Research Article

$q$-Parametric Bleimann Butzer and Hahn Operators

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Received 4 June 2008; Accepted 20 August 2008

Recommended by Vijay Gupta

We introduce a new $q$-parametric generalization of Bleimann, Butzer, and Hahn operators in $C_1^+_{[0,\infty)}$. We study some properties of $q$-BBH operators and establish the rate of convergence for $q$-BBH operators. We discuss Voronovskaja-type theorem and saturation of convergence for $q$-BBH operators for arbitrary fixed $0 < q < 1$. We give explicit formulas of Voronovskaja-type for the $q$-BBH operators for $0 < q < 1$. Also, we study convergence of the derivative of $q$-BBH operators.

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1. Introduction

$q$-Bernstein polynomials

\[ B_{n,q}(f)(x) := \sum_{k=0}^{n} f\left( \frac{k}{n}\right) \binom{n}{k} x^k \prod_{s=0}^{n-k-1} (1-q^s x) \]  

were introduced by Phillips in [1]. $q$-Bernstein polynomials form an area of an intensive research in the approximation theory, see survey paper [2] and references therein. Nowadays, there are new studies on the $q$-parametric operators. Two parametric generalizations of $q$-Bernstein polynomials have been considered by Lewanowicz and Woźni (cf. [3]), an analog of the Bernstein-Durrmeyer operator and Bernstein-Chlodowsky operator related to the $q$-Bernstein basis has been studied by Derriennic [4], Gupta [5] and Karsli and Gupta [6], respectively, a $q$-version of the Szasz-Mirakjan operator has been investigated by Aral and Gupta in [7]. Also, some results on $q$-parametric Meyer-König and Zeller operators can be found in [8–11].

In [12], Bleimann et al. introduced the following operators:

\[ H_n(f)(x) = \frac{1}{(1+x)^n} \sum_{k=0}^{n} f\left( \frac{k}{n-k+1}\right) \binom{n}{k} x^k, \quad x > 0, \quad n \in \mathbb{N}. \]
There are several studies related to approximation properties of Bleimann, Butzer, and Hahn operators (or, briefly, BBH), see, for example, [12–18]. Recently, Aral and Doğru [19] introduced a $q$-analog of Bleimann, Butzer, and Hahn operators and they have established some approximation properties of their $q$-Bleimann, Butzer, and Hahn operators in the subspace of $C_q[0, \infty)$. Also, they showed that these operators are more flexible than classical BBH operators, that is, depending on the selection of $q$, rate of convergence of the $q$-BBH operators is better than the classical one. Voronovskaja-type asymptotic estimate and the monotonicity properties for $q$-BBH operators are studied in [20].

In this paper, we propose a different $q$-analog of the Bleimann, Butzer, and Hahn operators in $C^{*}_{1+\mathbf{x}}[0, \infty)$. We use the connection between classical BBH and Bernstein operators suggested in [16] to define new $q$-BBH operators as follows:

$$H_{n,q}(f)(x) := (\Phi^{-1}B_{n+1,q}\Phi)(f)(x),$$  

(1.3)

where $B_{n+1,q}$ is a $q$-Bernstein operator, $\Phi$ and $\Phi^{-1}$ will be defined later. Thanks to (1.3), different properties of $B_{n+1,q}$ can be transferred to $H_{n,q}$ with a little extra effort. Thus the limiting behavior of $H_{n,q}$ can be immediately derived from (1.3) and the well-known properties of $B_{n+1,q}$. It is natural that even in the classical case, when $q = 1$, to define $H_{n}$ in the space $C^{*}_{1+\mathbf{x}}[0, \infty)$, the limit $l_{f}$ of $f(x)/(1 + x)$ as $x \to \infty$ has to appear in the definition of $H_{n}$. Thus in $C^{*}_{1+\mathbf{x}}[0, \infty)$ the classical BBH operator has to be modified as follows:

$$H_{n}(f)(x) = \frac{1}{(1 + x)^{n}} \sum_{k=0}^{n} f\left(\frac{k}{n - k + 1}\right) \binom{n}{k} x^{k + l_{f}} \frac{x^{n+1}}{(1 + x)^{n}}, \quad x > 0, \ n \in \mathbb{N}. \quad (1.4)$$

The paper is organized as follows. In Section 2, we give construction of $q$-BBH operators and study some elementary properties. In Section 3, we investigate convergence properties of $q$-BBH, Voronovskaja-type theorem and saturation of convergence for $q$-BBH operators for arbitrary fixed $0 < q < 1$, and also we study convergence of the derivative of $q$-BBH operators.

2. Construction and some properties of $q$-BBH operators

Before introducing the operators, we mention some basic definitions of $q$ calculus.

