Research Article

Conditions for Carathéodory Functions

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The purpose of the present paper is to derive some sufficient conditions for Carathéodory functions in the open unit disk. Our results include several interesting corollaries as special cases.

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1. Introduction

Let $P$ be the class of functions $p$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$  \hspace{1cm} (1.1)

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. If $p$ in $P$ satisfies

$$\text{Re}\{p(z)\} > 0 \quad (z \in U),$$  \hspace{1cm} (1.2)

then we say that $p$ is the Carathéodory function.

Let $A$ denote the class of all functions $f$ analytic in the open unit disk $U = \{z : |z| < 1\}$ with the usual normalization $f(0) = f'(0) - 1 = 0$. If $f$ and $g$ are analytic in $U$, we say that $f$ is subordinate to $g$, written $f < g$ or $f(z) < g(z)$, if $g$ is univalent, $f(0) = g(0)$ and $f(U) \subset g(U)$.

For $0 < \alpha \leq 1$, let $ST_{\alpha}C$ and $ST_{\alpha}S$ denote the classes of functions $f \in A$ which are strongly convex and starlike of order $\alpha$; that is, which satisfy

$$1 + \frac{zf''(z)}{f'(z)} < \left(\frac{1 + z}{1 - z}\right)^{\alpha} \quad (z \in U),$$  \hspace{1cm} (1.3)

and

$$zf'(z) \frac{f''(z)}{f(z)} < \left(\frac{1 + z}{1 - z}\right)^{\alpha} \quad (z \in U),$$  \hspace{1cm} (1.4)
respectively. We note that (1.3) and (1.4) can be expressed, equivalently, by the argument functions. The classes \( \mathcal{STC}(\alpha) \) and \( \mathcal{STS}(\alpha) \) were introduced by Brannan and Kirwan [1] and studied by Mocanu [2] and Nunokawa [3, 4]. Also, we note that if \( \alpha = 1 \), then \( \mathcal{STS}(\alpha) \) coincides with \( \mathcal{S}^* \), the well-known class of starlike (univalent) functions with respect to origin, and if \( 0 < \alpha < 1 \), then \( \mathcal{STS}(\alpha) \) consists only of bounded starlike functions [1], and hence the inclusion relation \( \mathcal{STS}(\alpha) \subset \mathcal{S}^* \) is proper. Furthermore, Nunokawa and Thomas [4] (see also [5]) found the value \( \beta(\alpha) \) such that \( \mathcal{STC}(\beta(\alpha)) \subset \mathcal{STS}(\alpha) \).

In the present paper, we consider general forms which cover the results by Mocanu [6] and Nunokawa and Thomas [4]. An application of a certain integral operator is also considered. Moreover, we give some sufficient conditions for univalent (close-to-convex) and (strongly) starlike functions (of order \( \beta \)) as special cases of main results.

2. Main Results

To prove our results, we need the following lemma due to Nunokawa [3].

**Lemma 2.1.** Let \( p \) be analytic in \( U, p(0) = 1 \) and \( p(z) \neq 0 \) in \( U \). Suppose that there exists a point \( z_0 \in U \) such that

\[
\begin{align*}
|\arg p(z)| &< \frac{\pi}{2} \alpha \quad \text{for } |z| < |z_0|, \\
|\arg p(z_0)| &< \frac{\pi}{2} \alpha \quad (0 < \alpha \leq 1).
\end{align*}
\]

Then we have

\[
\frac{z_0 p'(z_0)}{p(z_0)} = i \alpha k,
\]

where

\[
\begin{align*}
k &\geq \frac{1}{2} \left( x + \frac{1}{x} \right) \quad \text{when } \arg p(z_0) = \frac{\pi}{2} \alpha, \\
k &\leq -\frac{1}{2} \left( x + \frac{1}{x} \right) \quad \text{when } \arg p(z_0) = -\frac{\pi}{2} \alpha, \\
& \{ p(z_0) \}^{1/\alpha} = \pm ix \quad (x > 0).
\end{align*}
\]

With the help of Lemma 2.1, we now derive the following theorem.

