Research Article
On $k$-Quasiclass A Operators

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An operator $T \in B(H)$ is called $k$-quasiclass A if

$$T^k(|T^2| - |T|^2)T^k \geq 0$$

for a positive integer $k$, which is a common generalization of quasiclass A. In this paper, firstly we prove some inequalities of this class of operators; secondly we prove that if $T$ is a $k$-quasiclass A operator, then $T$ is isoloid and $T - \lambda$ has finite ascent for all complex number $\lambda$; at last we consider the tensor product for $k$-quasiclass A operators.

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1. Introduction

Throughout this paper let $\mathcal{H}$ be a separable complex Hilbert space with inner product $(\cdot, \cdot)$. Let $B(\mathcal{H})$ denote the $C^*$-algebra of all bounded linear operators on $\mathcal{H}$.

Let $T \in B(\mathcal{H})$ and let $\lambda_0$ be an isolated point of $\sigma(T)$. Here $\sigma(T)$ denotes the spectrum of $T$. Then there exists a small enough positive number $r > 0$ such that

$$\{ \lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq r \} \cap \sigma(T) = \{ \lambda_0 \}. \quad (1.1)$$

Let

$$E = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = r} (\lambda - T)^{-1} d\lambda. \quad (1.2)$$

$E$ is called the Riesz idempotent with respect to $\lambda_0$, and it is well known that $E$ satisfies $E^2 = E$, $TE = ET$, $\sigma(T |_{E\mathcal{H}}) = \{ \lambda_0 \}$, and $\ker((T - \lambda_0)^n) \subset E\mathcal{H}$ for all positive integers $n$. Stampfli [1] proved that if $T$ is hyponormal (i.e., operators such that $T^*T - TT^* \geq 0$), then

$$E$$

is self-adjoint and $E\mathcal{H} = \ker(T - \lambda_0) = \ker((T - \lambda_0)^*)$. \quad (1.3)
After that many authors extended this result to many other classes of operators. Cho and Tanahashi [2] proved that (1.3) holds if \( T \) is either \( p \)-hyponormal or log-hyponormal. In the case \( \lambda_0 \neq 0 \), the result was further shown by Tanahashi and Uchiyama [3] to hold for \( p \)-quasihyponormal operators, by Tanahashi et al. [4] to hold for \( (p,k) \)-quasihyponormal operators and by Uchiyama and Tanahashi [5] and Uchiyama [6] for class A and paranormal operators. Here an operator \( T \) is called \( p \)-hyponormal for \( 0 < p \leq 1 \) if \( (T^*T)^p - (TT^*)^p \geq 0 \), and log-hyponormal if \( T \) is invertible and \( \log T^*T \geq \log TT^* \). An operator \( T \) is called \( (p,k) \)-quasihyponormal if \( T^k((T^*T)^p - (TT^*)^p)T^k \geq 0 \), where \( 0 < p \leq 1 \) and \( k \) is a positive integer; especially, when \( p = 1 \), \( k = 1 \), and \( p = k = 1 \), \( T \) is called \( k \)-quasihyponormal, \( p \)-quasihyponormal, and log-hyponormal, respectively. And an operator \( T \) is called paranormal if \( \|Tx\|^2 \leq \|T^2x\|\|x\| \) for all \( x \in \mathcal{H} \); normaloid if \( \|T^k\| = \|T\|^n \) for all positive integers \( n \). \( p \)-hyponormal, log-hyponormal, \( (p,k) \)-quasihyponormal, and paranormal operators were introduced by Aluthge [7], Tanahashi [8], S. C. Arora and P. Arora [9], Kim [10], and Furuta [11, 12], respectively.

In order to discuss the relations between paranormal and \( p \)-hyponormal and log-hyponormal operators, Furuta et al. [13] introduced a very interesting class of bounded linear Hilbert space operators: class A defined by \( |T^2| - |T|^2 \geq 0 \), where \( |T| = (T^*T)^{1/2} \) which is called the absolute value of \( T \) and they showed that class A is a subclass of paranormal and contains \( p \)-hyponormal and log-hyponormal operators. Class A operators have been studied by many researchers, for example, [5, 14–19].