Let $q > 0$. For any $n \in \mathbb{N} \cup \{0\}$, the $q$-integer $[n] = [n]_{q}$ is defined by

$$[n] := 1 + q + \cdots + q^{n-1}, \quad [0] := 0; \quad (2.1)$$

and the $q$-factorial $[n]! = [n]_{q} !$ by

$$[n]! := [1][2] \cdots [n], \quad [0]! := 1. \quad (2.2)$$

For integers $0 \leq k \leq n$, the $q$-binomial is defined by

$$\begin{align*} 
\binom{n}{k} := \frac{[n]!}{[k]![n-k]!}. 
\end{align*} \quad (2.3)$$

Also, we use the following standard notations:

$$\begin{align*} 
(z; q)_{0} & := 1, \quad (z; q)_{n} := \prod_{j=0}^{n-1} (1 - q^{j}z), \quad (z; q)_{\infty} := \prod_{j=0}^{\infty} (1 - q^{j}z), \\
_{p, q}^{n,k}(q;x) & := \binom{n}{k} x^{k} \prod_{s=0}^{n-k-1} (1 - q^{s}x), \quad _{p, q}^{x}(q;x) := \frac{x^{k}}{(1 - q)^{k}[k]!} \prod_{s=0}^{\infty} (1 - q^{s}x). 
\end{align*} \quad (2.4)$$

It is agreed that an empty product denotes 1. It is clear that \( p_{nk}(q; x) \geq 0, \) \( p_{n0k}(q; x) \geq 0 \) \( \forall x \in [0,1] \) and
\[
\sum_{k=0}^{n} p_{nk}(q; x) = \sum_{k=0}^{\infty} p_{n0k}(q; x) = 1. \tag{2.5}
\]

Introduce the following spaces.
\[
B_{\rho}[0, \infty) = \{ f : [0, \infty) \rightarrow \mathbb{R} \mid \exists M_f > 0 \text{ such that } |f(x)| \leq M_f \rho(x) \ \forall x \in [0, \infty) \},
\]
\[
C_{\rho}[0, \infty) = \{ f \in B_{\rho}[0, \infty) \mid f \text{ is continuous} \},
\]
\[
C^*_{\rho}[0, \infty) = \left\{ f \in C_{\rho}[0, \infty) \mid \lim_{x \to \infty} \frac{f(x)}{\rho(x)} = l_f \text{ exists and is finite} \right\}, \tag{2.6}
\]
\[
C^0_{\rho}[0, \infty) = \left\{ f \in C_{\rho}[0, \infty) \mid \lim_{x \to \infty} \frac{f(x)}{\rho(x)} = 0 \right\}.
\]

It is clear that \( C^*_{\rho}[0, \infty) \subset C_{\rho}[0, \infty) \subset B_{\rho}[0, \infty) \). In each space, the norm is defined by
\[
\| f \|_{\rho} = \sup_{x \geq 0} \frac{|f(x)|}{\rho(x)}. \tag{2.7}
\]

We introduce the following auxiliary operators. Firstly, let us denote
\[
\psi(y) = \frac{y}{1-y}, \quad y \in [0,1), \quad \psi^{-1}(x) = \frac{x}{1+x}, \quad x \in [0, \infty). \tag{2.8}
\]

Secondly, let \( \Phi : C^*_{\rho}[0, \infty) \rightarrow C[0,1] \) be defined by
\[
\Phi(f)(y) := \begin{cases} 
\frac{f(\psi(y))}{\rho(\psi(y))}, & \text{if } y \in [0,1), \\
l_f = \lim_{x \to \infty} \frac{f(x)}{\rho(x)}, & \text{if } y = 1.
\end{cases} \tag{2.9}
\]

Then \( \Phi \) is a positive linear isomorphism, with positive inverse \( \Phi^{-1} : C[0,1] \rightarrow C^*_{\rho}[0, \infty) \) defined by
\[
\Phi^{-1}(g)(x) = \rho(x)g(\psi^{-1}(x)), \quad g \in C[0,1], \quad x \in [0, \infty). \tag{2.10}
\]

For \( f \in C[0,1], \) \( t > 0 \), we define the modulus of continuity \( \omega(f; t) \) as follows:
\[
\omega(f; t) := \sup \{|f(x) - f(y)| : |x - y| \leq t, \ x, y \in [0,1] \}. \tag{2.11}
\]

We introduce new Bleimann-, Butzer-, and Hahn- (BBH) type operators based on \( q \)-integers as follows.

**Definition 2.1.** For \( f \in C^*_{\rho}[0, \infty) \), the \( q \)-Bleimann, Butzer, and Hahn operators are given by
\[
H_{n,q}(f)(x) := (\Phi^{-1}B_{n+1,q}\Phi)(f)(x)
= \rho(x) \sum_{k=0}^{n} \frac{f(\psi([k]/[n+1]))}{\rho(\psi([k]/[n+1]))} p_{n+1,k}(q; \psi^{-1}(1)) + l_f \rho(x)(\psi^{-1}(1))^{n+1}, \quad n \in \mathbb{N}, \tag{2.12}
\]
where
\[
p_{n+1,k}(q; \psi^{-1}(1)) := \binom{n+1}{k} (\psi^{-1}(1))^{k} \prod_{s=0}^{n-k} (1 - q^s \psi^{-1}(1)), \quad k = 0, 1, \ldots, n. \tag{2.13}
\]
Note that for \( q = 1, \ \rho = 1 + x \) and \( l_f = 0 \), we recover the classical Bleimann, Butzer, and Hahn operators. If \( q = 1, \ \rho = 1 + x \) but \( l_f \neq 0 \), it is new Bleimann, Butzer, and Hahn operators with additional term \( l_f(x^{n+1}/(1 + x)^n) \). Thus if \( f \in C_{1+x}^0 [0, \infty) \) then

\[
H_{n,q}(f)(x) := \sum_{k=0}^{n} \binom{n}{k} \left( \frac{q^k}{1+k} \right) \left( \frac{q^n}{1+n} \right) \prod_{s=1}^{n-k} \left( 1 - q^k \frac{x}{1+x} \right).
\]

