We consider the oscillatory property of the following $p(t)$-Laplacian equations

$$-(|u'|^{p(t)-2}u')' = \frac{1}{t^{\theta(t)}} g(t, u), \quad t > 0.$$  

Since there is no Picone-type identity for $p(t)$-Laplacian equations, it is an unsolved problem that whether the Sturmian comparison theorems for $p(x)$-Laplacian equations are valid or not. We obtain sufficient conditions of the oscillatory of solutions for $p(t)$-Laplacian equations.

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1. Introduction

In recent years, the study of differential equations and variational problems with non-standard $p(x)$-growth conditions have been an interesting topic (see [1–6]). The study of such problems arise from nonlinear elasticity theory, electrorheological fluids (see [3, 6]). On the asymptotic behavior of solutions of $p(x)$-Laplacian equations on unbounded domain, we refer to [5].

In this paper, we consider the oscillation problem

$$-\Delta_{p(t)} u := -(|u'|^{p(t)-2}u')' = \frac{1}{t^{\theta(t)}} g(t, u), \quad t > 0,$$

where $p : \mathbb{R} \to (1, \infty)$ is a function, and $-\Delta_{p(t)}$ is called $p(t)$-Laplacian.

By an oscillatory solution we mean one having an infinite number of zeros on $0 < t < \infty$. Otherwise, the solution is said to be nonoscillatory. Hence, a nonoscillatory solution eventually keeps either positive or negative. It is called a positive (or negative) solution.

If $p(t) \equiv p$ is a constant, then $-\Delta_{p(t)}$ is the well-known $p$-Laplacian, and (1.1) is the usual $p$-Laplacian equation. But if $p(t)$ is a function, the $-\Delta_{p(t)}$ is more complicated...
than $-\Delta_p$, since it represents a nonhomogeneity and possesses more nonlinearity; for example, if $\Omega$ is bounded, the Rayleigh quotient
\[
\lambda_{p(t)} = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (1/p(t)) |\nabla u(t)|^p dt}{\int_{\Omega} (1/p(t)) |u(t)|^p dt},
\]
is zero in general, and only under some special conditions $\lambda_{p(t)}>0$ (see [2]), but the fact that $\lambda_p>0$ is very important in the study of $p$-Laplacian problems.

It is well known that, there exists Picone-type identity for $p$-Laplacian equations, and then it is easy to obtain Sturmian comparison theorems for $p$-Laplacian equations, which is very important in the study of the oscillation of the solutions of $p$-Laplacian equations. There are many papers about the oscillation problem of $p$-Laplacian equations (see [7–10]). On the typical $p$-Laplacian problem
\[
-\Delta_p u = \frac{\lambda}{t^p} |u|^{p-2} u, \quad t>0,
\]
when $\lambda>((p-1)/p)^p$, then all the solutions oscillation, but when $\lambda\leq((p-1)/p)^p$, then all the solutions are nonoscillation (see [10]). But there is no Picone-type identity for $p(t)$-Laplacian equations, it is an unsolved problem that whether the Sturmian comparison theorems for $p(t)$-Laplacian equations are valid or not. The results on the oscillation problem of $p(t)$-Laplacian equations are rare.

We say a function $f: \mathbb{R} \to \mathbb{R}$ possesses property $(H)$ if it is continuous and satisfies $\lim_{t \to +\infty} f(t) = f_\infty$, and $t|f(t)-f_\infty| \leq M^*$ for $t>0$.

Throughout the paper, we always assume that
$$(A_1) \quad \theta \in C(\mathbb{R}^+, \mathbb{R}), \, p \in C^1(\mathbb{R},(1,\infty)) \text{ and satisfies } 1<\inf_{x \in \mathbb{R}} p(x) \leq \sup_{x \in \mathbb{R}} p(x)<+\infty; \quad (1.4)$$

$$(A_2) \quad g \text{ is continuous on } \mathbb{R}^+ \times \mathbb{R}, \, g(t, \cdot) \text{ is increasing for any fixed } t>0, \, g(t,u)u > 0 \text{ for any } u \neq 0 \text{ and satisfies } 0 < \lim_{t \to +\infty} g(t,u)u \leq \lim_{t \to +\infty} g(t,u)u < +\infty, \quad \forall u \in \mathbb{R}\setminus\{0\}. \quad (1.5)$$

The main results of this paper are as follows.

**Theorem 1.1.** Assume that $\lim_{t \to +\infty} \theta(t) < \lim_{t \to +\infty} p(t)$, suppose that (1.1) has a positive solution $u$, then $u$ is increasing for $t$ sufficiently large, and $u$ tends to $+\infty$ as $t \to +\infty$.

