Research Article

Hardy-Littlewood and Caccioppoli-Type Inequalities for $A$-Harmonic Tensors

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We prove the new versions of the weighted Hardy-Littlewood inequality and Caccioppoli-type inequality for $A$-harmonic tensors. We also explore applications of our results to $K$-quasiregular mappings and $p$-harmonic functions in $\mathbb{R}^n$.

1. Introduction

The purpose of this paper is to prove the new versions of the weighted Hardy-Littlewood and Caccioppoli-type inequalities for the $A$-harmonic tensors. Our results may have applications in different fields, particularly, in the study of the integrability of solutions to the $A$-harmonic equation in some domains. Roughly speaking, the $A$-harmonic tensors are solutions of the $A$-harmonic equation, which is intimately connected to the fields, including potential theory, quasiconformal mappings, and the theory of elasticity. The investigation of the $A$-harmonic equation has developed rapidly in the recent years see [1–11].

In this paper, we still keep using the standard notations and symbols. All notations and definitions involved in this paper can be found in [1] cited in the paper. We always assume that $M$ is a bounded and convex domain in $\mathbb{R}^n$, $n \geq 2$. We write $\mathbf{R} = \mathbb{R}^1$. Let $e_1, e_2, \ldots, e_n$ be the standard unit basis of $\mathbb{R}^n$ and $\wedge^l = \wedge^l(\mathbb{R}^n)$ the linear space of $l$-vectors, generated by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered $l$-tuples $I = (i_1, i_2, \ldots, i_l)$, $1 \leq i_1 < i_2 < \cdots < i_l \leq n$, $l = 0, 1, \ldots, n$. The Grassman algebra $\wedge = \bigoplus \wedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha_I e_I \in \wedge$ and $\beta = \sum \beta_I e_I \in \wedge$, the inner product in $\wedge$ is given by $\langle \alpha, \beta \rangle = \sum \alpha_I \beta_I^\top$, with summation over all $l$-tuples $I = (i_1, i_2, \ldots, i_l)$ and all integers $l = 0, 1, \ldots, n$. We define the Hodge star operator $\ast: \wedge \to \wedge$ by the rule $\ast 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$
and \( \alpha \land \beta = \beta \land \alpha = \langle \alpha, \beta \rangle (\bullet 1) \) for all \( \alpha, \beta \in \land \). The norm of \( \alpha \in \land \) is given by the formula 
\[
|\alpha|^2 = \langle \alpha, \alpha \rangle = \langle \alpha \land \alpha \rangle \in \land_0 \subset \mathbb{R}.
\]
The Hodge star is an isometric isomorphism on \( \land \) with 
\[
\ast : \land^l \to \land^{n-l} \text{ and } \ast \ast (\ast l) = \land^l \to \land^l.
\]

It is well known that a differential \( l \)-form \( \omega \) on \( M \) is a de Rham current (see [12, Chapter III]) on \( M \) with values in \( \land^l (\mathbb{R}^n) \). Let \( \land^l M \) be the \( l \)th exterior power of the cotangent bundle. We use \( D'(M, \land^l) \) to denote the space of all differential \( l \)-forms and \( L^p(\land^l M) \) to denote the \( l \)-forms

\[
\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 \ldots i_l}(x) dx_{i_1} \land dx_{i_2} \land \ldots \land dx_{i_l}
\]

(1.1)
on \( M \) satisfying \( \int_M |\omega|^p < \infty \) for all ordered \( l \)-tuples \( I \), where \( I = (i_1, i_2, \ldots, i_l) \), \( 1 \leq i_1 < i_2 < \cdots < i_l \leq n \), and \( \omega_{i_1 \ldots i_l}(x) \) are differentiable functions. Thus, \( L^p(\land^l M) \) is a Banach space with norm \( \|\omega\|_{p,M} = (\int_M |\omega(x)|^p dx)^{1/p} = (\int_M (\sum_I |\omega_I(x)|^p dx)^{1/p} dx)^{1/p} \). Here, \( |\omega(x)| = (\sum_I |\omega_I(x)|^2)^{1/2} = (\sum_I |\omega_{i_1 \ldots i_l}(x)|^2)^{1/2} \). We denote the exterior derivative by \( d : D'(M, \land^l) \to D'(M, \land^{l+1}) \) for \( l = 0, 1, \ldots, n \). The Hodge codifferential operator \( d^* : D'(M, \land^{l+1}) \to D'(M, \land^l) \) is given by \( d^* = (-1)^{n-l} \ast d \ast \) on \( D'(M, \land^{l+1}) \), \( l = 0, 1, \ldots, n \). We use \( B \) to denote a ball and \( \sigma B, \sigma > 0 \), is the ball with the same center as \( B \) and with \( \text{diam}(\sigma B) = \sigma \text{ diam}(B) \). We do not distinguish the balls from cubes in this paper. For any measurable set \( E \subset \mathbb{R}^n \), we write \( |E| \) for the \( n \)-dimensional Lebesgue measure of \( E \). We call \( \omega \) a weight if \( \omega \in \land^1(\mathbb{R}^n) \) and \( \omega > 0 \) a.e. For \( 0 < p < \infty \), we write \( f \in L^p(\land^l \mathbb{R}^n, \omega^\alpha) \) if the weighted \( L^p \)-norm of \( f \) over \( E \) satisfies \( \|f\|_{p,\land^l \mathbb{R}^n} = (\int_E |f(x)|^p \omega(x)^{\alpha} dx)^{1/p} < \infty \), where \( \alpha \) is a real number. See [1] or [13] for more properties of differential forms.