(2.14)

To present an explicit form of the limit \( q \)-BBH operators, we consider

\[
p_{\infty,k}(q; q^{-1}(x)) := \left( \frac{q^{-1}(x)}{1-q^k} \right)^k \prod_{s=0}^{\infty} (1 - q^s q^{-1}(x)).
\]

(2.15)

**Definition 2.2.** Let \( 0 < q < 1 \). The linear operator defined on \( C_p^* [0, \infty) \) given by

\[
H_{\infty,q}(f)(x) := \rho(x) \sum_{k=0}^{\infty} f(q^{-1}(x)) p_{\infty,k}(q; q^{-1}(x))
\]

(2.16)

is called the limit \( q \)-BBH operator.

**Lemma 2.3.** \( H_{n,q}, H_{\infty,q} : C_p^* [0, \infty) \to C_p^* [0, \infty) \) are linear positive operators and

\[
\|H_{n,q}(f)\|_\rho \leq \|f\|_{\rho}, \quad \|H_{\infty,q}(f)\|_\rho \leq \|f\|_{\rho}.
\]

(2.17)

**Proof.** We prove the first inequality, since the second one can be done in a like manner. Thanks to the definition, we have

\[
|H_{n,q}(f)(x)| \leq \rho(x)\|f\|_\rho \sum_{k=0}^{n} p_{n+1,k}(q; q^{-1}(x)) + \rho(x)\|l_f\|(q^{-1}(x))^{n+1}
\]

\[
\leq \rho(x)\|f\|_\rho \sum_{k=0}^{n} p_{n+1,k}(q; q^{-1}(x)) + \rho(x)\|f\|_\rho (q^{-1}(x))^{n+1}
\]

(2.18)

\[
= \rho(x)\|f\|_\rho \sum_{k=0}^{n+1} p_{n+1,k}(q; q^{-1}(x)) = \rho(x)\|f\|_\rho.
\]

\[\square\]

**Lemma 2.4.** The following recurrence formula holds:

\[
H_{n,q}\left( \rho(t) \left( \frac{t}{1+t} \right)^m \right)(x) = \frac{1}{[n+1]^{m-1}} \frac{x}{1+x} \sum_{j=0}^{m-1} \binom{m-1}{j} q^j [n]! H_{n-1,q}\left( \rho(t) \left( \frac{t}{1+t} \right)^j \right)(x).
\]

(2.19)

In particular, we have

\[
H_{n,q}(\rho)(x) = \rho(x), \quad H_{n,q}\left( \rho(t) \frac{t}{1+t} \right)(x) = \rho(x) \frac{x}{1+x}, \quad H_{n,q}(1)(x) = 1,
\]

(2.20)

\[
H_{n,q}\left( \rho(t) \left( \frac{t}{1+t} \right)^2 \right)(x) = \rho(x) \left( \frac{x}{1+x} \right)^2 + \rho(x) \frac{x}{(1+x)^2} \frac{1}{[n+1]}.
\]
Proof. We prove only the recurrence formula, since the formulae (2.20) can easily be obtained by standard computations. Since \( l_f = 1 \) for \( f = \rho(t)(t/(1 + t))^m \), we have

\[
H_{n,q}(\rho(t) \left(\frac{t}{1+t}\right)^m)(x) = \rho(x) \sum_{k=0}^{n} \left(\frac{[k]}{[n+1]}\right)^m \rho_{n+1,k} \left( q^{-1}_m x \right) + \rho(x) \left( \frac{x}{1+x} \right)^{n+1}
\]

\[
= \rho(x) \sum_{k=0}^{n} \left(\frac{[k]}{[n+1]}\right)^m \binom{n+1}{k} \left( \frac{x}{1+x} \right) \prod_{s=0}^{k-n-k} \left( 1 - q^s x \right) + \rho(x) \left( \frac{x}{1+x} \right)^{n+1}
\]

\[
= \rho(x) \sum_{k=0}^{n} \left[ \frac{[k]^{m-1}}{[n+1]^{m-1}} \right] \binom{n}{k-1} \left( \frac{x}{1+x} \right) \prod_{s=0}^{k-n-k} \left( 1 - q^s x \right) + \rho(x) \left( \frac{x}{1+x} \right)^{n+1}
\]

\[
= \rho(x) \sum_{k=1}^{m-1} \binom{m-1}{j} q^j \left[ n \right]^j \left[ H_{n-1,q}(\rho(t) \left(\frac{t}{1+t}\right)^m)(x) - \rho(x) \left( \frac{x}{1+x} \right)^n \right] + \rho(x) \left( \frac{x}{1+x} \right)^{n+1}
\]

\[
= \frac{1}{[n+1]^{m-1}} \frac{x}{1+x} \sum_{j=0}^{m-1} \binom{m-1}{j} q^j \left[ n \right]^j \left[ 1 - \frac{1}{[n+1]^{m-1}} \sum_{j=0}^{m-1} \binom{m-1}{j} q^j \left[ n \right]^j \right]
\]

Next theorem shows the monotonicity properties of \( q \)-BBH operators.