**Theorem 2.2.** Let \( p \) be nonzero analytic in \( U \) with \( p(0) = 1 \) and let \( p \) satisfy the differential equation

\[
\eta z p'(z) + B(z) p(z) = a + i b A(z),
\]

where \( \eta > 0, a \in \mathbb{R}^+, \) \( 0 \leq b \leq a \tan(\pi/2) \alpha, \) \( 0 < \alpha < 1, \) \( A(z) = \text{sign}(\text{Im} p(z)) \) and \( B(z) \) is analytic in \( U \) with \( B(0) = a. \) If

\[
|\arg B(z)| < \frac{\pi}{2} \beta(\eta, \alpha, a, b) \quad (z \in U),
\]

then \( p(z) \) is analytic in \( U \) and \( p(z) = b(z) p \) for some bounded analytic function \( b(z) \).
where

\[
\beta(\eta, a, b) = \frac{2}{\pi} \tan^{-1} \left\{ \frac{S(\alpha)T(\alpha)(a \sin(\pi/2)\alpha - b \cos(\pi/2)\alpha) + \eta\alpha}{S(\alpha)T(\alpha)(a \cos(\pi/2)\alpha + b \sin(\pi/2)\alpha)} \right\},
\]

(2.6)

\[
S(\alpha) = (1 + \alpha)^{(1-a)/2}, \quad T(\alpha) = (1 - \alpha)^{(1-a)/2},
\]

(2.7)

then

\[
|\arg p(z)| < \frac{\pi}{2}\alpha \quad (z \in U).
\]

(2.8)

Proof. If there exists a point \( z_0 \in U \) such that the conditions (2.1) are satisfied, then (by Lemma 2.1) we obtain (2.2) under the restrictions (2.3). Then we obtain

\[
A(z_0) = \begin{cases} 
1, & \text{if } p(z_0) = (ix)^a, \\
-1, & \text{if } p(z_0) = (-ix)^a,
\end{cases}
\]

(2.9)

\[
B(z_0) = \frac{a + ibA(z_0)}{p(z_0)} - \frac{\eta z_0p'(z_0)}{p(z_0)}
\]

\[
= (a + ibA(z_0))(\pm ix)^{-a} - i\eta k
\]

\[
= \left( \frac{a}{x^a} \cos \frac{\pi}{2}\alpha + \frac{b}{x^a}A(z_0) \sin \left( \frac{\pm \pi}{2}\alpha \right) \right)
\]

\[
+ i \left( \frac{b}{x^a}A(z_0) \cos \frac{\pi}{2}\alpha - \frac{a}{x^a} \sin \left( \frac{\pm \pi}{2}\alpha \right) - \eta k \right).
\]

Now we suppose that

\[
[p(z_0)]^{1/a} = ix \quad (x > 0).
\]

(2.10)

Then we have

\[
\arg B(z_0) = -\tan^{-1} \left\{ \frac{a \sin(\pi/2)\alpha - b \cos(\pi/2)\alpha + \eta k}{a \cos(\pi/2)\alpha + b \sin(\pi/2)\alpha} \right\},
\]

(2.11)

where

\[
kx^a \geq \frac{1}{2} \left( x^{a+1} + x^{a-1} \right) \equiv g(x) \quad (x > 0).
\]

(2.12)
Then, by a simple calculation, we see that the function $g(x)$ takes the minimum value at $x = \sqrt{(1 - \alpha)/(1 + \alpha)}$. Hence, we have

$$\arg B(z_0) \leq -\tan^{-1} \left\{ \frac{(1 + \alpha)^{(1+\alpha)/2}(1 - \alpha)^{(1-\alpha)/2}(a \sin(\pi/2)\alpha - b \cos(\pi/2)\alpha) + \eta \alpha}{(1 + \alpha)^{(1+\alpha)/2}(1 - \alpha)^{(1-\alpha)/2}(a \cos(\pi/2)\alpha + b \sin(\pi/2)\alpha)} \right\}$$

$$= -\frac{\pi}{2} \beta(\eta, \alpha, a, b),$$

(2.13)

where $\beta(\eta, \alpha, a, b)$ is given by (2.6). This evidently contradicts the assumption of Theorem 2.2. Next, we suppose that

$$\{p(z_0)\}^{1/\alpha} = -ix \quad (x > 0).$$

(2.14)

