Research Article

On Linear Maps Preserving g-Majorization from $\mathbb{F}^n$ to $\mathbb{F}^m$

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Let $\mathbb{F}^n$ and $\mathbb{F}^m$ be the usual spaces of $n$-dimensional column and $m$-dimensional row vectors on $\mathbb{F}$, respectively, where $\mathbb{F}$ is the field of real or complex numbers. In this paper, the relations gs-majorization, lgw-majorization, and rgw-majorization are considered on $\mathbb{F}^n$ and $\mathbb{F}^m$. Then linear maps $T: \mathbb{F}^n \to \mathbb{F}^m$ preserving lgw-majorization or gs-majorization and linear maps $S: \mathbb{F}^n \to \mathbb{F}^m$, preserving rgw-majorization are characterized.

1. Introduction

Majorization is a topic of much interest in various areas of mathematics and statistics. If $x$ and $y$ are $n$-vectors of real numbers such that $x = Dy$ for some doubly stochastic matrix $D$, then we say that $x$ is (vector) majorized by $y$; see [1]. Marshall and Olkin’s text [2] is the standard general reference for majorization. Some kinds of majorization such as multivariate or matrix majorization were motivated by the concepts of vector majorization and were introduced in [3]. Let $V$ and $W$ be two vector spaces over a field $\mathbb{F}$, and let $\sim$ be a relation on both $V$ and $W$. We say that a linear map $T: V \to W$, preserves the relation $\sim$ if

$$Tx \sim Ty \quad \text{whenever} \quad x \sim y. \quad (1.1)$$

The problem of describing these preserving linear maps is one of the most studied linear preserver problems. A lot of effort has been done in [4–9] and [10–12] to characterize the structure of majorization preserving linear maps on certain spaces of matrices. A complex $n \times m$ matrix $R$ is said to be g-row (or g-column) stochastic, if $Re = e$ (or $R^te = e$), where $e = (1, \ldots, 1)^t \in \mathbb{F}^n$ (or $e = (1, \ldots, 1)^t \in \mathbb{F}^m$). A complex $n \times n$ matrix $D$ is said to be g-doubly stochastic if it is both g-row and g-column stochastic. The notions of generalized majorization (g-majorization) were motivated by the matrix majorization and were introduced in [4–6] as follows.
Definition 1.1. Let $x$ and $y$ be two vectors in $\mathbb{F}^n$. It is said that

1. $x$ is gs-majorized by $y$ if there exists an $n \times n$ g-doubly stochastic matrix $D$ such that $x = Dy$, and denoted by $y \succ_{gs} x$;
2. $x$ is lgw-majorized by $y$ if there exists an $n \times n$ g-row stochastic matrix $R$ such that $x = Ry$, and denoted by $y \succ_{lgw} x$;
3. $x'$ is rgw-majorized by $y'$ if there exists an $n \times n$ g-row stochastic matrix $R$ such that $x' = y'R$, and denoted by $y' \succ_{rgw} x'$ (here $z'$ is the transpose of $z$).

Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ that preserve left matrix majorization or weak majorization were already characterized in [10, 11]. In this paper we characterize all linear maps preserving $\succ_{rgw}$ from $\mathbb{F}_n$ to $\mathbb{F}_m$ and all linear maps preserving $\succ_{lgw}$ or $\succ_{gs}$ from $\mathbb{F}^n$ to $\mathbb{F}^m$.

Throughout this paper, the standard bases of $\mathbb{F}^n$ and $\mathbb{F}_m$ are denoted by $\{e_1, \ldots, e_n\}$ and $\{e_1, \ldots, e_m\}$, respectively. The notation $\text{tr}(x)$ is used for the sum of the components of a vector $x \in \mathbb{F}^n$ or $x \in \mathbb{F}_n$. The vector space of all $n \times m$ complex matrices is denoted by $\mathbb{M}_{n,m}$. The notations $[x_1/x_2/\cdots/x_n]$ and $[y_1 \mid y_2 \mid \cdots \mid y_m]$ are used for the $n \times m$ matrix with rows $x_1, x_2, \ldots, x_n \in \mathbb{F}_m$ and columns $y_1, y_2, \ldots, y_m \in \mathbb{F}^n$. The sets of g-row and g-column stochastic $m \times n$ matrices are denoted by $\text{GR}_{m,n}$ and $\text{GC}_{m,n}$, respectively. The set of g-doubly stochastic $n \times n$ matrices is denoted by $\text{GD}_n$. The symbol $J_n$ is used for the $n \times n$ matrix with all entries equal to one. The notation $[T]$ is used for the matrix representation of the linear map $T : V \to W$ with respect to the standard bases of $V$ and $W$ where $V, W \in \{\mathbb{F}^n, \mathbb{F}^m, \mathbb{F}_n, \mathbb{F}_m\}$.