**Theorem 2.5.** If \( f \in C^*_t [0, \infty) \) is convex and

\[
l_f + \left[ f \left( \frac{[n]}{q^n} \right) - f \left( \frac{[n+1]}{q^{n+1}} \right) \right] q^{n+1} \geq 0,
\]

then its \( q \)-BBH operators are nonincreasing, in the sense that

\[
H_{n,q}(f)(x) \geq H_{n+1,q}(f)(x), \quad n = 1, 2, \ldots, q \in (0, 1), \quad x \in [0, \infty).
\]
Proof. We begin by writing

\[ H_{n,q}(f)(x) - H_{n+1,q}(f)(x) \]
\[ = \sum_{k=0}^{n} f \left( \frac{[k]}{q^k[n-k+1]} \right) \frac{[n]}{[k]} q^k (q^{n-k} q^{n+1} q_{k+1}) \]
\[ - \sum_{k=0}^{n+1} f \left( \frac{[k]}{q^k[n-k+2]} \right) \frac{n+1}{k} q^n q_k + l_f \left( \frac{x}{1+x} \right)^{n+1} \]
\[ = \sum_{k=1}^{n} \frac{[n]}{k} a_k q^k q_k + \sum_{k=1}^{n+1} f \left( \frac{[k-1]}{q^k[n-k+2]} \right) \frac{n}{k} q^n q_k \]
\[ - \sum_{k=0}^{n+1} f \left( \frac{[k]}{q^k[n-k+2]} \right) \frac{n+1}{k} q^n q_k + l_f \left( \frac{x}{1+x} \right)^{n+1} \]
\[ = \sum_{k=1}^{n+1} \frac{n+1}{k} a_k q^k q_k + \left[ f \left( \frac{[n]}{q^n} \right) - f \left( \frac{[n+1]}{q^{n+1}} \right) \right] q^{n+1} \left( \frac{x}{1+x} \right)^{n+1} + l_f \left( \frac{x}{1+x} \right)^{n+1}, \]
\[ \text{(2.27)} \]

where

\[ a_k = \frac{[n-k+1]}{[n+1]} f \left( \frac{[k]}{q^k[n-k+1]} \right) + q^{n-k+1} \frac{[k]}{[n+1]} f \left( \frac{[k-1]}{q^k[n-k+2]} \right) - f \left( \frac{[k]}{q^k[n-k+2]} \right). \]
\[ \text{(2.28)} \]

By assumption, the sum of the last three terms of (2.27) is positive. Thus to show monotonicity of \( H_{n,q} \) it suffices to show nonnegativity of \( a_k \), \( 0 \leq k \leq n \). Let us write

\[ \alpha = \frac{[n-k+1]}{[n+1]}, \quad x_1 = \frac{[k]}{q^k[n-k+1]}, \quad x_2 = \frac{[k-1]}{q^k[n-k+2]}. \]
\[ \text{(2.29)} \]
Then it follows that
\[
1 - \alpha = \frac{q^{n-k+1}[k]}{[n+1]},
\]
\[
ax_1 + (1-\alpha)x_2 = \frac{[k]}{q^n[n+1]} \left( 1 + \frac{q^{n-k+2}[k-1]}{[n-k+2]} \right)
\]
\[
= \frac{[k]}{q^n[n+1]} \left( 1 - \frac{q^{n-k+2} + q^{n-k+2}(1-q^{-k})}{1 - q^{n-k+2}} \right) = \frac{[k]}{q^n[n-k+2]},
\]
and we see immediately that
\[
a_k = af(x_1) + (1-\alpha)f(x_2) - f(ax_1 + (1-\alpha)x_2) \geq 0,
\]
and so \(H_{n,q}(f)(x) - H_{n+1,q}(f)(x) \geq 0). \qedhere

Remark 2.6. It is easily seen that
\[
l_{f} + \left[ f\left(\frac{[n]}{q^n}\right) - f\left(\frac{[n+1]}{q^{n+1}}\right)\right]q^{n+1}
\]
\[
= [n+2]\left( \frac{1}{[n+2]}(\Phi f)(1) + \frac{q[n+1]}{[n+2]}(\Phi f)\left(\frac{[n]}{[n+1]}\right) - (\Phi f)\left(\frac{[n+1]}{n+2}\right)\right). \tag{2.32}
\]
The condition (2.22) follows from convexity of \(\Phi f\). On the other hand, \(\Phi f\) is convex if \(f\) is convex and nonincreasing, see [16].

3. Convergence properties

Theorem 3.1. \(\text{Let } q \in (0,1), \text{ and let } f \in C^*_\rho[0,\infty). \text{ Then}
\]
\[
\|H_{n,q}(f) - H_{\infty,q}(f)\|_{\rho} \leq C(q)\omega^\rho(\Phi f, q^{n+1}),
\]
where \(C(q) = (4/(q(1-q))) \ln(1/(1-q)) + 2\).