Recently Jeon and Kim [20] introduced quasi-class A (i.e., \( T^*(|T^2| - |T|^2)T \geq 0 \)) operators as an extension of the notion of class A operators, and they also proved that (1.3) holds for this class of operators when \( \lambda_0 \neq 0 \). It is interesting to study whether Stampli’s result holds for other larger classes of operators.

In [21], Tanahashi et al. considered an extension of quasi-class A operators, similar in spirit to the extension of the notion of \( p \)-quasihyponormality to \( (p,k) \)-quasihyponormality, and prove that (1.3) holds for this class of operators in the case \( \lambda_0 \neq 0 \).

**Definition 1.1.** \( T \in B(\mathcal{H}) \) is called a \( k \)-quasiclass A operator for a positive integer \( k \) if

\[
T^k \left( |T^2| - |T|^2 \right) T^k \geq 0. \tag{1.4}
\]

**Remark 1.2.** In [21], this class of operators is called quasi-class \((A, k)\).

It is clear that the class of quasi-class A operators \( \subseteq \) the class of \( k \)-quasiclass A operators and

\[
\text{the class of } k \text{-quasiclass A operators} \subseteq \text{the class of } (k+1) \text{-quasiclass A operators}. \tag{1.5}
\]

We show that the inclusion relation (1.5) is strict, by an example which appeared in [20].
Example 1.3. Given a bounded sequence of positive numbers \( \{a_i\}_{i=0}^{\infty} \), let \( T \) be the unilateral weighted shift operator on \( l^2 \) with the canonical orthonormal basis \( \{e_n\}_{n=0}^{\infty} \) by \( Te_n = a_ne_{n+1} \) for all \( n \geq 0 \), that is,

\[
T = \begin{pmatrix}
0 & a_0 & 0 & \ldots \\
a_1 & 0 & a_2 & \ldots \\
a_2 & 0 & \ldots & \\
& \ddots & \ddots & \ddots
\end{pmatrix}
\] (1.6)

Straightforward calculations show that \( T \) is a \( k \)-quasiclass A operator if and only if \( a_k \leq a_{k+1} \leq \cdots \leq a_{k+k} \leq \cdots \) and \( a_k > a_{k+1} \), then \( T \) is a \( k \)-quasiclass A operator, but not a \( k \)-quasiclass A operator.

In this paper, firstly we consider some inequalities of \( k \)-quasiclass A operators; secondly we prove that if \( T \) is a \( k \)-quasiclass A operator, then \( T \) is isoloid and \( T - \lambda \) has finite ascent for all complex number \( \lambda \); at last we give a necessary and sufficient condition for \( T \otimes S \) to be a \( k \)-quasiclass A operator when \( T \) and \( S \) are both non-zero operators.

2. Results

In the following lemma, Tanahashi, Jeon, Kim, and Uchiyama studied the matrix representation of a \( k \)-quasiclass A operator with respect to the direct sum of \( \text{ran}(T^k) \) and its orthogonal complement.

Lemma 2.1 (see [21]). Let \( T \in B(\mathcal{H}) \) be a \( k \)-quasiclass A operator for a positive integer \( k \) and let \( T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_1 \end{pmatrix} \) on \( \mathcal{H} = \text{ran}(T^k) \oplus \text{ker}T^{*k} \) be \( 2 \times 2 \) matrix expression. Assume that \( \text{ran}T^k \) is not dense, then \( T_1 \) is a class A operator on \( \text{ran}(T^k) \) and \( T_3 = 0 \). Furthermore, \( \sigma(T) = \sigma(T_1) \cup \{0\} \).