For any differential \( k \)-form \( u(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 \ldots i_k}(x) dx_{i_1} \land dx_{i_2} \land \cdots \land dx_{i_k}, \)
k = 1,2,\ldots,n, the vector-valued differential form \( \nabla u \) is defined by

\[
\nabla u = \left( \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right) = \left( \sum_I \frac{\partial u_I}{\partial x_1} dx_I, \sum_I \frac{\partial u_I}{\partial x_2} dx_I, \cdots, \sum_I \frac{\partial u_I}{\partial x_n} dx_I \right),
\]

(1.2)

Also, we all know that

\[
du(x) = \sum_{k=1}^n \sum_{1 \leq i_1 < i_2 < \cdots < i_k} \frac{\partial \omega_{i_1 \ldots i_k}(x)}{\partial x_k} dx_k \land dx_{i_1} \land dx_{i_2} \land \cdots \land dx_{i_k}, \quad k = 0, 1, \ldots, n-1,
\]

(1.3)

and

\[
|du(x)| = \left( \sum_{k=1}^n \sum_{1 \leq i_1 < i_2 < \cdots < i_k} \left| \frac{\partial \omega_{i_1 \ldots i_k}(x)}{\partial x_k} \right|^2 \right)^{1/2}.
\]

There has been remarkable work in the study of the \( A \)-harmonic equation

\[
d^* A(x, d\omega) = 0
\]

(1.4)
for differential forms, where $A : M \times \Lambda^l(\mathbb{R}^n) \rightarrow \Lambda^l(\mathbb{R}^n)$ satisfies the following conditions:

$$|A(x, \xi)| \leq a|\xi|^{p-1}, \quad \langle A(x, \xi), \xi \rangle \geq |\xi|^p$$

(1.5)

for almost every $x \in M$ and all $\xi \in \Lambda^l(\mathbb{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.4). A solution to (1.4) is an element of the Sobolev space $W^1_{p,\text{loc}}(\Omega, \Lambda^{l-1})$ such that $\int_{\Omega} \langle A(x, d\omega), d\varphi \rangle = 0$ for all $\varphi \in W^1_p(M, \Lambda^{l-1})$ with compact support.

**Definition 1.1.** We call $u$ an $A$-harmonic tensor on $M$ if $u$ satisfies the $A$-harmonic equation (1.4) on $M$.

A differential $l$-form $u \in D^l(M, \Lambda^l)$ is called a closed form if $du = 0$ on $M$. Similarly, a differential $l+1$-form $v \in D^l(M, \Lambda^{l+1})$ is called a coclosed form if $d^*v = 0$. The equation

$$A(x, du) = d^*v$$

(1.6)

is called the conjugate $A$-harmonic equation. Suppose that $u$ is a solution to (1.4) in $\Omega$. Then, at least locally in a ball $B$, there exists a form $v \in W^1_q(B, \Lambda^{l+1}), 1/p + 1/q = 1$, such that (1.6) holds.

**Definition 1.2.** When $u$ and $v$ satisfy (1.6) on $M$, and $A^{-1}$ exists on $M$, we call $u$ and $v$ conjugate $A$-harmonic tensors on $M$.

Let $Q \subset \mathbb{R}^n$ be a cube or a ball. To each $y \in Q$ there corresponds a linear operator $K_y : C^\infty(Q, \Lambda^l) \rightarrow C^\infty(Q, \Lambda^{l-1})$ defined by $(K_y \omega)(x; \xi_1, \ldots, \xi_l) = \int_Q \omega(tx + y - ty; x - y, \xi_1, \ldots, \xi_{l-1}) dt$ and the decomposition $\omega = d(K_y \omega) + K_y(d\omega)$. The linear operator $T_Q : C^\infty(Q, \Lambda^l) \rightarrow C^\infty(Q, \Lambda^{l-1})$ is defined by averaging $K_y$ over all points $y$ in $Q$: $T_Q \omega = \int_Q \varphi(y) K_y \omega dy$, where $\varphi \in C^\infty_0(Q)$ is normalized by $\int_Q \varphi(y) dy = 1$. See [1] for more property for the operator $T_Q$. We define the $l$-form $\omega_Q \in D^l(Q, \Lambda^l)$ by $\omega_Q = |Q|^{-1} \int_Q \varphi(y) dy, l = 0$, and $\omega_Q = d(T_Q \omega), l = 1, 2, \ldots, n$, for all $\omega \in L^p(Q, \Lambda^l), 1 \leq p < \infty$.