Proof. For all \(x \in [0,\infty)\), by the definitions of \(H_{n,q}(f)(x)\) and \(H_{\infty,q}(f)(x)\), we have that
\[
H_{n,q}(f) - H_{\infty,q}(f) = \rho(x) \sum_{k=0}^{n} f\left(\frac{q([k]/[n+1])}{\rho(q([k]/[n+1]))}\right)p_{n+1,k}(q; q^{-1}(x))
\]
\[
+ l_{f} \rho(x) \left( \frac{x}{1+x} \right)^{n+1} - \rho(x) \sum_{k=0}^{\infty} f\left(\frac{q(1-q^k)}{\rho(q(1-q^k))}\right)p_{n+1,k}(q; q^{-1}(x))
\]
\[
= \rho(x) \sum_{k=0}^{n+1} \left[ (\Phi f)\left(\frac{[k]}{[n+1]}\right) - (\Phi f)(1-q^k) \right]p_{n+1,k}(q; q^{-1}(x))
\]
\[
+ \rho(x) \sum_{k=n+2}^{\infty} \left[ (\Phi f)(1-q^k) - (\Phi f)(1) \right]p_{n+1,k}(q; q^{-1}(x))
\]
\[
- \rho(x) \sum_{k=n+2}^{\infty} \left[ (\Phi f)(1-q^k) - (\Phi f)(1) \right]p_{\infty,k}(q; q^{-1}(x))
\]
\[
:= I_1 + I_2 + I_3. \tag{3.2}
\]
First, we estimate $I_1$, $I_3$. By using the following inequalities:

\[
0 \leq \frac{k}{n+1} - (1 - q^k) = \frac{1 - q^k}{1 - q^{n+1}} - (1 - q^k) = \frac{q^{n+1} - q^k}{1 - q^{n+1}} \leq q^{n+1},
\]

\[
0 \leq 1 - (1 - q^k) = q^k \leq q^{n+1}, \quad k \geq n + 2,
\]

we get

\[
|I_1| \leq \rho(x)\omega(\Phi f, q^{n+1}) \sum_{k=0}^{n+1} p_{n+1, k}(q; \varphi^{-1}(x)) = \rho(x)\omega(\Phi f, q^{n+1}),
\]

\[
|I_3| \leq \rho(x) \sum_{k=n+2}^{\infty} \omega(\Phi f, q^k) p_{\infty k}(q; \varphi^{-1}(x)) \leq \rho(x)\omega(\Phi f, q^{n+1}).
\]

Next, we estimate $I_2$. Using the well-known property of modulus of continuity

\[
\omega(g, \lambda t) \leq (1 + \lambda)\omega(g, t), \quad \lambda > 0,
\]

we get

\[
|I_2| \leq \rho(x) \sum_{k=0}^{n+1} \omega(\Phi f, q^k)|p_{n+1, k}(q; \varphi^{-1}(x)) - p_{\infty k}(q; \varphi^{-1}(x))|
\]

\[
\leq \rho(x)\omega(\Phi f, q^{n+1}) \sum_{k=0}^{n+1} (1 + q^{k-1})|p_{n+1, k}(q; \varphi^{-1}(x)) - p_{\infty k}(q; \varphi^{-1}(x))|
\]

\[
\leq 2\rho(x)\omega(\Phi f, q^{n+1}) \frac{1}{q^{n+1}} \sum_{k=0}^{n+1} q^k|p_{n+1, k}(q; \varphi^{-1}(x)) - p_{\infty k}(q; \varphi^{-1}(x))|
\]

\[
= \rho(x) \frac{2}{q^{n+1}} \omega(\Phi f, q^{n+1}) J_{n+1}(\varphi^{-1}(x)),
\]

where

\[
J_{n+1}(\varphi^{-1}(x)) = \sum_{k=0}^{n+1} q^k|p_{n+1, k}(q; \varphi^{-1}(x)) - p_{\infty k}(q; \varphi^{-1}(x))|.
\]

Now, using the estimation (2.9) from [21], we have

\[
J_{n+1}(\varphi^{-1}(x)) \leq \frac{q^{n+1}}{q(1 - q)} \ln \frac{1}{1 - q} \sum_{k=0}^{n+1} (p_{n+1, k}(q; \varphi^{-1}(x)) + p_{\infty k}(q; \varphi^{-1}(x)))
\]

\[
\leq \frac{2q^{n+1}}{q(1 - q)} \ln \frac{1}{1 - q}.
\]

From (3.6) and (3.8), it follows that

\[
|I_2| \leq \rho(x) \frac{4}{q(1 - q)} \ln \frac{1}{1 - q} \omega(\Phi f, q^{n+1}).
\]

From (3.4), and (3.9), we obtain the desired estimation.
Theorem 3.2. Let $0 < q < 1$ be fixed and let $f \in C_{1+\varepsilon}^*[0, \infty)$. Then $H_{\omega,q}(f)(x) = f(x) \forall x \in [0, \infty)$ if and only if $f$ is linear.

Proof. By definition of $H_{\omega,q}$ we have

$$H_{\omega,q}(f)(x) = (\Phi^{-1} B_{\omega,q} \Phi)(f)(x).$$

(3.10)

Assume that $H_{\omega,q}(f)(x) = f(x)$. Then $(B_{\omega,q} \Phi)(f)(x) = (\Phi f)(x)$. From [22], we know that $B_{\omega,q}(g) = g$ if and only if $g$ is linear. So $(B_{\omega,q} \Phi)(f)(x) = (\Phi f)(x)$ if and only if $(\Phi f)(x) = (1 - x) f(x)/(1 - x)) = Ax + B$. It follows that $f(x) = (1 + x)(A(x/(1 + x)) + B) = (A + B)x + B$. The converse can be shown in a similar way.

Remark 3.3. Let $0 < q < 1$ be fixed and let $f \in C_{1+\varepsilon}^*[0, \infty)$. Then the sequence $\{H_{n,q}(f)(x)\}$ does not approximate $f(x)$ unless $f$ is linear. It is completely in contrast to the classical case.