Proof. Consider the matrix representation of \( T \) with respect to the decomposition \( \mathcal{H} = \text{ran}(T^k) \oplus \text{ker}T^{*k} \): \( T = \begin{pmatrix} T_1 \\ 0 \end{pmatrix} \). Let \( P \) be the orthogonal projection of \( \mathcal{H} \) onto \( \text{ran}(T^k) \). Then \( T_1 = TP = PTP \). Since \( T \) is a \( k \)-quasiclass A operator, we have

\[
P \left( \left| T^2 \right| - \left| T \right|^2 \right) P \geq 0.
\] (2.1)

Then

\[
\left| T_1 \right| = (PT^*PT^*TPTP)^{1/2} = (PT^*T^*TTP)^{1/2} = \left( P \left| T^2 \right| P \right)^{1/2} \geq P \left| T^2 \right| P
\] (2.2)

by Hansen’s inequality [22]. On the other hand

\[
\left| T_1 \right|^2 = T_1^*T_1 = PT^*TP = P\left| T^2 \right| P \leq P \left| T^2 \right| P.
\] (2.3)
Hence
\[ |T_1^2| \geq |T_1|^2. \tag{2.4} \]

That is, \( T_1 \) is a class A operator on \( \mathcal{A} \).

For any \( x = (x_1, x_2) \in \mathcal{A} \),
\[ \langle T_3^k x_2, x_2 \rangle = \langle T^k (I - P)x, (I - P)x \rangle = \langle (I - P)x, T^*k (I - P)x \rangle = 0, \tag{2.5} \]

which implies \( T_3^k = 0 \).

Since \( \sigma(T) \cup \mathcal{G} = \sigma(T_1) \cup \sigma(T_3) \), where \( \mathcal{G} \) is the union of the holes in \( \sigma(T) \) which happen to be subset of \( \sigma(T_1) \cap \sigma(T_3) \) by [23, Corollary 7], and \( \sigma(T_3) = 0 \) and \( \sigma(T_1) \cap \sigma(T_3) \) has no interior points, we have \( \sigma(T) = \sigma(T_1) \cup \{0\} \).

**Theorem 2.2.** Let \( T \in \mathcal{B}(\mathcal{A}) \) be a k-quasiclass A operator for a positive integer \( k \). Then the following assertions hold.

1. \( \| T^{n+2} x \| \| T^n x \| \geq \| T^{n+1} x \|^2 \) for all \( x \in \mathcal{A} \) and all positive integers \( n \geq k \).
2. If \( T^n = 0 \) for some positive integer \( n \geq k \), then \( T^{k+1} = 0 \).
3. \( \| T^{n+1} \| \leq \| T^n \| r(T) \) for all positive integers \( n \geq k \), where \( r(T) \) denotes the spectral radius of \( T \).

To give a proof of Theorem 2.2, the following famous inequality is needful.

**Lemma 2.3** (Hölder-McCarthy’s inequality [24]). Let \( A \geq 0 \). Then the following assertions hold.

1. \( \langle A^r x, x \rangle \geq \langle Ax, x \rangle^r \| x \|^{2(1-r)} \) for \( r > 1 \) and all \( x \in \mathcal{A} \).
2. \( \langle A^r x, x \rangle \leq \langle Ax, x \rangle^r \| x \|^{2(1-r)} \) for \( r \in [0, 1] \) and all \( x \in \mathcal{A} \).

**Proof of Theorem 2.2.** (1) Since it is clear that k-quasiclass A operators are \((k + 1)\)-quasiclass A operators, we only need to prove the case \( n = k \). Since
\[
\langle T^k |T^2 T^k x, x \rangle = \langle T^{k+1} T^* T^{k+1} x, x \rangle = \| T^{k+1} x \|^2, \\
\langle T^k |T^2 T^k x, x \rangle = \langle T^2 |T^k x, T^k x \rangle \\
\leq \langle T^* T^2 T T^k x, T^k x \rangle^{1/2} \| T^{k+1} x \|^{2(1-1/2)} \\
= \| T^{k+2} x \| \| T^k x \| \\
\tag{2.6}
\]

by Hölder-McCarthy’s inequality, we have
\[
\| T^{k+2} x \| \| T^k x \| \geq \| T^{k+1} x \|^2 \tag{2.7}
\]

for \( T \) is a k-quasiclass A operator.
If $n = k, k + 1$, it is obvious that $T^{k+1} = 0$. If $T^{k+2} = 0$, then $T^{k+1} = 0$ by (1). The rest of the proof is similar.