### 2. The Local Hardy-Littlewood Inequality

We first introduce the following two-weight class which is an extension of $A_r$-weight and $A_r(\lambda)$-weights.

**Definition 2.1.** We say the weight $(w_1(x), w_2(x))$ satisfies the $A_r(\lambda, M)$ condition for $r > 1$ and $0 < \lambda < \infty$, write $(w_1, w_2) \in A_r(\lambda, M)$, if $w_1(x) > 0, w_2(x) > 0$ a.e., and

$$\sup_B \left( \frac{1}{|B|} \int_B w_1^{1/\lambda} dx \right) \left( \frac{1}{|B|} \int_B \left( \frac{1}{w_2} \right)^{1/(r-1)} dx \right)^{(r-1)}$$

(2.1)

for any ball $B \subset M$.

If we choose $w_1 = w_2$ in Definition 2.1, we obtain the usual $A_r(\lambda)$-weights introduced in [7]. Also, if $\lambda = 1$ and $w_1 = w_2$, the above weight reduces to the well-known $A_r$-weight.
See [1, 14, 15] for more properties of weights. We will also need the following generalized Hölder inequality.

**Lemma 2.2.** Let $0 < \alpha < \infty$, $0 < \beta < \infty$, and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If $f$ and $g$ are measurable functions on $\mathbb{R}^n$, then

$$\|fg\|_{s,M} \leq \|f\|_{\alpha,M} \cdot \|g\|_{\beta,M}$$

(2.2)

for any $M \subset \mathbb{R}^n$.

The following two versions of the Hardy-Littlewood integral inequality (Theorem A and Theorem B) appear in [16] and [9], respectively.

**Theorem A.** For each $p > 0$, there is a constant $C$ such that

$$\int_D |u - u(0)|^p \, dx \, dy \leq C \int_D |v - v(0)|^p \, dx \, dy$$

(2.3)

for all analytic functions $f = u + iv$ in the unit disk $D$.

**Theorem B.** Let $u$ and $v$ be conjugate $A$-harmonic tensors in $M \subset \mathbb{R}^n$, $\sigma > 1$, and $0 < s, t < \infty$. Then there exists a constant $C$, independent of $u$ and $v$, such that

$$\|u - u_B\|_{s,B} \leq C |B|^\beta \|v - c\|_{t,\sigma B}^{q/p}$$

(2.4)

for all balls $B$ with $\sigma B \subset M$. Here $c$ is any form in $W^1_{p,\text{loc}}(M, \Lambda)$ with $d^*c = 0$ and $\beta = 1/s + 1/n - (1/t + 1/n)q/p$.

Now we prove the following local two-weight Hardy-Littlewood integral inequality.

**Theorem 2.3.** Let $u$ and $v$ be conjugate $A$-harmonic tensors on $M \subset \mathbb{R}^n$ and $(\omega_1, \omega_2) \in A_r(\lambda, M)$ for some $r > 1$ and $\lambda > 0$. Let $0 < s, t < \infty$. Then there exists a constant $C$, independent of $u$ and $v$, such that

$$\left( \int_B |u - u_B|^{s} \omega_1^{1/s} \, dx \right)^{1/s} \leq C |B|^{\gamma} \left( \int_{\sigma B} |v - c|^{t} \omega_2^{pt/\alpha q s} \, dx \right)^{q/pt}$$

(2.5)

for all balls $B$ with $\sigma B \subset M \subset \mathbb{R}^n$, $\sigma > 1$ and $\alpha > 1$. Here $c$ is any form in $W^1_{q,\text{loc}}(M, \Lambda)$ with $d^*c = 0$ and $\gamma = 1/s + 1/n - (1/t + 1/n)q/p$.

Note that (2.5) can be written as the following symmetric form:

$$\left( \frac{1}{|B|} \int_B |u - u_B|^{s} \omega_1^{1/s} \, dx \right)^{1/s} \leq C |B|^{(1/q - 1/p)/n} \left( \frac{1}{|B|} \int_{\sigma B} |v - c|^{t} \omega_2^{pt/\alpha q s} \, dx \right)^{1/pt}$$

(2.6)
Proof. Let \( k = \alpha s / (\alpha - 1) \). Since \( \alpha > 1 \), then \( k > 0 \) and \( k > s \). Applying the Hölder inequality, we have

\[
\left( \int_B |u - u_B|^s \omega_1^{\lambda/a} \, dx \right)^{1/s} = \left( \int_B (|u - u_B| \omega_1^{\lambda/\alpha})^s \, dx \right)^{1/s} \\
\leq \|u - u_B\|_{k,B} \left( \int_B \omega_1^{k/(a(k-s))} \, dx \right)^{(k-s)/ks} \\
= \|u - u_B\|_{k,B} \left( \int_B \omega_1^s \, dx \right)^{1/as}.
\]  