Theorem 3.4. Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \rightarrow 1$ as $n \rightarrow \infty$. For any $x \in [0, \infty)$ and for any $f \in C_1^*(0, \infty)$, the following inequality holds:

$$\frac{1}{\rho(x)}|H_{n,q_n}(g)(x) - f(x)| \leq 2\omega(\Phi f, \sqrt{\lambda_n(x)}),$$

(3.11)

where $\lambda_n(x) = (x/(1 + x^2))(1/[n + 1]_{q_n})$.

Proof. Positivity of $B_{n+1,q_n}$ implies that for any $g \in C[0,1]$

$$|B_{n+1,q_n}(g)(x) - g(x)| \leq B_{n+1,q_n}(|g(t) - g(x)|)(x).$$

(3.12)

On the other hand,

$$|(\Phi f)(t) - (\Phi f)(x)| \leq \omega(\Phi f, |t - x|)$$

$$\leq \omega(\Phi f, \delta) \left(1 + \frac{1}{\delta}|t - x|\right), \quad \delta > 0.$$
Corollary 3.5. Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. For any $f \in C^*_1[0, \infty)$ it holds that

$$
\lim_{n \to \infty} \| H_{n,q_n} (f)(x) - f(x) \|_p = 0. \tag{3.15}
$$

Next, we study Voronovskaja-type formulas for the $q$-BBH operators. For the $q$-Bernstein operators, it is proved in [23] that for any $f \in C^1[0,1],

$$
\lim_{n \to \infty} \frac{[n]}{q^n} [B_{n,q}(f)(x) - B_{\infty,q}(f)(x)] = L_q(f,x) \tag{3.16}
$$

uniformly in $x \in [0,1]$, where

$$
L_q(f,x) := \begin{cases} 
\sum_{k=0}^{\infty} [k] \left( f'(1 - q^k) - f(1 - q^k) - f(1 - q^{k-1}) \right) \frac{x^k}{(q;q)_k}, & 0 \leq x < 1, \\
0, & x = 1.
\end{cases} \tag{3.17}
$$

Similarly, we have the following Voronovskaja-type theorem for the $q$-BBH operators for fixed $q \in (0,1)$. Before stating the theorem we introduce an analog of $L_q(f,x)$ for $q$-BBH operators

$$
V_q(f,x) := (\Phi^{-1} L_q \Phi)(f)(x) = \left( \frac{x}{1 + x} ; q \right) \sum_{k=0}^{\infty} [k] \times \left( f' \left( \frac{1 - q^k}{q^k} \right) \frac{1}{q^k} - f \left( \frac{1 - q^k}{q^k} \right) - \frac{1}{q} f \left( \frac{(1 - q^k)/q^k - q^{-1} f \left( (1 - q^{k-1})/q^{k-1} \right)}{1 - q^k - (1 - q^{k-1})} \right) \right) \\
\times \frac{1}{(q;q)_k} \frac{x^k}{(1 + x)^{k-1}} \\
= \left( \frac{x}{1 + x} ; q \right) \sum_{k=0}^{\infty} [k] \left( f' \left( \frac{1 - q^k}{q^k} \right) \frac{1}{q^k} - \frac{1}{q^{k-1}} f \left( \frac{(1 - q^k)/q^k - f \left( (1 - q^{k-1})/q^{k-1} \right)}{q^{k-1} - q^k} \right) \right) \\
\times \frac{1}{(q;q)_k} \frac{x^k}{(1 + x)^{k-1}}. \tag{3.18}
$$

Theorem 3.6. Let $0 < q < 1$, $f \in C^1_{1+x}[0, \infty) \cap C^1[0, \infty)$, and $\Phi f$ is differentiable at $x = 1$. Then

$$
\lim_{n \to \infty} \frac{[n+1]}{q^{n+1}} [H_{n,q}(f)(x) - H_{\infty,q}(f)(x)] = V_q(f,x), \tag{3.19}
$$
in $C^1_{1+x}[0, \infty)$.

Proof. We estimate the difference

$$
\Delta(x) := \left| \frac{[n+1]}{q^{n+1}} (H_{n,q}(f)(x) - H_{\infty,q}(f)(x)) - V_q(f,x) \right| \\
= \left| \frac{[n+1]}{q^{n+1}} ((\Phi^{-1} B_{n+1,q} \Phi)(f)(x) - (\Phi^{-1} B_{\infty,q} \Phi)(f)(x)) - (\Phi^{-1} L_q \Phi)(f)(x) \right| \\
= \left| (\Phi^{-1} \left[ \frac{[n+1]}{q^{n+1}} (B_{n+1,q} - B_{\infty,q}) \right] (f)(x) \right| \\
= (1+x) \left| \left[ \frac{[n+1]}{q^{n+1}} (B_{n+1,q} - B_{\infty,q}) \right] (\Phi f)(q^{r-1}(x)) \right|. \tag{3.20}
$$
Since $\Phi f$ is well defined on whole $[0,1]$, from [23, Theorem 1], we get that

$$\lim_{n \to \infty} \|\Delta\|_{1+x} \leq \lim_{n \to \infty} \sup_{0 \leq u \leq 1} \left\{ \frac{n + 1}{q^{n+1}} (B_{n+1,q} - B_{\infty,q}) - L \right\} (\Phi f)(u) = 0. \quad (3.21)$$

Theorem is proved.