(3) We only need to prove the case $n = k$, that is,

$$\|T^{k+1}\| \leq \|T^k\| r(T). \tag{2.8}$$

If $T^n = 0$ for some $n \geq k$, then $T^{k+1} = 0$ by (2) and in this case $r(T) = (r(T^{k+1}))^{1/(k+1)} = 0$. Hence (3) is clear. Therefore we may assume $T^n \neq 0$ for all $n \geq k$. Then

$$\begin{align*}
\frac{\|T^{k+1}\|}{\|T^k\|} &\leq \frac{\|T^{k+2}\|}{\|T^k\|} \leq \frac{\|T^{k+3}\|}{\|T^{k+2}\|} \leq \cdots \leq \frac{\|T^{mk}\|}{\|T^{mk-1}\|}
\end{align*} \tag{2.9}$$

by (1), and we have

$$\left(\frac{\|T^{k+1}\|}{\|T^k\|}\right)^{mk-k} \leq \frac{\|T^{k+1}\|}{\|T^k\|} \times \frac{\|T^{k+2}\|}{\|T^{k+1}\|} \times \cdots \times \frac{\|T^{mk}\|}{\|T^{mk-1}\|} = \frac{\|T^{mk}\|}{\|T^k\|}. \tag{2.10}$$

Hence

$$\left(\frac{\|T^{k+1}\|}{\|T^k\|}\right)^{k-(k/m)} \leq \left(\frac{\|T^{mk}\|}{\|T^k\|}\right)^{1/m}. \tag{2.11}$$

By letting $m \to \infty$, we have

$$\|T^{k+1}\|^k \leq \|T^k\|^k (r(T))^k, \tag{2.12}$$

that is,

$$\|T^{k+1}\| \leq \|T^k\| r(T). \tag{2.13}$$

Lemma 2.4 (see [21]). Let $T \in B(\mathcal{H})$ be a $k$-quasiclass $A$ operator for a positive integer $k$. If $\lambda \neq 0$ and $(T - \lambda)x = 0$ for some $x \in \mathcal{H}$, then $(T - \lambda)^{-1}x = 0$.

Proof. We may assume that $x \neq 0$. Let $\mathcal{M}_0$ be a span of $\{x\}$. Then $\mathcal{M}_0$ is an invariant subspace of $T$ and

$$T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \mathcal{M}_0 \oplus \mathcal{M}_0^1. \tag{2.14}$$
Let $P$ be the orthogonal projection of $H$ onto $M_0$. It suffices to show that $T_2 = 0$ in (2.14). Since $T$ is a $k$-quasiclass A operator, and $x = T^k(x/\lambda^k) \in \text{ran}(T^k)$, we have

$$P\left(|T^2| - |T|^2\right)P \geq 0. \quad (2.15)$$

We remark

$$P|T^2|^2 P = PT^*T^*TP = PT^*PTTP = \begin{pmatrix} |\lambda|^4 & 0 \\ 0 & 0 \end{pmatrix}. \quad (2.16)$$

Then by Hansen’s inequality and (2.15), we have

$$\begin{pmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{pmatrix} = \left(P|T^2|^2 P \right)^{1/2} \geq P|T^2|P \geq P|T|^2P = \begin{pmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{pmatrix}. \quad (2.17)$$

Hence we may write

$$|T^2| = \begin{pmatrix} |\lambda|^2 & A \\ A^* & B \end{pmatrix}. \quad (2.18)$$