(2.6)

Choose \( m = a\alpha s / (a\alpha s + pt(r - 1)) \), then \( m < t \). By Theorem B we have

\[
\|u - u_B\|_{k,B} \leq C_1 |B|^\beta \|v - c\|_{m,\sigma B}^{q/p}.
\]

(2.7)

where \( \beta = 1/k + 1/n - (1/m + 1/n)q/p \). Since \( 1/m = 1/t + (t-m)/mt \), by the Hölder inequality again, we obtain

\[
\|v - c\|_{m,\sigma B} = \left( \int_{\sigma B} \left( |v - c| \omega_2^{p/\alpha s} \omega_2^{-p/\alpha s} \right)^m \, dx \right)^{1/m} \\
\leq \left( \int_{\sigma B} |v - c|^{t} \omega_2^{p/\alpha s} \, dx \right)^{1/t} \left( \int_{\sigma B} \left( \frac{1}{\omega_2} \right)^{pmt/\alpha s(t-m)} \, dx \right)^{(t-m)/mt} \\
= \left( \int_{\sigma B} |v - c|^{t} \omega_2^{p/\alpha s} \, dx \right)^{1/t} \left( \int_{\sigma B} \left( \frac{1}{\omega_2} \right)^{1/(r-1)} \, dx \right)^{p(r-1)/\alpha s}.
\]

(2.8)

Hence

\[
\|v - c\|_{m,\sigma B}^{q/p} \leq \left( \int_{\sigma B} \left( \frac{1}{\omega_2} \right)^{1/(r-1)} \, dx \right)^{(r-1)/as} \left( \int_{\sigma B} |v - c|^{t} \omega_2^{p/\alpha s} \, dx \right)^{q/p}.
\]

(2.9)

Combining (2.6), (2.7), and (2.9) yields

\[
\left( \int_B |u - u_B|^{s} \omega_1^{\lambda/a} \, dx \right)^{1/s} \\
\leq C_1 |B|^\beta \left( \int_B \omega_1^s \, dx \right)^{1/as} \left( \int_{\sigma B} \left( \frac{1}{\omega_2} \right)^{1/(r-1)} \, dx \right)^{(r-1)/as} \left( \int_{\sigma B} |v - c|^{t} \omega_2^{p/\alpha s} \, dx \right)^{q/p}.
\]

(2.10)
Using the condition that \((w_1, w_2) \in A_r(\lambda, M)\), we obtain

\[
\left( \int_B w_1^1 \, dx \right)^{1/as} \left( \int_{\sigma B} \left( \frac{1}{w_2} \right)^{1/(r-1)} \, dx \right)^{(r-1)/as} \\
\leq |\sigma B|^{r/as} \left( \left( \frac{1}{|\sigma B|} \int_B w_1^1 \, dx \right) \left( \frac{1}{|\sigma B|} \int_{\sigma B} \left( \frac{1}{w_2} \right)^{1/(r-1)} \, dx \right) \right)^{1/as} \\
\leq C_2 |\sigma B|^{r/as} \\
= C_3 |B|^{r/as}.
\]  

(2.11)

Putting (2.11) into (2.10) and noting that \(\beta + r/as = 1/k + 1/n - (1/m + 1/n)q/p + r/as = 1/s + 1/n - (1/t + 1/n)q/p\), we have

\[
\left( \int_B |u - u_B|^s w_1^{1/a} \, dx \right)^{1/s} \leq C |B|^\gamma \left( \int_{\sigma B} |v - c|^t w_2^{pt/\lambda q s} \, dx \right)^{q/pt},
\]

(2.12)

where \(\gamma = 1/s + 1/n - (1/t + 1/n)q/p\). We have completed the proof of Theorem 2.3.

Note that in Theorem 2.3, \(\alpha > 1\) is arbitrary. Hence, if we choose \(\alpha\) to be some special values, we will have some different versions of the Hardy-Littlewood inequality. For example, if we let \(\alpha = \lambda, \lambda > 1\). By Theorem 2.3, we have

\[
\left( \int_B |u - u_B|^s w_1^{1/a} \, dx \right)^{1/s} \leq C |B|^\gamma \left( \int_{\sigma B} |v - c|^t w_2^{pt/\lambda q s} \, dx \right)^{q/pt}
\]

(2.13)

for all balls \(B\) with \(\sigma B \subset M \subset \mathbb{R}^n\), \(\sigma > 1\), and \(\gamma = 1/s + 1/n - (1/t + 1/n)q/p\).