Remark 3.7. It is clear that $\Phi f$ is differentiable in $[0,1]$ if $f \in C^1[0,\infty)$. If $\Phi f$ is not differentiable at $x = 1$, then

$$\lim_{n \to \infty} \frac{n + 1}{q^{n+1}} [H_{n,q}(f)(x) - H_{\infty,q}(f)(x)] = V_{q}(f,x), \quad (3.22)$$

uniformly on any $[0,A] \subset [0,\infty)$.

**Theorem 3.8.** If $f \in C^2[0,\infty)$ and $q_n \to 1$ as $n \to \infty$, then

$$\lim_{n \to \infty} \frac{n + 1}{q^{n+1}} \{H_{n,q_n}(f)(x) - f(x)\} = \frac{1}{2} f''(x)(1 + x)^2 x \quad (3.23)$$

uniformly on any $[0,A] \subset [0,\infty)$.

**Proof.** By definition of $H_{n,q_n}$,

$$H_{n,q_n}(f)(x) - f(x) = (\Phi^{-1}B_{n+1,q_n} \Phi)(f)(x) - (\Phi^{-1} \Phi)(f)(x) = (\Phi^{-1}[B_{n+1,q_n} - I] \Phi)(f)(x) = (1 + x)([B_{n+1,q_n} - I] \Phi)(f)(q^{-1}(x)), \quad (3.24)$$

and if $L := (1/2)f''(x)(1 - x)x$, then

$$\frac{1}{2} f''(x)(1 + x)^2 x = (\Phi^{-1}L \Phi)(f)(x) = (1 + x)(L \Phi)(f)(q^{-1}(x)) = \frac{1}{2} (1 + x)(\Phi f)''(q^{-1}(x))q^{-1}(x)(1 - q^{-1}(x)). \quad (3.25)$$

On the other hand, by [24, Corollary 5.2] we have that

$$\lim_{n \to \infty} \sup_{0 \leq u \leq 1} \left\{ \frac{n + 1}{q^{n+1}} ([B_{n+1,q_n} - I] \Phi)(f)(u) - \frac{1}{2} (\Phi f)''(u)u(1 - u) \right\} = 0. \quad (3.26)$$

Now, the result follows from the following inequality:

$$\left| \frac{n + 1}{q^{n+1}} \{H_{n,q_n}(f)(x) - f(x)\} - \frac{1}{2} f''(x)(1 + x)^2 x \right|$$

$$\leq (1 + A) \sup_{0 \leq u \leq A/(1+A)} \left| \frac{n + 1}{q^{n+1}} ([B_{n+1,q_n} - I] \Phi)(f)(u) - \frac{1}{2} (\Phi f)''(u)u(1 - u) \right|. \quad (3.27)$$

The theorem is proved.
From Theorem 3.6, we have the following saturation of convergence for the $q$-BBH operators for fixed $q \in (0, 1)$.

**Corollary 3.9.** Let $0 < q < 1$ and $f \in C^1_{1+\infty}[0, \infty) \cap C^1[0, \infty)$. Then

$$\|H_{n,q}(f)(x) - H_{\infty,q}(f)(x)\|_{1+x} = o(q^{n+1})$$

(3.28)

if and only if $V_q(f, x) \equiv 0$, and this is equivalent to

$$f'(\frac{1 - q^k}{q^k})(\frac{1}{q^k} - \frac{1}{q^{k-1}}) = f'(\frac{1 - q^k}{q^k}) - f'(\frac{1 - q^{k-1}}{q^{k-1}}), \quad k = 1, 2, \ldots.$$  

(3.29)

**Theorem 3.10.** Let $0 < q < 1$ and $f \in C^1_{1+\infty}[0, \infty) \cap C^1[0, \infty)$. If $f$ is a convex function, then $\|H_{n,q}(f)(x) - H_{\infty,q}(f)(x)\|_{1+x} = o(q^{n+1})$ if and only if $f$ is a linear function.

Proof. If $\|H_{n,q}(f) - H_{\infty,q}(f)\|_{1+x} = o(q^{n+1})$, then by Corollary 3.9

$$f'(\frac{1 - q^k}{q^k})q^{k-1} - q^k = f'(\frac{1 - q^k}{q^k}) - f'(\frac{1 - q^{k-1}}{q^{k-1}}), \quad k = 1, 2, \ldots.$$  

(3.30)

Hence for $k = 1, 2, \ldots$

$$\int^{(1-q^k)/q^k}_{(1-q^{k-1})/q^{k-1}} f'(\frac{1 - q^k}{q^k}) - f'(t) \, dt = 0.$$  

(3.31)

Since $f$ is convex and $f'$ is continuous on $[0, \infty)$, we get $f'(t) = f'(1 - q^k/q^k) \forall t \in [(1 - q^{k-1})/q^{k-1}, (1 - q^k)/q^k]$. Hence $f'(t) \equiv f'(0)$, and therefore $f(t) = At + B$. Conversely, if $f$ is linear, then $\|H_{n,q}(f)(x) - H_{\infty,q}(f)(x)\|_{1+x} = 0$. \hfill \Box

One of the remarkable properties of the $q$-Bernstein approximation is that derivatives of $B_n(f)$ of any order converge to corresponding derivatives of $f$, see [25]. Next theorem shows the same property for $H_{n,q}$ for the first derivative.