Applying the same method as the above, we have

$$\arg B(z_0) \geq \tan^{-1} \left\{ \frac{(1 + \alpha)^{(1+\alpha)/2}(1 - \alpha)^{(1-\alpha)/2}(a \sin(\pi/2)\alpha - b \cos(\pi/2)\alpha) + \eta \alpha}{(1 + \alpha)^{(1+\alpha)/2}(1 - \alpha)^{(1-\alpha)/2}(a \cos(\pi/2)\alpha + b \sin(\pi/2)\alpha)} \right\}$$

$$= \frac{\pi}{2} \beta(\eta, \alpha, a, b),$$

(2.15)

where $\beta(\eta, \alpha, a, b)$ is given by (2.6), which is a contradiction to the assumption of Theorem 2.2. Therefore, we complete the proof of Theorem 2.2. \hfill \Box

**Corollary 2.3.** Let $f \in \mathcal{A}$ and $\eta > 0$, $0 < \alpha < 1$. If

$$\left| \arg \left\{ (1 - \eta) \frac{zf'(z)}{f(z)} + \eta \left(1 + \frac{zf''(z)}{f'(z)}\right) \right\} \right| < \frac{\pi}{2} \beta(\eta, \alpha) \quad (z \in U),$$

(2.16)

where $\beta(\eta, \alpha)$ is given by (2.6) with $a = 1$ and $b = 0$, then $f \in \mathcal{S}\mathcal{C}(\alpha)$.

**Proof.** Taking

$$p(z) = \frac{f(z)}{zf'(z)}, \quad B(z) = (1 - \eta) \frac{zf'(z)}{f(z)} + \eta \left(1 + \frac{zf''(z)}{f'(z)}\right)$$

(2.17)

in Theorem 2.2, we can see that (2.4) is satisfied. Therefore, the result follows from Theorem 2.2. \hfill \Box

**Corollary 2.4.** Let $f \in \mathcal{A}$ and $0 < \alpha < 1$. Then $\mathcal{S}\mathcal{C}(\beta(a)) \subset \mathcal{S}\mathcal{C}(\alpha)$, where $\beta(a)$ is given by (2.6) with $\eta = a = 1$ and $b = 0$. 

By a similar method of the proof in Theorem 2.2, we have the following theorem.

**Theorem 2.5.** Let \( p \) be nonzero analytic in \( U \) with \( p(0) = 1 \) and let \( p \) satisfy the differential equation

\[
\frac{zp'(z)}{p(z)} + B(z) = a + ibA(z),
\]

where \( a \in \mathbb{R}^+, b \in \mathbb{R}^- \cup \{0\}, A(z) = \text{sign}(\text{Im } p(z)) \), and \( B(z) \) is analytic in \( U \) with \( B(0) = a \). If

\[
|\arg B(z)| < \frac{\pi}{2} a(\delta, a, b) \quad (z \in U), \tag{2.19}
\]

where

\[
a(\delta) := a(\delta, a, b) = \frac{2}{\pi} \tan^{-1} \frac{\delta - b}{a} \quad (\delta > 0), \tag{2.20}
\]

then

\[
|\arg p(z)| < \frac{\pi}{2} \delta \quad (z \in U). \tag{2.21}
\]

**Corollary 2.6.** Let \( f \in S_{\alpha}(\delta) \), where \( \alpha(\delta) \) is given by (2.20) with \( a = 1 \) and \( b = 0 \). Then

\[
|\arg \frac{f(z)}{z}| < \frac{\pi}{2} \delta \quad (z \in U). \tag{2.22}
\]

**Proof.** Letting

\[
p(z) = \frac{z}{f(z)}, \quad B(z) = \frac{zf'(z)}{f(z)} \tag{2.23}
\]

in Theorem 2.5, we have Corollary 2.6 immediately. \( \square \)

If we combine Corollaries 2.4 and 2.6, then we obtain the following result obtained by Nunokawa and Thomas [4].

**Corollary 2.7.** Let \( f \in S_{\alpha}(\delta) \), where

\[
\beta(\delta) = \frac{2}{\pi} \tan^{-1} \left\{ \tan \frac{\pi}{2} a(\delta) + \frac{a(\delta)}{(1 + a(\delta))^{1-a(\delta)/2}(1 - a(\delta))^{1-a(\delta)/2}} \cos\left(\frac{\pi}{2} a(\delta)\right) \right\} \tag{2.24}
\]

and \( a(\delta) \) is given by (2.20). Then

\[
|\arg \frac{f(z)}{z}| < \frac{\pi}{2} \delta \quad (z \in U). \tag{2.25}
\]
Corollary 2.8. Let $f \in \mathcal{A}$, $0 < \alpha < 1$ and $\beta, \gamma$ be real numbers with $\beta \neq 0$ and $\beta + \gamma > 0$. If