If we choose \(\alpha = p\) in Theorem 2.3, we obtain the following result:

\[
\left( \int_B |u - u_B|^s w_1^{1/p} \, dx \right)^{1/s} \leq C |B|^\gamma \left( \int_{\sigma B} |v - c|^t w_2^{pt/\lambda q s} \, dx \right)^{q/pt}
\]

(2.14)

for all balls \(B\) with \(\sigma B \subset M \subset \mathbb{R}^n\), \(\sigma > 1\), and \(\gamma = 1/s + 1/n - (1/t + 1/n)q/p\).

As an application of Theorem 2.3, we have the following example.

**Example 2.4.** Let \(f(x) = (f^1, f^2, \ldots, f^n)\) be \(K\)-quasiregular in \(\mathbb{R}^n\), then

\[
u = f^1 df^1 \wedge df^2 \wedge \cdots \wedge df^{l-1}, \quad u = * f^{l+1} df^{l+2} \wedge \cdots \wedge df^n,
\]

(2.15)
l = 1, 2, ..., n - 1, are conjugate $A$-harmonic tensors with $p = n/l$ and $q = n/(n - l)$, where $A$ is some operator satisfying (1.5). Then by Theorem 2.3, we obtain

$$
\left( \int_B \left| f^1 \right|^q \left( \int_B \left| f^2 \right|^q \cdots \left| f^{l-1} \right|^q \right) \right)^{1/q} \leq C |B|^{1/p} \left( \int_{\sigma B} \left| f^{l+2} \right| \cdots \left| f^n \right| - c \left| f^l \right|^q \right)^{1/q}
$$

(2.16)

where $C$ is independent of $f, \gamma = 1/s + 1/n - (1/t + 1/n)q/p$ and $d^* c = 0$.

For more examples of conjugate harmonic tensors, see [3]. We will have different versions of the global two-weight Hardy-Littlewood inequality if we choose $\alpha$ and $\lambda$ to be some special values as we did in the local case. Recently, Xing and Ding introduced the following $A(\alpha, \beta, \gamma; E)$-weights in [17].

Definition 2.5. We say that a measurable function $g(x)$ defined on a subset $E \subset \mathbb{R}^n$ satisfies the $A(\alpha, \beta, \gamma; E)$-condition for some positive constants $\alpha, \beta, \gamma$, write $g(x) \in A(\alpha, \beta, \gamma; E)$ if $g(x) > 0$ a.e., and

$$
\sup_B \left( \frac{1}{|B|} \int_B g^\alpha dx \right)^{\gamma/\beta} \left( \frac{1}{|B|} \int_B g^{-\beta} dx \right)^{\gamma/\beta} < \infty,
$$

(2.17)

where the supremum is over all balls $B \subset E$. We say $g(x)$ satisfies the $A(\alpha, \beta; E)$-condition if (2.17) holds for $\gamma = 1$ and write $g(x) \in A(\alpha, \beta; E) = A(\alpha, \beta, 1; E)$.

We should notice that there are three parameters in the definition of the $A(\alpha, \beta, \gamma; E)$-weights. If we choose some special values for these parameters, we may obtain some existing weighted classes. For example, it is easy to see that the $A(\alpha, \beta, \gamma; E)$-class reduces to the usual $A_\alpha(E)$-class if $\alpha = \gamma = 1$ and $\beta = 1/(r - 1)$. Moreover, it has been proved in [17] that the $A_\alpha(E)$-weight is a proper subset of the $A(\alpha, \beta, \gamma; E)$-weight. Using the similar method to the proof of Theorem 1.5.5 in [1], we can prove the following version of the Hardy-Littlewood inequality. Considering the length of the paper, we do not include the proof here.

Theorem 2.6. Let $u$ and $v$ be conjugate $A$-harmonic tensors on $M \subset \mathbb{R}^n$ and $g(x) \in A(\alpha, \beta, \gamma; M)$ with $\alpha > 1$ and $\beta > 0$. Let $0 < s, t < \infty$. Then, there exists a constant $C$, independent of $u$ and $v$, such that

$$
\left( \int_B \left| u - u_B \right|^q g dx \right)^{1/q} \leq C |B|^t \left( \int_{\sigma B} \left| v - c \right|^q g^{s/t} dx \right)^{1/q}
$$

(2.18)

for all balls $B$ with $\sigma B \subset M \subset \mathbb{R}^n$ and $\sigma > 1$. Here $c$ is any form in $W^{1,q}_{\text{loc}}(M, \Lambda)$ with $d^* c = 0$ and $\gamma = 1/s + 1/n - (1/t + 1/n)q/p$. 