**Theorem 1.2.** Assume that $p$ possesses property $(H)$ and $g(t,u) = |u|^{q(t)}-2 u$, where $\theta$ satisfies
\[
\lim_{t \to +\infty} \theta(t) < \lim_{t \to +\infty} q(t), \quad (1.6)
\]
where $q$ satisfies
\[
1 < \lim_{t \to +\infty} q(t) < \lim_{t \to +\infty} p(t), \quad (1.7)
\]
or \( \lim_{t \to +\infty} q(t) = \lim_{t \to -\infty} p(t) \) and \( q(t) \) possesses property \( (H) \), then all the solutions of (1.1) are oscillatory.

2. Proofs of main results

In the following, we denote \( -(\varphi(t,u'))' = -(|u'|^{p(t)-2}u')' \), and use \( C_i \) and \( c_i \) to denote positive constants.

Proof of Theorem 1.1. Let \( u(t) \) be a positive solution of (1.1), then there exists a \( T > 0 \) such that \( u(t) > 0 \) for \( t \geq T \). Hence, by (A2), we have

\[
(\varphi(t,u'))' = -\frac{1}{t^{\theta(t)}}g(t,u) < 0 \quad \text{for } t > T. \tag{2.1}
\]

We first show that \( u' > 0 \) for \( t > T \). If it is false, we suppose that there exists a \( t_1 \geq T \) such that \( u'(t_1) \leq 0 \). Since \( ug(t,u) > 0 \) when \( u \neq 0 \), by (2.1), we have

\[
\varphi(t,u'(t)) < \varphi(t_1,u'(t_1)) \leq 0 \quad \text{for } t > t_1. \tag{2.2}
\]

Hence we can find a \( t_2 > t_1 \) such that \( u'(t_2) < 0 \). Integrating both sides of (2.1) from \( t_2 \) to \( t \), we get \( \varphi(t,u'(t)) \leq \varphi(t_2,u'(t_2)) < 0 \) for \( t > t_2 \), and therefore

\[
u'(t) \leq -|u'(t_2)|^\frac{(p(t_2)-1)/(p(t)-1)}{\left|u'(t_2)\right|^\frac{(p(t_2)-1)/(p(t)-1)}} \leq -\min_{t \geq t_2} \left|u'(t)\right|^\frac{(p(t_2)-1)/(p(t)-1)} := -a < 0. \tag{2.3}
\]

Integrate this inequality to obtain \( u(t) \leq -a(t-t_2) + u(t_2) \rightarrow -\infty \), as \( t \rightarrow +\infty \). It is a contradiction. Thus, \( u(t) \) is increasing for \( t \geq T \).

We next suppose that there exists a \( K > 0 \) such that \( u(t) \leq K \) for \( t \geq T \). Since \( u(t) \) is increasing, then \( u(t) \geq u(T) \) for \( t \geq T \). From (2.1), we have

\[
0 < \varphi(t,u'(t)) = \varphi(T,u'(T)) - \int_t^T \frac{1}{t^{\theta(t)}}g(t,u)dt. \tag{2.4}
\]

Since \( u \) is a bounded positive solution, then it is easy to see that

\[
0 = \lim_{t \to -\infty} \varphi(t,u'(t)) = \varphi(T,u'(T)) - \lim_{t \to -\infty} \int_t^T \frac{1}{t^{\theta(t)}}g(t,u)dt,
\]

\[
\varphi(t,u'(t)) = \int_t^{+\infty} \frac{1}{t^{\theta(t)}}g(t,u)dt. \tag{2.5}
\]

Denote \( \theta_* = \{\lim_{t \to +\infty} p(t) + \max\{1,\lim_{t \to +\infty} \theta(t)\}\}/2 \), when \( t \) is large enough, we have \( u'(t) \geq u^{-1}(t,\int_t^{+\infty} (1/t^{\theta_*})c)dt \), then

\[
u(t) - u(T) \geq \int_T^t \varphi^{-1} \left( t, \int_t^{+\infty} \frac{1}{t^{\theta_*}}c \right)dt \rightarrow +\infty. \tag{2.6}
\]

It is a contradiction, thereby completing the proof.
Proof of Theorem 1.2. If it is false, then we may assume that (1.1) has a positive solution \( u \). From Theorem 1.1, we can see that \( u \) is increasing, then

\[
0 \leq \lim_{t \to +\infty} \varphi(t, u'(t)) = \varphi(T, u'(T)) - \lim_{t \to +\infty} \int_{T}^{t} \frac{1}{t^{\theta(t)}} g(t, u) dt. \tag{2.7}
\]