2. Main Results

In this section we state some preliminary lemmas to describe the linear maps preserving $\succ_{rgw}$ from $\mathbb{F}_n$ to $\mathbb{F}_m$ and the linear maps preserving $\succ_{lgw}$ or $\succ_{gs}$ from $\mathbb{F}^n$ to $\mathbb{F}^m$.

Lemma 2.1. Let $T : \mathbb{F}_n \to \mathbb{F}_m$ be a linear map. Then $T$ preserves the subspace $\{x \in \mathbb{F}_n : \text{tr}(x) = 0\}$ if and only if $[T] \in \text{GR}_{m,n}$.

Proof. Let $B = [b_{ij}] := [T]$. Assume that $Be = \lambda e$ for some $\lambda \in \mathbb{F}$. If $x \in \mathbb{F}_n$ and $\text{tr}(x) = 0$, then $0 = xe = x(\lambda e) = x(\lambda e) = (xB)e = \text{tr}(xB) = \text{tr}(Tx)$, so $T$ preserves the subspace $\{x \in \mathbb{F}_n : \text{tr}(x) = 0\}$. Conversely, assume that $T$ preserves the subspace $\{x \in \mathbb{F}_n : \text{tr}(x) = 0\}$. Then $\text{tr}(T(e_i - e_i)) = \text{tr}((e_i - e_i)B) = 0$ for every $i (1 \leq i \leq n)$. Therefore $Be = \lambda e$ where $\lambda = \sum_{i=1}^n b_{ik} = \sum_{k=1}^n b_{ik}$ for every $i (1 \leq i \leq n).$ 

The following lemma gives an equivalent condition for $\succ_{rgw}$ on $\mathbb{F}_m$.

Lemma 2.2 (see [4, Lemma 2.2]). Let $x, y \in \mathbb{F}_n$ and let $x \neq 0$. Then $x \succ_{rgw} y$ if and only if $\text{tr}(x) = \text{tr}(y)$.

The following theorem characterizes all linear maps which preserve $\succ_{rgw}$ from $\mathbb{F}_n$ to $\mathbb{F}_m$. It is clear that every $T : \mathbb{F}_1 \to \mathbb{F}_m$ preserves $\succ_{rgw}$, so assume that $n \geq 2$.

Theorem 2.3. A nonzero linear map $T : \mathbb{F}_n \to \mathbb{F}_m$ preserves $\succ_{rgw}$ if and only if $[T] \in \text{GR}_{m,n}$ and $\{x \in \mathbb{F}_n : x[T] = 0\} = \{x \in \mathbb{F}_n : \text{tr}(x) = 0\}$ or $\{0\}$. 

Proof. Put $B := [T]$. Let $Be = λe$ for some $λ ∈ ℂ$. If $\{x ∈ ℂ^m : xB = 0\} = \{x ∈ ℂ^m : \text{tr}(x) = 0\}$ it is clear that $T$ preserves $\succ_{rgw}$. If $\{x ∈ ℂ^m : xB = 0\} = \{0\}$, $x \succ_{rgw} y$ and $x ≠ 0$ then $Tx ≠ 0$ and by Lemma 2.2, $\text{tr}(x) = \text{tr}(y)$. So $\text{tr}(x - y) = 0$ and hence $\text{tr}(T(x - y)) = 0$ by Lemma 2.1. Therefore $Tx \succ_{rgw} Ty$ by Lemma 2.2 and so $T$ preserves $\succ_{rgw}$. Now, we prove the necessity of the conditions. Let $T : ℂ^m → ℂ^m$ be a linear preserver of $\succ_{rgw}$. If $\text{tr}(x) = 0$, then $x \succ_{rgw} 0$ by Lemma 2.2. So $Tx \succ_{rgw} Ty = 0$ and hence $\text{tr}(Tx) = 0$ by Lemma 2.2. Therefore $T$ preserves the subspace $\{x ∈ ℂ^m : xL = 0\}$ and so $B ∈ \text{GR}_{m,n}$ by Lemma 2.1. If $\{x ∈ ℂ^m : xB = 0\} ≠ \{0\}$, then there exists a nonzero vector $a ∈ ℂ^m$ such that $Ta = aB = 0$. If $\text{tr}(a) = δ ≠ 0$ then $a \succ_{rgw} δe_j$ for every $j (1 ≤ j ≤ n)$, by Lemma 2.2. Then $Ta = 0 \succ_{rgw} δTe_j$ for every $j (1 ≤ j ≤ n)$ and hence $T = 0$ which is a contradiction. Therefore $\text{tr}(a) = 0$ and hence $a \succ_{rgw} (e_1 - e_j)$ for every $j (1 ≤ j ≤ n)$, by Lemma 2.2. Then $Ta = 0 \succ_{rgw} T(e_1 - e_j)$ and so $Te_1 = Te_j$ for every $j (1 ≤ j ≤ n)$. Put $b := Te_1 = e_1B$. Thus $B = [b/⋯/b]$ and hence $\{x ∈ ℂ^m : xL = 0\} = \{x ∈ ℂ^m : \text{tr}(x) = 0\}$.