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Example 2.7. Let

\[ u(x) = \frac{3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \]  

(2.19)

be a harmonic function in \( \mathbb{R}^3 \) and \( v \) a 2-form in \( \mathbb{R}^3 \) defined by

\[ v = v_3 dx_1 \wedge dx_2 + v_2 dx_1 \wedge dx_3 + v_1 dx_2 \wedge dx_3, \]  

(2.20)

where \( v_1, v_2, \) and \( v_3 \) are defined as follows:

\[ v_1 = \frac{x_2 x_3}{\sqrt{\sum x_i^2 \prod_{i<j} (x_i^2 + x_j^2)}} \frac{x_2^4 - x_3^4}{\sqrt{x_1^2 + x_2^2 + x_3^2} (x_1^2 + x_2^2)(x_2^2 + x_3^2)}, \]

\[ v_2 = \frac{x_1 x_3}{\sqrt{\sum x_i^2 \prod_{i<j} (x_i^2 + x_j^2)}} \frac{x_1^4 - x_3^4}{\sqrt{x_1^2 + x_2^2 + x_3^2} (x_1^2 + x_2^2)(x_2^2 + x_3^2)}, \]

\[ v_3 = \frac{x_1 x_2}{\sqrt{\sum x_i^2 \prod_{i<j} (x_i^2 + x_j^2)}} \frac{x_1^4 - x_2^4}{\sqrt{x_1^2 + x_2^2 + x_3^2} (x_1^2 + x_2^2)(x_2^2 + x_3^2)}. \]  

(2.21)

Then \( u \) and \( v \) are a pair of conjugate harmonic tensors; see [3]. Hence, the Hardy-Littlewood inequality is applicable. Using inequality (2.5) with \( w_1 = w_2 = 1 \) and \( c = 0 \) over any ball \( B \), we can obtain the norm comparison inequality for \( u \) and \( v \) defined by (2.19) and (2.20), respectively.

3. The Local Caccioppoli-Type Inequality

The purpose of this section is to obtain some estimates which give upper bounds for the \( L^p \)-norm of \( \nabla u \) or \( du \) in terms of the corresponding norm \( u \) or \( u - c \), where \( u \) is a differential form satisfying the \( A \)-harmonic equation (1.4) and \( c \) is any closed form. These kinds of estimates are called the Caccioppoli-type estimates or the Caccioppoli inequalities. From [9], we can obtain the following Caccioppoli-type inequality.

Theorem C. Let \( u \) be an \( A \)-harmonic tensor on \( M \) and let \( \sigma > 1 \). Then there exists a constant \( C \), independent of \( u \), such that

\[ \|du\|_{s,B} \leq C \text{diam } (B)^{-1} \|u - c\|_{s,\sigma B} \]  

(3.1)

for all balls or cubes \( B \) with \( \sigma B \subset M \) and all closed forms \( c \). Here \( 1 < s < \infty \).

The following weak reverse Hölder inequality appears in [9].
Theorem D. Let $u$ be an $A$-harmonic tensor in $\Omega$, $\sigma > 1$ and $0 < s, t < \infty$. Then there exists a constant $C$, independent of $u$, such that

$$\|u\|_{s,B} \leq C |B|^{(t-s)/st} \|u\|_{t,\sigma B}$$

(3.2)

for all balls or cubes $B$ with $\sigma B \subset \Omega$.

Now, we prove the following local two-weight Caccioppoli-type inequality for $A$-harmonic tensors.

Theorem 3.1. Let $u \in D'(M, \mathcal{N})$, $l = 0, 1, \ldots, n$, be an $A$-harmonic tensor on $M \subset \mathbb{R}^n$, $\rho > 1$ and $0 < \alpha < 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the $A$-harmonic equation and $(\omega_1, \omega_2) \in A_r(\lambda, M)$ for some $r > 1$ and $\lambda > 0$. Then there exists a constant $C$, independent of $u$, such that

$$\left( \int_B |du|^s \omega_1^{a1} \right)^{1/s} \leq \frac{C}{\text{diam}(B)} \left( \int_{\rho B} |u - c|^s \omega_2^a \right)^{1/s}$$

(3.3)

for all balls $B$ with $\rho B \subset M$ and all closed forms $c$.

Proof. Choose $t = s/(1 - \alpha)$, then $1 < s < t$. Since $1/s = 1/t + (t - s)/st$, by Hölder inequality and Theorem C, we have

$$\left( \int_B |du|^s \omega_1^{a1} \right)^{1/s} = \left( \int_B \left( |du|^a \omega_1^{a1/s} \right)^s \right)^{1/s} \leq \left( \int_B |du|^a dx \right)^{1/t} \left( \int_B \left( \omega_1^{a1/s} \right)^{s/(t-s)} dx \right)^{(t-s)/st} \leq \|du\|_{t,B} \cdot \left( \int_B \omega_1^{a1} dx \right)^{a/s} = C_1 \text{diam}(B)^{-1} \|u - c\|_{t,\sigma B} \left( \int_B \omega_1^{a1} dx \right)^{a/s}$$

(4.3)

for all balls $B$ with $\sigma B \subset \Omega$ and all closed forms $c$. Since $c$ is a closed form and $u$ is an $A$-harmonic tensor, then $u - c$ is still an $A$-harmonic tensor. Taking $m = s/(1 + \alpha(r - 1))$, we find that $m < s < t$. Applying Theorem D yields

$$\|u - c\|_{t,\sigma B} \leq C_2 |B|^{(m-t)/mt} \|u - c\|_{m,\sigma^2 B} = C_2 |B|^{(m-t)/mt} \|u - c\|_{m,\rho B},$$

