). Hence \( \|(1, 1)\|_{\psi} = \|(1, 1)\|_{\psi, \varphi} = 1 \), that is, \( \psi(1/2) = 1/2 \). Then \( \psi = \psi_\infty \) and so \( \ell_\psi - \ell_\varphi = \ell_\infty - \ell_\varphi \) is not uniformly nonsquare. By Corollary 3.11(i), we have \( \varphi = \psi_\infty = \psi \); a contradiction. \( \square \)
The James constant of normalized norms on $\mathbb{R}^2$

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