We use the following lemmas to find the structure of linear preservers of $\succ_{lgw}$-majorization.

Remark 2.4 (see [7, Lemma 2.2]). If $x \notin \text{Span}\{e\}$, then $x \succ_{lgw} y$, for all $y ∈ ℂ^m$.

Lemma 2.5. Let $T : ℂ^m → ℂ^m$ be a linear map. If $x \notin \text{Span}\{e\}$ implies $Tx \notin \text{Span}\{e\}$, then $T$ preserves $\succ_{lgw}$.

Proof. Let $x, y ∈ ℂ^m$ and $x \succ_{lgw} y$. If $x ∈ \text{Span}\{e\}$ then $y = x$ and it is clear that $Tx \succ_{lgw} Ty$. If $x \notin \text{Span}\{e\}$ so $Tx \notin \text{Span}\{e\}$ by the hypothesis and hence $Tx \succ_{lgw} Ty$, by Remark 2.4. Therefore $T$ preserves $\succ_{lgw}$.

Lemma 2.6. Let $T : ℂ^m → ℂ^m$ be a nonzero singular linear map. Then $T$ preserves $\succ_{lgw}$ if and only if $\text{Ker}(T) = \text{Span}\{e\}$ and $e \notin \text{Im}(T)$.

Proof. Let $T$ be a linear preserver of $\succ_{lgw}$. If $x ∈ \text{Ker}(T)$ and $x \notin \text{Span}\{e\}$, then $Tx = 0$ and $x \succ_{lgw} y$, for all $y ∈ ℂ^m$ by Remark 2.4. So $Ty = 0$, for all $y ∈ ℂ^m$, which is a contradiction. Therefore $\text{Ker}(T) ⊂ \text{Span}\{e\}$ and since $\text{Ker}(T) ≠ \{0\}$, $\text{Ker}(T) = \text{Span}\{e\}$. If $e ∈ \text{Im}(T)$, then there exists $x ∈ ℂ^m$ such that $Tx = e$ and $x \notin \text{Span}\{e\}$. Therefore $x \succ_{lgw} y$, for all $y ∈ ℂ^m$, and hence $Ty = e$ for all $y ∈ ℂ^m$, which is a contradiction. So $e \notin \text{Im}(T)$. The converse follows from Lemma 2.5.

Proposition 2.7. Let $T : ℂ^m → ℂ^m$ be a nonzero linear preserver of $\succ_{lgw}$. Then $n ≤ m$.

Proof. If $T$ is injective, then $n ≤ m$. If $T$ is not injective, we obtain $\text{Ker}(T) = \text{Span}\{e\}$ by Lemma 2.6 and $e \notin \text{Im}(T)$. Therefore $n ≤ m$, by the rank and nullity theorem.

Theorem 2.8. Let $T : ℂ^m → ℂ^m$ be a nonzero linear map and $A := [T]$. Then $T$ preserves $\succ_{lgw}$ if and only if one of the following holds:

(i) $\{x : Ax ∈ \text{Span}\{e\}\} = \{0\}$,

(ii) $A ∈ \text{Span}\{\text{GR}_{m,n}\}$ and $\{x : Ax ∈ \text{Span}\{e\}\} = \text{Span}\{e\}$.