$$\left| \arg \left( \frac{\beta z f'(z)}{f(z)} + \gamma \right) \right| < \frac{\pi}{2} \delta(\alpha, \beta, \gamma) \quad (z \in U), \quad (2.26)$$

where

$$\delta(\alpha, \beta, \gamma) = \frac{2}{\pi} \tan^{-1} \left\{ \tan \frac{\pi}{2} \alpha + \frac{\alpha}{(\beta + \gamma)(1 + \alpha)(1 + \alpha/2)(1 - \alpha)(1 - \alpha/2) \cos(\pi/2) \alpha} \right\}, \quad (2.27)$$

then

$$\left| \arg \left( \frac{\beta z F'(z)}{F(z)} + \gamma \right) \right| < \frac{\pi}{2} \alpha \quad (z \in U), \quad (2.28)$$

where $F$ is the integral operator defined by

$$F(z) = \left( \frac{\beta + \gamma}{z^\gamma} \int_0^z f(t) t^{t-1} dt \right)^{1/\beta} \quad (z \in U). \quad (2.29)$$

Proof. Let

$$B(z) = \frac{1}{\beta + \gamma} \left( \frac{\beta z f'(z)}{f(z)} + \gamma \right), \quad (2.30)$$

$$p(z) = \frac{\beta + \gamma}{z^\gamma f(z)} \int_0^z f(t) t^{t-1} dt. \quad (2.31)$$

Then $B(z)$ and $p(z)$ are analytic in $U$ with $B(0) = p(0) = 1$. By a simple calculation, we have

$$\frac{1}{\beta + \gamma} z p'(z) + B(z)p(z) = 1. \quad (2.32)$$

Using a similar method of the proof in Theorem 2.2, we can obtain that

$$\left| \arg p(z) \right| < \frac{\pi}{2} \alpha \quad (z \in U). \quad (2.33)$$

From (2.29) and (2.31), we easily see that

$$F(z) = f(z) \{ p(z) \}^{1/\beta}. \quad (2.34)$$
Since

\[ \frac{\beta z F'(z)}{F(z)} + \gamma = \frac{\beta + \gamma}{p(z)}, \]

the conclusion of Corollary 2.8 immediately follows.  \( \square \)

**Remark 2.9.** Letting \( \alpha \rightarrow 1 \) in Corollary 2.8, we have the result obtained by Miller and Mocanu [7].

The proof of the following theorem below is much akin to that of Theorem 2.2 and so we omit for details involved.

**Theorem 2.10.** Let \( p \) be nonzero analytic in \( U \) with \( p(0) = 1 \) and let \( p \) satisfy the differential equation

\[ \frac{zp'(z)}{p(z)} + B(z)p(z) = a + ibA(z), \]

where \( a \in \mathbb{R}^+, b \in \mathbb{R}^- \cup \{0\}, A(z) = \text{sign}(\text{Im } p(z)) \) and \( B(z) \) is analytic in \( U \) with \( B(0) = a \). If

\[ |\arg B(z)| < \frac{\pi}{2} \beta(\alpha, a, b) \quad (z \in U), \]

where

\[ \beta(\alpha, a, b) = \alpha + \frac{2}{\pi} \arctan \frac{\alpha - b}{a} \quad (0 < \alpha \leq 1), \]

then

\[ |\arg p(z)| < \frac{\pi}{2} \alpha \quad (z \in U). \]

**Corollary 2.11.** Let \( f \in \mathcal{A} \) with \( f'(z) \neq 0 \) in \( U \) and \( 0 < \alpha \leq 1 \). If

\[ |\arg (f'(z) + zf''(z))| < \frac{\pi}{2} \beta(\alpha) \quad (z \in U), \]

where \( \beta(\alpha) \) is given by (2.38) with \( a = 1 \) and \( b = 0 \), then

\[ |\arg f'(z)| < \frac{\pi}{2} \alpha \quad (z \in U), \]

that is, \( f \) is univalent (close-to-convex) in \( U \).
Proof. Let
\[ p(z) = \frac{1}{f'(z)}, \quad B(z) = f'(z) + zf''(z) \] (2.42)
in Theorem 2.10. Then (2.36) is satisfied and so the result follows. □

By applying Theorem 2.10, we have the following result obtained by Mocanu [6].