We have

$$\begin{pmatrix} |\lambda|^4 & 0 \\ 0 & 0 \end{pmatrix} = P|T^2||T^2|P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |\lambda|^2 & A \\ A^* & B \end{pmatrix} \begin{pmatrix} |\lambda|^2 & A \\ A^* & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} |\lambda|^4 + AA^* & 0 \\ 0 & 0 \end{pmatrix}. \quad (2.19)$$

This implies $A = 0$ and $|T^2|^2 = \begin{pmatrix} |\lambda|^4 & 0 \\ 0 & 0 \end{pmatrix}$. On the other hand,

$$|T^2|^2 = T^*T^*TT$$

$$= \begin{pmatrix} \bar{\lambda} & 0 \\ T_2^*T_3^* & T_2^*T_3 \end{pmatrix} \begin{pmatrix} \bar{\lambda} & 0 \\ T_2^* & T_3 \end{pmatrix} \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix} = \begin{pmatrix} |\lambda|^4 & \bar{\lambda}^2(\lambda T_2 + T_2T_3) \\ \lambda^2(\lambda T_2 + T_2T_3)^* & |\lambda T_2 + T_2T_3|^2 + |T_3|^2 \end{pmatrix}. \quad (2.20)$$
Hence \( \lambda T_2 + T_2 T_3 = 0 \) and \( B = |T_3^2| \). Since \( T \) is a \( k \)-quasiclass \( A \) operator, by a simple calculation we have

\[
0 \leq T^k \left( |T^2| - |T|^2 \right) T^k \leq \left( \begin{array}{c}
0 \n
\begin{array}{c}
(-1)^{k+1} |\lambda|^{2k} T_2 \\
(-1)^{k+1} |\lambda|^{2k} T_2^* + T_3^k |T_2^*|^2 + T_3^k |T_3^k - |T_3^k|^2 |^2 \end{array}
\end{array} \right).
\]

Recall that \( \left( \begin{array}{c}
X \\
Y^*, Z
\end{array} \right) \geq 0 \) if and only if \( X, Z \geq 0 \) and \( Y = X^{1/2} W Z^{1/2} \) for some contraction \( W \). Thus we have \( T_2 = 0 \). This completes the proof. \( \square \)

**Lemma 2.5** (see [25]). If \( T \) satisfies \( \ker(T - \lambda) \subseteq \ker(T - \lambda)^* \) for some complex number \( \lambda \), then \( \ker(T - \lambda) = \ker(T - \lambda)^n \) for any positive integer \( n \).

**Proof.** It suffices to show \( \ker(T - \lambda) = \ker(T - \lambda)^2 \) by induction. We only need to show \( \ker(T - \lambda)^2 \subseteq \ker(T - \lambda) \) since \( \ker(T - \lambda)^2 \) is clear. In fact, if \( (T - \lambda)^2 x = 0 \), then we have \( (T - \lambda)^* (T - \lambda)x = 0 \) by hypothesis. So we have \( \| (T - \lambda)x \|^2 = \| (T - \lambda)^* (T - \lambda)x,x \| = 0 \), that is, \( (T - \lambda)x = 0 \). Hence \( \ker(T - \lambda)^2 \subseteq \ker(T - \lambda). \) \( \square \)

An operator is said to have finite ascent if \( \ker T^n = \ker T^{n+1} \) for some positive integer \( n \).

**Theorem 2.6.** Let \( T \in B(\mathcal{H}) \) be a \( k \)-quasiclass \( A \) operator for a positive integer \( k \). Then \( T - \lambda \) has finite ascent for all complex number \( \lambda \).

**Proof.** We only need to show the case \( \lambda = 0 \) because the case \( \lambda \neq 0 \) holds by Lemmas 2.4 and 2.5.