(3.5)

where $\rho = \sigma^2$. Substituting (3.5) in (4.3), we have

$$\left( \int_B |du|^s \omega_1^{a1} \right)^{1/s} \leq C_3 \text{diam}(B)^{-1} |B|^{(m-t)/mt} \|u - c\|_{m,\rho B} \left( \int_B \omega_1^{a1} dx \right)^{a/s}.$$  

(3.6)
Now $1/m = 1/s + (s - m)/sm$, by the Hölder inequality again, we obtain

$$\|u - c\|_{m,\rho B} = \left( \int_{\rho B} |u - c|^m dx \right)^{1/m}$$

$$= \left( \int_{\rho B} |u - c|^m w_2^{\alpha/s} w_{2-a/s}^m dx \right)^{1/m}$$

$$\leq \left( \int_{\rho B} |u - c|^m w_2^\alpha dx \right)^{1/s} \left( \frac{1}{w_2} \right)^{1/(r-1)} \left( \int_{\rho B} \frac{1}{w_2} dx \right)^{\frac{a}{1/(r-1) - a/s}} \left( \int_{\rho B} |u - c|^m w_2^\alpha dx \right)^{1/s}$$

for all balls $B$ with $\rho B \subset \Omega$ and all closed forms $c$. Combining (3.6) and (3.7), we obtain

$$\left( \int_B |d\mu|^{r} w_1^{\alpha/s} dx \right)^{1/s} \leq C_3 \text{diam}(B)^{-1} |B|^{(m-1)/mt} \|w_1\|_{\lambda,B}^{\alpha/s} \left( \frac{1}{w_2} \right)^{1/(1/(r-1) - a/s)} \left( \int_{\rho B} |u - c|^m w_2^\alpha dx \right)^{1/s}$$

(3.8)

Since $(w_1, w_2) \in A_r(\lambda, M)$, then we have

$$\|w_1\|_{\lambda,B}^{\alpha/s} \cdot \left( \frac{1}{w_2} \right)^{1/(1/(r-1) - a/s)} \leq \left( \left( \int_{\rho B} w_1^{\alpha/s} dx \right) \left( \int_{\rho B} \left( \frac{1}{w_2} \right)^{1/(r-1)} dx \right)^{r-1} \frac{a}{s} \right)^{a/s}$$

$$= \left[ \frac{\rho B}{|\rho B|} \left( \int_{\rho B} w_1^{\alpha/s} dx \right) \left( \frac{1}{|\rho B|} \left( \int_{\rho B} \left( \frac{1}{w_2} \right)^{1/(r-1)} dx \right)^{r-1} \frac{a}{s} \right)^{a/s} \right] \leq C_4 |B|^{ar/s}.$$

(3.9)

Substituting (3.9) in (3.8), we find that

$$\left( \int_B |d\mu|^{r} w_1^{\alpha/s} dx \right)^{1/s} \leq \frac{C}{\text{diam}(B)} \left( \int_{\rho B} |u - c|^m w_2^\alpha dx \right)^{1/s}$$

(3.10)

for all balls $B$ with $\rho B \subset M$ and all closed forms $c$. This ends the proof of Theorem 3.1. □
Note that if $\lambda = 1$, then $A_r(\lambda, M) = A_r(1, M)$ becomes the usual $A_r(M)$ weight. See [14] for the properties of $A_r(M)$ weights. Thus, choosing $\lambda = 1$ and $w_1 = w_2$ in Theorem 3.1, we have the following $A_r(M)$-weighted Caccioppoli-type inequality.

**Theorem 3.2.** Let $u \in D'(M, \lambda^l), l = 0, 1, \ldots, n$, be an $A$-harmonic tensor in a domain $M \subset \mathbb{R}^n$, $\rho > 0$ and $0 < \alpha < 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the $A$-harmonic equation and $w \in A_r(M)$ for some $r > 1$. Then there exists a constant $C$, independent of $u$, such that

$$
\left( \int_B |du|^s w^s dx \right)^{1/s} \leq \frac{C}{\text{diam}(B)} \left( \int_{\rho B} |u - c|^s w^s dx \right)^{1/s}
$$

(3.11)

for all balls $B$ with $\rho B \subset M$ and all closed forms $c$.

We also need to note that in Theorem 3.1 $\alpha$ is a parameter with $0 < \alpha < 1$. Thus, we will obtain different versions of the Caccioppoli-type inequality if we let $\alpha$ be some particular values. For example, putting $\alpha = 1/s$, we have the following result.