Corollary 2.12. Let \( f \in \mathcal{M} \) with \( f(z)/z \neq 0 \) and \( a_0 \) be the solution of the equation given by
\[ 2a + \frac{2}{\pi} \tan^{-1}a = 1 \quad (0 < a < 1). \] (2.43)

If
\[ |\arg f'(z)| < \frac{\pi}{2} \left(1 - a_0\right) \quad (z \in \mathbb{U}), \] (2.44)
then \( f \in S^* \).

Proof. Let
\[ p(z) = \frac{z}{f(z)}, \quad B(z) = f'(z). \] (2.45)

Then, by Theorem 2.10, condition (2.44) implies that
\[ \left| \arg \frac{z}{f(z)} \right| < \frac{\pi}{2} a_0. \] (2.46)

Therefore, we have
\[ \left| \arg \frac{zf'(z)}{f(z)} \right| \leq |\arg f'(z)| + \left| \arg \frac{z}{f(z)} \right| < \frac{\pi}{2}, \] (2.47)
which completes the proof of Corollary 2.12. □

Corollary 2.13. Let \( f \in \mathcal{M} \) with \( f(z)f'(z)/z \neq 0 \) in \( \mathbb{U} \) and \( 0 < a \leq 1 \). If
\[ \left| \arg \frac{zf''(z)}{f(z)} \left(2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \right| < \frac{\pi}{2} \beta(a) \quad (z \in \mathbb{U}), \] (2.48)
where \( \beta(a) \) is given by (2.38), then \( f \in \mathcal{S} \mathcal{T} \mathcal{S}(a) \).
Finally, we have the following result.

**Theorem 2.14.** Let $p$ be nonzero analytic in $\mathbb{U}$ with $p(0) = 1$. If

$$|\arg((1 - \lambda)p(z) + \lambda z p'(z))| < \frac{\pi}{2} \beta(\lambda, \alpha), \quad (\text{2.49})$$

$$\beta(\lambda, \alpha) = \alpha + \frac{2}{\pi} \tan^{-1} \frac{\lambda \alpha}{1 - \lambda} \quad (0 \leq \lambda < 1; \ 0 < \alpha < 1), \quad (\text{2.50})$$

then

$$|\arg p(z)| < \frac{\pi}{2} \alpha \quad (z \in \mathbb{U}). \quad (\text{2.51})$$

**Proof.** If there exists a point $z_0 \in \mathbb{U}$ satisfying the conditions of Lemma 2.1, then we have

$$(1 - \lambda)p(z_0) + \lambda z_0 p'(z_0) = (\pm ix)^{\alpha}(1 - \lambda + i\lambda k). \quad (\text{2.52})$$

Now we suppose that

$$\{p(z_0)\}^{1/\alpha} = ix \quad (x > 0). \quad (\text{2.53})$$

Then we have

$$\arg((1 - \lambda)p(z_0) + \lambda z_0 p'(z_0)) = \frac{\pi}{2} \alpha + \tan^{-1} \frac{\lambda \alpha}{1 - \lambda}$$

$$\geq \frac{\pi}{2} \left( \alpha + \frac{2}{\pi} \tan^{-1} \frac{\lambda \alpha}{1 - \lambda} \right) \quad (\text{2.54})$$

$$= \frac{\pi}{2} \beta(\lambda, \alpha),$$

where $\beta(\lambda, \alpha)$ is given by (2.50). Also, for the case

$$\{p(z_0)\}^{1/\alpha} = -ix \quad (x > 0), \quad (\text{2.55})$$

we obtain

$$\arg((1 - \lambda)p(z_0) + \lambda z_0 p'(z_0)) \leq -\frac{\pi}{2} \left( \alpha + \frac{2}{\pi} \tan^{-1} \frac{\lambda \alpha}{1 - \lambda} \right)$$

$$= -\frac{\pi}{2} \beta(\lambda, \alpha), \quad (\text{2.56})$$

where $\beta(\lambda, \alpha)$ is given by (2.50). These contradict the assumption of Theorem 2.14 and so we complete the proof of Theorem 2.14. □
Corollary 2.15. Let \( f \in \mathcal{A} \) with \( f(z)/z \neq 0 \) in \( U \) and \( 0 < \alpha < 1 \). If

\[
\left| \arg \left( \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right) \right| < \frac{\pi}{2} (\alpha + 1) \quad (z \in U),
\]

then \( f \in \mathcal{S}_{\mathcal{T}_{\mathcal{S}}} (\alpha) \).

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