If \( \lim_{t \to +\infty} \varphi(t, u'(t)) > 0 \), then there exists a positive constant \( a \) such that

\[
\varphi(t, u'(t)) = \varphi(T, u'(T)) - \int_{T}^{t} \frac{1}{t^{\theta(t)}} g(t, u) dt = a + \int_{t}^{+\infty} \frac{1}{t^{\theta(t)}} g(t, u) dt, \tag{2.8}
\]

then there exists a positive constant \( k \) such that \( u(t) \geq kt \) for \( t \geq T \). From (1.6), when \( t \) is large enough, we have

\[
\varphi(T, u'(T)) \geq \varphi(t, u'(t)) = a + \int_{t}^{+\infty} \frac{1}{t^{\theta(t)}} (kt)^{q(t)-1} dt = +\infty. \tag{2.9}
\]

It is a contradiction. Then we have

\[
\lim_{t \to +\infty} \varphi(t, u'(t)) = 0, \tag{2.10}
\]

\[
\varphi(t, u'(t)) = \int_{t}^{+\infty} \frac{1}{t^{\theta(t)}} g(t, u) dt. \tag{2.11}
\]

There are two cases.

(i) Equation (1.7) is satisfied. From (1.6) and (1.7), there exists a \( T_1 > T \) which is large enough such that

\[
\theta^+ := \sup_{t \geq T_1} \theta(t) < q^- := \inf_{t \geq T_1} q(t),
\]

\[
q^+ := \sup_{t \geq T_1} q(t) < p^- := \inf_{t \geq T_1} p(t). \tag{2.12}
\]

If \( \theta^+ \leq 1 \), since \( u \) is increasing, then

\[
\varphi(t, u'(t)) = \int_{t}^{+\infty} \frac{1}{t^{\theta(t)}} g(t, u) dt \geq \int_{t}^{+\infty} \frac{1}{t^{\theta^+}} c_1 dt = +\infty, \quad \forall t \geq T_1. \tag{2.13}
\]

It is a contradiction to (2.10). Thus \( 1 < \theta^+ < p^- \). Since \( u \) is increasing, then

\[
\varphi(t, u'(t)) = \int_{t}^{+\infty} \frac{1}{t^{\theta(t)}} g(t, u) dt \geq \int_{t}^{+\infty} \frac{1}{t^{\theta^+}} c_1 dt = \frac{c_1}{\theta^+ - 1} \int_{t}^{+\infty} \frac{1}{t^{\theta^+ - 1}} dt, \quad \forall t \geq T_1, \tag{2.14}
\]