Proof. If (i) or (ii) holds, it is easy to show that $T$ preserves $\succ_{lgw}$ by Lemmas 2.5 and 2.6. Conversely, assume that $T$ preserves $\succ_{lgw}$. If (i) does not hold, we show that (ii) holds. Since (i) does not hold, there exists a nonzero vector $b ∈ ℂ^m$ such that $Tb = Ab = μe$ for some $μ ∈ ℂ$. If $b \notin \text{Span}\{e\}$, then $b \succ_{lgw} x$, for all $x ∈ ℂ^m$ by Remark 2.4. So $Tb \succ_{lgw} Tx$, for all $x ∈ ℂ^m$. If (ii) does not hold, there exists a nonzero vector $b ∈ ℂ^m$ such that $Tb = Ab = μe$ for some $μ ∈ ℂ$. If $b \notin \text{Span}\{e\}$, then $b \succ_{lgw} x$, for all $x ∈ ℂ^m$ by Remark 2.4. So $Tb \succ_{lgw} Tx$, for all $x ∈ ℂ^m$. If (ii) holds, then $A ∈ \text{Span}\{\text{GR}_{m,n}\}$ and $\{x : Ax ∈ \text{Span}\{e\}\} = \text{Span}\{e\}$.
and hence \( T = 0 \), which is a contradiction. Then \( b = \lambda e \) for some nonzero \( \lambda \in \mathbb{F} \), and hence \( Ae = (\mu/\lambda)e \). Therefore, \( A \in \text{Span}\{GR_{n,m}\} \) and \( \{x : Ax \in \text{Span}\{e\}\} = \text{Span}\{e\} \).

The following examples show that Proposition 2.7 does not hold for \( >_{gs} \) or \( >_{rgw} \).

**Example 2.9.** For any positive integer \( n \), the linear map \( T : \mathbb{F}^n \rightarrow \mathbb{F} \) defined by \( Tx = \text{tr}(x) \), preserves \( >_{gs} \).

**Example 2.10.** The linear map \( T : F_3 \rightarrow F_2 \) defined by \( Tx = xB \), where \( B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \), preserves rgw-majorization.

We use the following statements to find the structure of linear preservers of gs-majorization.

**Lemma 2.11** (see [6, Proposition 2.1]). Let \( x \) and \( y \) be two distinct vectors in \( \mathbb{F}^n \). Then \( y >_{gs} x \) if and only if \( y \notin \text{Span}\{e\} \) and \( \text{tr}(x) = \text{tr}(y) \).

**Lemma 2.12.** If a linear map \( T : \mathbb{F}^n \rightarrow \mathbb{F}^m \) preserves \( >_{gs} \), then \( [T] \in \text{Span}\{GC_{m,n}\} \).

**Proof.** Let \( A := [T] \). For every \( i, j \) \( (1 \leq i \neq j \leq n) \), it is clear that \( (e_i - e_j) >_{gs} 0 \) by Lemma 2.11. Then \( A(e_i - e_j) >_{gs} 0 \) and hence there exists \( D \in GD_m \) such that \( DA(e_i - e_j) = 0 \). So \( J_mA(e_i - e_j) = J_mD(Ae_i - Ae_j) = 0 \) and therefore \( A \in \text{Span}\{GC_{m,n}\} \).

**Theorem 2.13.** Let \( T : \mathbb{F}^n \rightarrow \mathbb{F}^m \) be a linear map. Then \( T \) preserves \( >_{gs} \) if and only if one of the following holds:

1. there exists some \( a \in \mathbb{F}^m \) such that \( Tx = \text{tr}(x)a \) for all \( x \in \mathbb{F}^n \),
2. \( \lambda[T] \in GR_{m,n} \cap \text{Span}\{GC_{m,n}\} \) for some \( 0 \neq \lambda \in \mathbb{F} \) and \( \text{Ker}(T) \subset \text{Span}\{e\} \),
3. \( [T] \in \text{Span}\{GC_{m,n}\} \) and \( e \not\in \text{Im}([T]) \).

**Proof.** Let \( A := [T] \). Assume that \( T \) preserves \( >_{gs} \). So \( A \in \text{Span}\{GC_{m,n}\} \) by Lemma 2.12. Now, we consider two cases.

