Research Article

A Note on the $q$-Genocchi Numbers and Polynomials

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We discuss new concept of the $q$-extension of Genocchi numbers and give some relations between $q$-Genocchi polynomials and $q$-Euler numbers.

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1. Introduction

The Genocchi numbers $G_n$, $n = 0, 1, 2, \ldots$, which can be defined by the generating function

$$
\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad |t| < \pi,
$$

have numerous important applications in number theory, combinatorics, and numerical analysis, among other areas, [1–13]. It is easy to find the values $G_1 = 1$, $G_3 = G_5 = G_7 = \cdots = 0$, and even coefficients are given by $G_{2n} = 2(1 - 2^{2n})B_{2n} = 2nE_{2n-1}(0)$, where $B_n$ is a Bernoulli number and $E_n(x)$ is an Euler polynomial. The first few Genocchi numbers for $n = 2, 4, \ldots$ are $-1, -3, 17, -155, 2073, \ldots$. The Euler polynomials are well known as

$$
\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad \text{(see [1, 3, 7–9])}.
$$

By (1.1) and (1.2) we easily see that

$$
E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{G_{k+1}}{k+1} x^{n-k}, \quad \text{where} \quad \binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k!} \quad \text{(cf. [4–6])}.
$$

(1.3)
For $m, n \geq 1$ and, $m$ odd, we have

\[(n^m - n) G_m = \sum_{k=1}^{m-1} \binom{m}{k} n^k Z_{m-k}(n-1), \tag{1.4}\]

where $Z_m(n) = 1^m - 2^m + 3^m - \cdots + (-1)^{n-1} n^m$, see [3, 13]. From (1.15) we derive

\[2t = \sum_{n=0}^{\infty} \left( (G + 1)^n + G_n \right) \frac{t^n}{n!}, \tag{1.5}\]

where we use the technique method notation by replacing $G_m$ by $G_m(m \geq 0)$, symbolically. By comparing the coefficients on both sides in (1.5), we see that

\[G_0 = 0, \quad (G + 1)^n + G_n = \begin{cases} 2 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{1.6}\]

Let $p$ be a fixed odd prime, and let $\mathbb{C}_p$ denote the $p$-adic completion of the algebraic closure of $\mathbb{Q}_p (= p$-adic number field $)$. For $d$ is a fixed positive integer with $(p, d) = 1$, let

\[X = X_d = \lim_{N \to \infty} \frac{\mathbb{Z}}{dN \mathbb{Z}}, \quad X_1 = \mathbb{Z}_p, \quad X^* = \bigcup_{0 < a < dp} (a + dp \mathbb{Z}_p), \tag{1.7}\]

\[a + dp \mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{d} \}, \quad \text{where } a \in \mathbb{Z} \text{ lies in } 0 \leq a < dp^N. \]

Ordinary $q$-calculus is now very well understood from many different points of view. Let us consider a complex number $q \in \mathbb{C}$ with $|q| < 1$ (or $q \in \mathbb{C}_p$ with $|1 - q|_p < p^{-1/(p-1)}$) as an indeterminate. The $q$-basic numbers are defined by

\[[x]_q = \frac{q^x - 1}{q - 1}, \quad [x]_{-q} = \frac{(-q)^x + 1}{q + 1}, \quad \text{for } x \in \mathbb{R}. \tag{1.8}\]

We say that $f$ is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$, if the difference quotients

\[F_f(x, y) = \frac{f(x) - f(y)}{x - y} \tag{1.9}\]

have a limit $l = f'(a)$ as $(x, y) \to (a, a)$.

For $f \in UD(\mathbb{Z}_p)$, let us start with the expression

\[\frac{1}{[p^N]} \sum_{0 \leq j < p^N} q^j f(j) = \sum_{0 \leq j < p^N} f(j) \mu_q(j + p^N \mathbb{Z}_p) \tag{1.10}\]
representing a $q$-analogue of Riemann sums for $f$, (cf. [5]). The integral of $f$ on $\mathbb{Z}_p$ will be defined as limit ($n \to \infty$) of those sums, when it exists. The $p$-adic $q$-integral of the function $f \in UD(\mathbb{Z}_p)$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[pN]_q} \sum_{0 \leq x < pN} f(x) q^x, \text{ (see [5, 10–12]).} \quad (1.11)$$

In the previous paper [4, 9], the author constructed the $q$-extension of Euler polynomials by using $p$-adic $q$-fermionic integral on $\mathbb{Z}_p$ as follows:

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} [t+x]_q^n d\mu_{-q}(t), \quad \text{where } \mu_{-q}(x + pN \mathbb{Z}_p) = \frac{(-q)^x}{[pN]_{-q}}. \quad (1.12)$$

From (1.12), we note that

$$E_{n,q}(x) = \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{1+q^{2l+1}} q^{xl}, \text{ see [4].} \quad (1.13)$$

The $q$-extension of Genocchi numbers is defined as

$$g^*_q(t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]_q t} = \sum_{n=0}^{\infty} G^*_n \frac{t^n}{n!}, \text{ see [4].} \quad (1.14)$$

The following formula is well known in [4, 7]:

$$E_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} q^{[x]_q^n-k} [x]_q^{k+1} G^*_{k+1,q} \frac{k+x}{k+1}. \quad (1.15)$$

The modified $q$-Euler numbers are defined as

$$\xi_{0,q} = \frac{[2]_q}{2}, \quad (q\xi+1)^k + \xi_{k,q} = \begin{cases} [2]_q & \text{if } k = 0, \\ 0 & \text{if } k \neq 0, \end{cases} \quad (1.16)$$

with the usual convention of replacing $\xi^i$ by $\xi_{i,q}$, see [10]. Thus, we derive the generating function of $\xi_{n,q}$ as follows:

$$F_q(t) = [2]_q \sum_{k=0}^{\infty} (-1)^k e^{[k]_q t} = \sum_{n=0}^{\infty} \xi_{n,q} \frac{t^n}{n!}. \quad (1.17)$$

Now we also consider the $q$-Euler polynomials $\xi_{n,q}(x)$ as

$$F_q(t,x) = [2]_q \sum_{k=0}^{\infty} (-1)^k e^{[k+x]_q t} = \sum_{n=0}^{\infty} \xi_{n,q}(x) \frac{t^n}{n!}. \quad (1.18)$$

From (1.18) we note that

$$\xi_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} \xi_{l,q} q^{[x]_q^n-l}, \text{ see [10].} \quad (1.19)$$
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In the recent, several authors studied the $q$-extension of Genocchi numbers and polynomials (see [1, 2, 5–7, 12]). In this paper we discuss the new concept of the $q$-extension of Genocchi numbers and give the same relations between $q$-Genocchi numbers and $q$-Euler numbers.

2. $q$-extension of Genocchi numbers

In this section we assume that $q \in \mathbb{C}$ with $|q| < 1$. Now we consider the $q$-extension of Genocchi numbers as follows:

$$g_q(t) = [2]_q t \sum_{k=0}^{\infty} (-1)^k e^{[k]_q t} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}. \quad (2.1)$$

In (2.1), it is easy to show that $\lim_{q \to 1} g_q(t) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n(t^n/n!)$. From (2.1) we derive

$$g_q(t) = [2]_q t \sum_{k=0}^{\infty} (-1)^k \sum_{m=0}^{\infty} \frac{[k]_q^m}{m!} = [2]_q t \sum_{k=0}^{\infty} (-1)^k \sum_{m=1}^{\infty} \frac{m[k]_q^{m-1} t^m}{m!} \quad (2.2)$$

By (2.2), we easily see that

$$g_q(t) = [2]_q \sum_{m=0}^{\infty} \left( m \left( \frac{1}{1-q} \right)^m \sum_{l=0}^{m-1} \frac{(m-1)_l}{l!} \frac{1}{1+q^l} \right) \frac{t^m}{m!}. \quad (2.3)$$

From (2.1) and (2.3) we note that

$$\sum_{m=0}^{\infty} G_{m,q} \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left( m[2]_q \left( \frac{1}{1-q} \right)^m \sum_{l=0}^{m-1} \frac{(m-1)_l}{l!} \frac{1}{1+q^l} \right) \frac{t^m}{m!}. \quad (2.4)$$

By comparing the coefficients on both sides in (2.4), we have the following theorem.

**Theorem 2.1.** For $m \geq 0$,

$$G_{m,q} = m[2]_q \left( \frac{1}{1-q} \right)^m \sum_{l=0}^{m-1} \frac{(m-1)_l}{l!} \frac{1}{1+q^l}. \quad (2.5)$$

From Theorem 2.1, we easily derive the following corollary.

**Corollary 2.2.** For $k \in \mathbb{N}$,

$$G_{0,q} = 0, \quad (qG + 1)^k + G_{k,q} = \begin{cases} [2]_q^2 & \text{if } k = 1, \\ 0 & \text{if } k > 1, \end{cases} \quad (2.6)$$

with the usual convention of replacing $G^i$ by $G_{i,q}$. 
Remark 2.3. We note that Corollary 2.2 is the \(q\)-extension of (1.6). By (1.15)–(1.19) and Corollary 2.2, we obtain the following theorem.