\[
u(t) \geq \varphi^{-1}\left(t, \frac{c_1}{\theta^+ - 1} \int_{t}^{+\infty} \frac{1}{t^{\theta^+ - 1}} dt\right), \quad \forall t \geq T_1. \tag{2.15}
\]
Thus, there exist $T_2 > T_1$ and positive constants $C_1$ and $c_2$ such that
\[
u'(t) \geq c_2 \left( \frac{1}{t^{\theta_1-1}} \right)^{(p^-)^{-1}}, \quad u(t) \geq C_1 t^{-(\theta_1-1)/(p^-)-1} = C_1 t^{(p^- - \theta_1)/(p^-)-1}, \quad \forall t > T_2.
\]
\[
(2.16)
\]
From (2.11), when $t > T_2$, we have
\[
\varphi(t, u'(t)) \geq \int_t^{+\infty} \frac{1}{t^{\theta_1}} \big( C_1 t^{(p^- - \theta_1)/(p^-)-1} \big) dt = \int_t^{+\infty} \frac{(C_1)^{(q^-)-1}}{t^{\theta_1} - (p^- - \theta_1)/(p^-)-1)} dt.
\]
\[
(2.17)
\]
Denote $\theta_0 = \theta^+, \theta_1 = \theta^+ - ((p^- - \theta_0)/(p^- - 1))(q^- - 1)$. If $\theta_1 \leq 1$, then we have
\[
\varphi(t, u'(t)) \geq \int_t^{+\infty} \frac{(C_1)^{(q^-)-1}}{t^{\theta_1} - 1]} dt = +\infty.
\]
\[
(2.18)
\]
It is a contradiction to (2.10). Thus $1 < \theta_1 < p^-$, and we have
\[
u'(t) \geq \varphi^{-1} \left( t, \frac{(C_1)^{(q^-)-1}}{t^{\theta_1} - 1]} \right), \quad \forall t > T_2,
\]
\[
(2.19)
\]
then, there exists $T_3 > T_2$ and positive constant $c_3$ and $C_2$ such that
\[
u'(t) \geq c_3 \left( \frac{1}{t^{\theta_1-1}} \right)^{(p^-)^{-1}}, \quad u(t) \geq C_2 t^{-(\theta_1-1)/(p^-)-1} = C_2 t^{(p^- - \theta_1)/(p^-)-1}, \quad \forall t > T_3.
\]
\[
(2.20)
\]
Thus
\[
\varphi(t, u'(t)) = \int_t^{+\infty} \frac{1}{t^{\theta_1}} \varphi(t, u) dt \geq \int_t^{+\infty} \frac{(C_2)^{(q^-)-1}}{t^{\theta_1} - (p^- - \theta_1)/(p^-)-1)} dt.
\]
\[
(2.21)
\]
Denote $\theta_2 = \theta^+ - ((p^- - \theta_1)/(p^- - 1))(q^- - 1)$. If $\theta_2 \leq 1$, then
\[
\varphi(t, u'(t)) \geq \int_t^{+\infty} \frac{(C_3)^{(q^-)-1}}{t^{\theta_1} - 1]} dt = +\infty.
\]
\[
(2.22)
\]
It is a contradiction to (2.10). Thus $1 < \theta_2 < p^-$. So, we get a sequence $\theta_n > 1$ and satisfy $\theta_{n+1} = \theta^+ - ((p^- - \theta_n)/(p^- - 1))(q^- - 1), \quad n = 0, 1, 2, \ldots$. Then
\[
\theta_{n+1} = \theta_0 + \sum_{k=0}^{n} \left( \frac{q^- - 1}{p^- - 1} \right) \theta_1 - \theta_0), \quad n = 1, 2, \ldots.
\]
\[
(2.23)
\]
Since (1.7) is valid, then $q^- < p^-$, thus
\[
\lim_{n \to +\infty} \theta_{n+1} = \theta_0 - \frac{p^- - q^-}{p^- - q^-} (q^- - 1) \leq \theta_0 - (q^- - 1) < 1.
\]
\[
(2.24)
\]
It is a contradiction to $\theta_n > 1$. 

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Let \( \Omega \) be nonempty and open. Let us consider a radial solution \( u \geq 0 \) of \( (3.1) \). Then \( (3.1) \) can be transformed into

\[-(t^{N-1}|u'|^{p(t)-2}u')' = \frac{t^{N-1}}{t^{\theta(t)}}|u|^{q(t)-2}u, \quad t > r_0.\]
Theorem 3.1. Assume that \( p(t) \) satisfies \( N < \inf p(x) \), and \( \lim_{t \to +\infty} p(t) = p, p(t), q(t) \), and \( \theta(t) \) satisfies the conditions of Theorem 1.2, then every radial solution of (3.1) is oscillatory.

Proof. Denote \( s = \int_0^t \frac{t^{(1-N)/(p(t)-1)}}{d\tau} d\tau \), then \( ds/dt = t^{(1-N)/(p(t)-1)} \), and \( s \to +\infty \) if and only if \( t \to +\infty \). It is easy to see that (3.2) can be transformed into

\[
-\frac{d}{ds} \left( \frac{ds}{d\tau} \right)^{p(s)-2} \frac{ds}{d\tau} = t^{(N-1)/(p(t)-1)} t^{N-1} \frac{\theta(t)}{t^{\theta(t)}} g(t,u), \quad t > r_0.
\]

(3.3)

It is easy to see that

\[
0 < \lim_{t \to +\infty} \left[ \frac{t^{(N-1)/(p(t)-1)+N-1-\theta(t)}}{s^{(p-1)/(p-N) \theta(t)-(N-1)p/(p-1))}} \right] = +\infty.
\]

(3.4)

Since \( \lim_{t \to +\infty} \theta(t) < \lim_{t \to +\infty} q(t) \), it is easy to see that

\[
\frac{p-1}{p-N} \left( \lim_{s \to +\infty} \theta(s) - \frac{(N-1)p}{p-1} \right) < \lim_{s \to +\infty} q(s).
\]

(3.5)

According to Theorem 1.2, then every radial solution of (3.1) is oscillatory. \( \square \)

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References


Qihu Zhang: Information and Computation Science Department, Zhengzhou University of Light Industry, Zhengzhou, Henan 450002, China

*Email address*: zhangqh1999@yahoo.com.cn