**Case 1.** Suppose there exists \( b \in \mathbb{F}^n \setminus \text{Span}\{e\} \) such that \( Tb = Ab = \lambda e \) for some \( \lambda \in \mathbb{F} \). If \( \text{tr}(b) = 0 \), then \( 0 = \text{tr}(b)e = J_mb = (J_mA)b = J_m(AB) = J_m(\lambda e) \). So \( \lambda = 0 \) and hence \( Ab = 0 \). For every \( i, j \) \( (1 \leq i \neq j \leq n) \), \( b >_{gs} (e_i - e_j) \) by Lemma 2.11. Then \( 0 = Ab >_{gs} A(e_i - e_j) \) and hence \( Ae_i = Ae_j \) for all \( i, j \) \( (1 \leq i, j \leq n) \). Then \( A = [a | \cdots | a] \), for some \( a \in \mathbb{F}^m \) and hence \( T(x) = \text{tr}(x)a \) for all \( x \in \mathbb{F}^n \). If \( \text{tr}(b) = \delta \neq 0 \), consider the basis \( \{de_1, \ldots, de_n\} \) for \( \mathbb{F}^n \). For every \( i \) \( (1 \leq i \leq n) \), \( b >_{gs} (de_i) \), by Lemma 2.11. Consequently \( Te_i = (\lambda/\delta)e \) for every \( i \) \( (1 \leq i \leq n) \) and hence \( Tx = \text{tr}(x)a \) for all \( x \in \mathbb{F}^n \), where \( a = (\lambda/\delta)e \). Therefore, (i) holds in this case.

**Case 2.** Assume that \( xe \) \( \not\in \text{Span}\{e\} \) implies \( Tx \not\in \text{Span}\{e\} \). Since \( e_1 >_{gs} e_i \), we have \( T(e_1) >_{gs} T(e_i) \) for every \( i \) \( (1 \leq i \leq n) \). Thus it follows that \( \text{tr}(A_i) = \text{tr}(Te_i) = \text{tr}(Te_1) = \text{tr}(A_1) \) for every \( i \) \( (1 \leq i \leq n) \), where \( A_i \) is the \( i \)-th column of \( A \) and hence \( A \in \text{Span}\{GC_{m,n}\} \). If \( e \in \text{Im}(A) \), then there exists \( 0 \neq \lambda \in \mathbb{F} \) such that \( A(\lambda e) = e \) and hence \( \lambda A \in GR_{m,n} \cap \text{Span}\{GC_{m,n}\} \). By the hypothesis of this case, \( \text{Ker}(T) \subset \text{Span}\{e\} \). Then (ii) holds. If \( e \not\in \text{Im}(A) \) it is clear (iii) holds.
Conversely, if (i) or (iii) holds it is easy to show that $T$ preserves $gs$-majorization. Suppose that (ii) holds. Then there exists $z \in \text{Span}\{e\}$ such that $Tz = e$. Assume that $x \succ_{gs} y$. If $Tx \notin \text{Span}\{e\}$ then $Tx \succ_{gs} Ty$ by Lemma 2.11. If $Tx \in \text{Span}\{e\}$, then there exists $\mu \in \mathbb{F}$ such that $Tx = \mu e$ and hence $T(x - \mu z) = 0$. Therefore, $x - \mu z \in \text{Span}\{e\}$, and hence $x \in \text{Span}\{e\}$. Then $x = y$ and hence $T$ preserves $gs$-majorization.

**Corollary 2.14.** If $T : \mathbb{F}^n \to \mathbb{F}^m$ preserves $\succ_{gs}$ and $	ext{rank}(T) > 1$ then $n \leq m$.

**Proof.** If $T$ is injective it is clear that $n \leq m$. Assume that $T$ is not injective, so there exists a nonzero vector $b \in \mathbb{F}^n$ such that $Tb = 0$. If $b \notin \text{Span}\{e\}$, then by Case 1 in the proof of Theorem 2.13, $Tx = \text{tr}(x)a$ for some $a \in \mathbb{F}^m$. Therefore, $	ext{rank}(T) \leq 1$, which is a contradiction. So $b \in \text{Span}\{e\}$ and hence $\text{Ker}(T) = \text{Span}\{e\}$. It is clear that $e \notin \text{Im}(T)$, from which and the rank and nullity theorem, we obtain $n \leq m$, completing the proof.

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**References**