In the case \( \lambda = 0 \), we shall show that \( \ker T^{k+1} = \ker T^{k+2} \). It suffices to show that \( \ker T^{k+2} \subseteq \ker T^{k+1} \) since \( \ker T^{k+1} \subseteq \ker T^{k+2} \) is clear. Now assume that \( T^{k+2} x = 0 \). We may assume \( T^k x \neq 0 \) since if \( T^k x = 0 \), it is obvious that \( T^{k+1} x = 0 \). By Holder-McCarthy’s inequality, we have

\[
0 = \| T^{k+2} x \|^2 = \langle T^{k+2} x, T^{k+2} x \rangle = \langle |T^{k+2}|^2 T^k x, T^k x \rangle^{1/2} \geq \langle |T^2| T^k x, T^k x \rangle |T^k x|^{-1} \geq \langle |T^2| T^k x, T^k x \rangle |T^k x|^{-1} = |T^{k+1} x|^2 |T^k x|^{-1}.
\]

So we have \( T^{k+1} x = 0 \), which implies \( \ker T^{k+2} \subseteq \ker T^{k+1} \). Therefore \( \ker T^{k+1} = \ker T^{k+2}. \) \( \square \)
In the following lemma, Tanahashi, Jeon, Kim, and Uchiyama extended the result (1.3) to \( k \)-quasiclass A operators in the case \( \lambda_0 \neq 0 \).

**Lemma 2.7 (see [21]).** Let \( T \in B(\mathcal{A}) \) be a \( k \)-quasiclass A operator for a positive integer \( k \). Let \( \lambda_0 \) be an isolated point of \( \sigma(T) \) and \( E \) the Riesz idempotent for \( \lambda_0 \). Then the following assertions hold.

1. If \( \lambda_0 \neq 0 \), then \( E \) is self-adjoint and
   \[
   E\mathcal{A} = \ker(T - \lambda_0) = \ker((T - \lambda_0)^*).
   \]
2. If \( \lambda_0 = 0 \), then \( E\mathcal{A} = \ker(T^{k+1}) \).

An operator \( T \) is said to be isoloid if every isolated point of \( \sigma(T) \) is an eigenvalue of \( T \).

**Theorem 2.8.** Let \( T \in B(\mathcal{A}) \) be a \( k \)-quasiclass A operator for a positive integer \( k \). Then \( T \) is isoloid.

**Proof.** Let \( \lambda \in \sigma(T) \) be an isolated point. If \( \lambda \neq 0 \), by (1) of Lemma 2.7, \( \ker(T - \lambda) = E\mathcal{A} \neq \{0\} \) for \( E \neq 0 \). Therefore \( \lambda \) is an eigenvalue of \( T \). If \( \lambda = 0 \), by (2) of Lemma 2.7, \( \ker(T^{k+1}) = E\mathcal{A} \neq \{0\} \) for \( E \neq 0 \). So we have \( \ker(T) \neq \{0\} \). Therefore 0 is an eigenvalue of \( T \). This completes the proof. \( \square \)

Let \( T \otimes S \) denote the tensor product on the product space \( \mathcal{A} \otimes \mathcal{A} \) for nonzero \( T, S \in B(\mathcal{A}) \). The following theorem gives a necessary and sufficient condition for \( T \otimes S \) to be a \( k \)-quasiclass A operator, which is an extension of [20, Theorem 4.2].

**Theorem 2.9.** Let \( T, S \in B(\mathcal{A}) \) be nonzero operators. Then \( T \otimes S \) is a \( k \)-quasiclass A operator if and only if one of the following assertions holds.

1. \( T^{k+1} = 0 \) or \( S^{k+1} = 0 \).
2. \( T \) and \( S \) are \( k \)-quasiclass A operators.