**Theorem 3.11.** Let $f \in C^1_{1+\infty}[0, \infty) \cap C^1[0, \infty)$ and let $\{q_n\}$ be a sequence chosen so that the sequence

$$\varepsilon_n = \frac{n}{1 + q_n + q_n^2 + \cdots + q_n^n} - 1$$

(3.32)

converges to zero from above faster than $\{1/3^n\}$. Then

$$\lim_{n \to \infty} [H_{n,q_n}(f)(x)]' = f'(x)$$

(3.33)

uniformly on any $[0, A] \subset [0, \infty)$.

Proof. By definition

$$H_{n,q_n}(f)(x) = (1 + x)(B_{n+1,q_n}\Phi)f\left(\frac{x}{1+x}\right).$$

(3.34)
Since $H_{n,q_n}(f)(x)$ is a composition of differentiable functions, it is differentiable at any $x \in [0,A]$ and

$$
\frac{d}{dx} H_{n,q_n}(f)(x) = \frac{d}{dx} \left[ (1 + x)(B_{n+1,q_n} \Phi)f \left( \frac{x}{1 + x} \right) \right] = (B_{n+1,q_n} \Phi)f \left( \frac{x}{1 + x} \right) + \frac{1}{1 + x} \frac{d}{dx} (B_{n+1,q_n} \Phi)f \left( \frac{x}{1 + x} \right).
$$

(3.35)

By [24, Theorem 4.1]

$$
\left| (B_{n+1,q_n} \Phi)f \left( \frac{x}{1 + x} \right) - (\Phi f) \left( \frac{x}{1 + x} \right) \right| \leq 2\omega \left( \Phi f, \sqrt{B_{n+1,q_n} \left( t - \left( \frac{x}{1 + x} \right) \right)^2 \left( \frac{x}{1 + x} \right)} \right),
$$

(3.36)

and by [25, Theorem 3]

$$
\lim_{n \to \infty} \sup_{0 \leq t \leq A} \left| \frac{d}{dx} (B_{n+1,q_n} \Phi)f \left( \frac{x}{1 + x} \right) - (\Phi f)' \left( \frac{x}{1 + x} \right) \right| = 0.
$$

(3.37)

Thus the desired limit follows from the following inequality:

$$
\left| \frac{d}{dx} H_{n,q_n}(f)(x) - \frac{d}{dx} f(x) \right| \leq \left| (B_{n+1,q_n} \Phi)f \left( \frac{x}{1 + x} \right) - (\Phi f) \left( \frac{x}{1 + x} \right) \right| + \frac{1}{1 + x} \left| \frac{d}{dx} (B_{n+1,q_n} \Phi)f \left( \frac{x}{1 + x} \right) - (\Phi f)' \left( \frac{x}{1 + x} \right) \right|
$$

$$
\leq 2\omega \left( \Phi f, \sqrt{B_{n+1,q_n} \left( t - \left( \frac{x}{1 + x} \right) \right)^2 \left( \frac{x}{1 + x} \right)} \right) + \left| \frac{d}{dx} (B_{n+1,q_n} \Phi)f \left( \frac{x}{1 + x} \right) - (\Phi f)' \left( \frac{x}{1 + x} \right) \right|
$$

$$
= 2\omega \left( \Phi f, \sqrt{\frac{x}{(1 + x)^2 [n + 1]_{q_n}}} \right) + \left| \frac{d}{dx} (B_{n+1,q_n} \Phi)f \left( \frac{x}{1 + x} \right) - (\Phi f)' \left( \frac{x}{1 + x} \right) \right|
$$

$$
\leq 2\omega \left( \Phi f, \sqrt{\frac{A}{[n + 1]_{q_n}}} \right) + \left| \frac{d}{dx} (B_{n+1,q_n} \Phi)f \left( \frac{x}{1 + x} \right) - (\Phi f)' \left( \frac{x}{1 + x} \right) \right|
$$

(3.38)

\[\square\]

Remark 3.12. In [1], it is shown that

$$
B_{n+1,q_n}(f)(x) = \sum_{k=0}^{n+1} \binom{n+1}{k} \Delta^k f_0 x^k,
$$

(3.39)

where

$$
f_i = f \left( \frac{i}{n + 1} \right), \quad \Delta^0 f_i = f_i, \quad \Delta^k f_i = \Delta^k f_{i+1} - q^k \Delta^k f_i,
$$

$$
\Delta^k f_i = \sum_{j=0}^{k} (-1)^{i} q^{(j-1)/2} \binom{k}{j} f \left( \frac{i + k - j}{n + 1} \right).
$$

(3.40)
Immediately from the definition of $H_{n,q}$ we get an analog of (3.39) for $H_{n,q}$:

$$H_{n,q}(f)(x) = (\Phi^{-1}B_{n+1,q}\Phi)(f)(x)$$

$$= \Phi^{-1} \sum_{k=0}^{n+1} \binom{n+1}{k} \Delta^k(\Phi f)_0 x^k$$

$$(3.41)$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} \Delta^k(\Phi f)_0 \frac{x^k}{(1+x)^{k-1}}.$$

**Acknowledgment**

The research is supported by the Research Advisory Board of Eastern Mediterranean University under project BAP-A-08-04.

**References**