Theorem 2.4. For \(n \in \mathbb{N}\)

\[
\xi_{n,q} = \frac{G_{n+1,q}}{n+1}.
\]  

(2.7)

From (1.18) we derive

\[
F_q(x,t) = [2]_q \sum_{n=0}^{\infty} (-1)^n e^{[n+x]_q t} = q^x t \frac{[2]_q}{q^x t} \sum_{n=0}^{\infty} (-1)^n q^{n[n]_q t}
\]

\[
= e^{[x]_q t} \sum_{n=0}^{\infty} q^{nx} \frac{G_{n+1,q}}{n+1} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} [x]_q \sum_{k=0}^{n-k} \frac{q^{[k+x]_q} G_{k+1,q}}{k+1} \right) \frac{t^n}{n!}.
\]  

(2.8)

By (2.8), we easily see that

\[
\xi_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} [x]_q \sum_{k=0}^{n-k} \frac{q^{[k+x]_q} G_{k+1,q}}{k+1}.
\]  

(2.9)

This formula can be considered as the \(q\)-extension of (1.3). Let us consider the \(q\)-analogue of Genocchi polynomials as follows:

\[
g_q(x,t) = [2]_q \sum_{k=0}^{\infty} (-1)^k e^{[k+x]_q t} = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}.
\]  

(2.10)

Thus, we note that \(\lim_{q \to 1} g_q(x,t) = (2t/(e^t + 1))e^xt = \sum_{n=0}^{\infty} G_n(x)(t^n/n!)\). From (2.10), we easily derive

\[
G_{n,q}(x) = [2]_q n! \frac{1}{1-q} \sum_{l=0}^{n-1} \frac{(-1)^l}{1+q^l q^{[l]}_q} \left( \sum_{k=0}^{n} \binom{n}{k} [x]_q \sum_{k=0}^{n-k} \frac{q^{[k+x]_q} G_{k+1,q}}{k+1} \right).
\]  

(2.11)

By (2.10) we also see that

\[
\sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!} = [2]_q \sum_{k=0}^{\infty} (-1)^k e^{[k+x]_q t} = [2]_q t \sum_{a=0}^{\infty} (-1)^a \frac{[m]_{q\times} [2]_{q\times}}{q^{[a+x]/m}_{q\times}} \sum_{k=0}^{\infty} (-1)^k e^{[k+(a+x)/m]_qx [m]_{q\times}}
\]

\[
= \frac{[2]_q}{[m]_{q\times} [2]_{q\times}} \sum_{a=0}^{m-1} (-1)^a \left( [m]_{q\times} \frac{2}{q^{[a+b]/m}_{q\times}} \sum_{k=0}^{\infty} (-1)^k e^{[k+(a+x)/m]_qx [m]_{q\times}} \right) \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{[2]_q}{[m]_{q\times} [2]_{q\times}} \sum_{a=0}^{m-1} (-1)^a G_{n,q_{m\times}}(x/m) \right) \frac{t^n}{n!}, \quad \text{where } m \in \mathbb{N} \text{ odd.}
\]  

(2.12)

Therefore, we obtain the following theorem.
Theorem 2.5. Let \( m (= \text{odd}) \in \mathbb{N} \). Then the distribution of the \( q \)-Genocchi polynomials will be as follows:

\[
G_{n, q}(x) = \frac{[2]_q}{[2]_q^m} [m]_q^{n-1} \sum_{a=0}^{m-1} (-1)^a G_{n, q^a} \left( \frac{x + a}{m} \right),
\]

where \( n \) is positive integer.

Theorem 2.5 will be used to construct the \( p \)-adic \( q \)-Genocchi measures which will be treated in the next section. Let\( \chi \) be a primitive Dirichlet character with a conductor \( d (= \text{odd}) \in \mathbb{N} \). Then the generalized \( q \)-Genocchi numbers attached to \( \chi \) are defined as

\[
g_{\chi, q}(t) = \frac{[2]_q t}{[2]_q^d} [d]_q^{n-1} \sum_{a=0}^{d-1} (-1)^a \chi(a) G_{n, q^a} \left( \frac{a}{d} \right).
\]

From (2.14), we derive

\[
G_{n, \chi, q} = \frac{[2]_q}{[2]_q^d} [d]_q^{n-1} \sum_{a=0}^{d-1} (-1)^a \chi(a) G_{n, q^a} \left( \frac{a}{d} \right).
\]