**Proof.** It is clear that \( T \otimes S \) is a \( k \)-quasiclass A operator if and only if

\[
(T \otimes S)^k \left( |T \otimes S|^2 - |T \otimes S|^2 \right) (T \otimes S)^k \geq 0
\]

\[
\iff T^k \left( |T|^2 - |T|^2 \right) T^k \otimes T^k \left| S^2 \right| S^k + T^k |T|^2 T^k \otimes S^k \left( |S^2| - |S^2| \right) S^k \geq 0
\]

\[
\iff T^k \left( |T|^2 - |T|^2 \right) T^k \otimes S^k \left( |S^2| - |S^2| \right) S^k + T^k \left( |T|^2 - |T|^2 \right) T^k \otimes S^k \left| S^2 \right| S^k \geq 0.
\]

Therefore the sufficiency is clear.

To prove the necessary, suppose that \( T \otimes S \) is a \( k \)-quasiclass A operator. Let \( x, y \in \mathcal{A} \) be arbitrary. Then we have

\[
\langle T^k \left( |T|^2 - |T|^2 \right) T^k x, x \rangle \langle S^k \left| S^2 \right| S^k y, y \rangle + \langle T^k |T|^2 T^k x, x \rangle \langle S^k \left| S^2 \right| S^k y, y \rangle \geq 0.
\]
It suffices to prove that if (1) does not hold, then (2) holds. Suppose that $T^{k+1} \neq 0$ and $S^{k+1} \neq 0$. To the contrary, assume that $T$ is not a $k$-quasiclass A operator, then there exists $x_0 \in \mathcal{H}$ such that

$$\langle T^k \left( |T|^2 - |S|^2 \right) T^k x_0, x_0 \rangle = \alpha < 0, \quad \langle T^k |T|^2 T^k x_0, x_0 \rangle = \beta > 0. \quad (2.26)$$

From (2.25) we have

$$a \langle S^k |S|^2 S^k y, y \rangle + \beta \langle S^k \left( |S|^2 - |S|^2 \right) S^k y, y \rangle \geq 0 \quad \forall y \in \mathcal{H}, \quad (2.27)$$

that is,

$$(a + \beta) \langle S^k |S|^2 S^k y, y \rangle \geq \beta \langle S^k |S|^2 S^k y, y \rangle \quad (2.28)$$

for all $y \in \mathcal{H}$. Therefore $S$ is a $k$-quasiclass A operator. As the proof in Theorem 2.2 (1), we have

$$\langle S^k |S|^2 S^k y, y \rangle = \| S^{k+1} y \|^2, \quad \langle S^k |S|^2 S^k y, y \rangle \leq \| S^{k+2} y \| \| S^k y \|. \quad (2.29)$$

So we have

$$(a + \beta) \| S^{k+2} y \| \| S^k y \| \geq \beta \| S^{k+1} y \|^2 \quad (2.30)$$

for all $y \in \mathcal{H}$ by (2.28). Because $S$ is a $k$-quasiclass A operator, from Lemma 2.1 we can write $S = \left( \begin{smallmatrix} S_1 & S_2 \\ S_3 & S_4 \end{smallmatrix} \right)$ on $\mathcal{H} = \text{ran}(S^k) \oplus \ker S^k$, where $S_1$ is a class A operator (hence it is normaloid). By (2.30) we have

$$(a + \beta) \| S_1^2 \eta \| \| \eta \| \geq \beta \| S_1 \eta \|^2 \quad \forall \eta \in \text{ran}(S^k). \quad (2.31)$$

So we have

$$(a + \beta) \| S_1 \|^2 = (a + \beta) \| S_1^2 \| \geq \beta \| S_1 \|^2, \quad (2.32)$$

where equality holds since $S_1$ is normaloid.

This implies that $S_1 = 0$. Since $S^{k+1} y = S_1 S^k y = 0$ for all $y \in \mathcal{H}$, we have $S^{k+1} = 0$. This contradicts the assumption $S^{k+1} \neq 0$. Hence $T$ must be a $k$-quasiclass A operator. A similar argument shows that $S$ is also a $k$-quasiclass A operator. The proof is complete. \qed

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