3. \( p \)-adic \( q \)-Genocchi measures

In this section we assume that \( q \in \mathbb{C}_p \) with \( |1 - q|_p < p^{-1/(p-1)} \) so that \( q^x = \exp(x \log q) \). Let \( \chi \) be a primitive Dirichlet’s character with a conductor \( d (= \text{odd}) \in \mathbb{N} \). For any positive integers \( N, k, \) and \( d (= \text{odd}) \), let \( \mu_k = \mu_{k, q; G} \) be defined as

\[
\mu_k(a + dp^N \mathbb{Z}_p) = (-1)^a \left[ dp^N \right]_q^{k-1} \frac{[2]_q}{[2]_q^{d+N}} G_{k, dp^N} \left( \frac{a}{dp^N} \right).
\]

By using Theorem 2.5 and (3.1), we show that

\[
\sum_{i=0}^{p-1} \mu_k(a + idp^N + dp^{N+1} \mathbb{Z}_p) = \mu_k(a + dp^N \mathbb{Z}_p).
\]

Therefore, we obtain the following theorem.

Theorem 3.1. Let \( d \) be an odd positive integer. For any positive integers \( N, k, \) and let \( \mu_k = \mu_{k, q; G} \) be defined as

\[
\mu_k(a + dp^N \mathbb{Z}_p) = (-1)^a \left[ dp^N \right]_q^{k-1} \frac{[2]_q}{[2]_q^{d+N}} G_{k, dp^N} \left( \frac{a}{dp^N} \right).
\]

Then \( \mu_k \) can be extended to a distribution on \( X \).

From the definition of \( \mu_k \) and (2.15) we note that

\[
\int_X \chi(x) d\mu_k(x) = G_{k, \chi, q}.
\]
By (2.1) and (2.3), it is not difficult to show that

\[ G_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} [x]_{q}^{n-k} q^{kx} G_{k,q}. \]  

(3.5)

From (3.1) and (3.5) we derive

\[ d\mu_k(a) = \lim_{N \to \infty} \mu_k(a + dp^N \mathbb{Z}_p) = k[a]_{q}^{k-1} d\mu_{-q}(a). \]  

(3.6)

Therefore, we obtain the following corollary.

**Corollary 3.2.** Let \( k \) be a positive integer. Then,

\[ G_{k,x,q} = \int_{X} \chi(x) d\mu_k(x) = k \int_{X} \chi(x) [x]_{q}^{k-1} d\mu_{-q}(x). \]  

Moreover,

\[ G_{k,q} = k \int_{X} [x]_{q}^{k-1} d\mu_{-q}(x). \]  

(3.7)

(3.8)

**Remark 3.3.** In the recent paper (see [1]), Cenkci et al. have studied \( q \)-Genocchi numbers and polynomials and \( p \)-adic \( q \)-Genocchi measures. Starting from T. Kim, L.-C. Jang, and H. K. Pak's construction of \( q \)-Genocchi numbers [7], they employed the method developed in a series of papers by Kim [see, e.g., [5, 14–16]] and they consider another \( q \)-analogue of Genocchi numbers \( G_{k}(q) \) as

\[ G_{k}(q) = q(1 + q) \frac{k}{(1 - q)^{k-1}} \sum_{m=0}^{k} \binom{k}{m} \frac{m(-1)^{m+1}}{1 + q^m}, \]  

(3.9)

which is easily derived from the generating function

\[ F_{q}^{(G)}(t) = \sum_{k=0}^{\infty} G_{k}(q) \frac{t^k}{k!} = q(1 + q) t \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]t}. \]  

(3.10)

However, these \( q \)-Genocchi numbers and generating function do not seem to be natural ones; in particular, these numbers cannot be represented as a nice Witt’s type formula for the \( p \)-adic invariant integral on \( \mathbb{Z}_p \) and the generating function does not seem to be simple and useful for deriving many interesting identities related to \( q \)-Genocchi numbers. By this reason, we consider \( q \)-Genocchi numbers and polynomials which are different. Our \( q \)-Genocchi numbers and polynomials to treat in this paper can be represented by \( p \)-adic \( q \)-fermionic integral on \( \mathbb{Z}_p \) [9, 13] and this integral representation also can be considered as Witt’s type formula for \( q \)-Genocchi numbers. These formulae are useful to study congruences and worthwhile identities for \( q \)-Genocchi numbers. By using the generating function of our \( q \)-Genocchi numbers, we can derive many properties and identities as same as ordinary Genocchi numbers which were well known.
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