**Theorem 3.3.** Let $u \in D'(M, \lambda^l), l = 0, 1, \ldots, n$, be an $A$-harmonic tensor in a domain $M \subset \mathbb{R}^n$ and $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the $A$-harmonic equation and $(\omega_1, \omega_2) \in A_r(\lambda, M)$ for some $r > 1$ and $\lambda > 0$. Then there exists a constant $C$, independent of $u$, such that

$$
\left( \int_B |du|^s w_1^{\lambda/1/s} dx \right)^{1/s} \leq \frac{C}{\text{diam}(B)} \left( \int_{\rho B} |u - c|^s w_2^{1/s} dx \right)^{1/s}
$$

(3.12)

for all balls $B$ with $\rho B \subset M$ and all closed forms $c$.

If we choose $\alpha = 1/s$ in Theorem 3.2, then $0 < \alpha < 1$ since $1 < s < \infty$. Thus, Theorem 3.2 reduces to the following version.

**Theorem 3.4.** Let $u \in D'(M, \lambda^l), l = 0, 1, \ldots, n$, be an $A$-harmonic tensor in a domain $M \subset \mathbb{R}^n$ and $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the $A$-harmonic equation and $w \in A_r(M)$ for some $r > 1$. Then there exists a constant $C$, independent of $u$, such that

$$
\left( \int_B |du|^s w^{1/s} dx \right)^{1/s} \leq \frac{C}{\text{diam}(B)} \left( \int_{\rho B} |u - c|^s w^{1/s} dx \right)^{1/s}
$$

(3.13)

for all balls $B$ with $\rho B \subset M$ and all closed forms $c$.

**Example 3.5.** Let $A : M \times \lambda^l(\mathbb{R}^n) \rightarrow \lambda^l(\mathbb{R}^n)$ be an operator defined by $A(x, \xi) = \xi|\xi|^{p-2}$. Then $A$ satisfies the condition (1.5). Equation (1.4) reduces to the $p$-harmonic equation

$$
d^*\left( du |u|^{p-2} \right) = 0
$$

(3.14)
and (1.6) reduces to the conjugate $p$-harmonic equation

$$du|u|^{p-2} = d^*v$$

(3.15)

for differential forms, respectively. If $u$ is a function (0-form), (3.14) reduces to the usual $p$-harmonic equation

$$\text{div} \left( \nabla u |\nabla u|^{p-2} \right) = 0.$$  

(3.16)

Also, (3.16) becomes the usual Laplace equation if we let $p = 2$ in (3.16). Now assume that $u$ is a solution to (3.14). By theorems obtained above, we know that $u$ satisfies (3.3), (3.11), (3.12), and (3.13), respectively.

The following example appeared in [18] which shows us how to use the Caccioppoli inequality to estimate the norm of the harmonic function $u$ in $\mathbb{R}^2$.

**Example 3.6.** Let $u(x, y)$ be a function (0-form) defined in $\mathbb{R}^2$ by

$$u(x, y) = \frac{1}{\pi} \left( \arctan \frac{y}{x-1} - \arctan \frac{y}{x+1} \right).$$

(3.17)

It is easy to check that $u(x, y)$ satisfies the Laplace equation $u_{xx}(x, y) + u_{yy}(x, y) = 0$ in the upper half-plane; that is, $u(x, y)$ is a harmonic function in the upper half-plane. Let $r > 0$ be a constant, $(x_0, y_0)$ be a fixed point with $y_0 > r$, and $B = \{(x, y) : (x-x_0)^2 + (y-y_0)^2 \leq r^2\}$. To obtain the upper bound for the $L^s$-norm $\|du(x, y)\|_{s, B}$ with $s > 1$, it would be very complicated if we evaluate the integral $(\int_B |du(x, y)|^s \, dx \wedge dy)^{1/s}$ directly. However, using Caccioppoli inequality (3.11) with $w(x) = 1$ and $n = 2$, we can easily obtain the upper bound of the norm $\|du(x, y)\|_{s, B}$ as follows. First, we know that $|B| = \pi r^2$ and

$$|u(x, y)| \leq \frac{1}{\pi} \left| \arctan \frac{y}{x-1} - \arctan \frac{y}{x+1} \right|$$

$$\leq \frac{1}{\pi} \left| \arctan \frac{y}{x-1} \right| + \left| \arctan \frac{y}{x+1} \right|$$

$$\leq \frac{1}{\pi} \left( \frac{\pi}{2} + \frac{\pi}{2} \right) = 1.$$  

(3.18)
Applying (3.11) and (3.18), we have

\[ \|du(x, y)\|_{s,B} = \left( \int_B |du(x, y)|^s \, dx \, dy \right)^{1/s} \]
\[ \leq C|B|^{-1/2} \left( \int_{\sigma B} |u(x, y)|^s \, dx \, dy \right)^{1/s} \]
\[ \leq C \pi^{-1/2} r^{-1} \left( \int_{\sigma B} dx \, dy \right)^{1/s} \]
\[ = C \pi^{-1/2} r^{-1} \left( \pi(rr)^2 \right)^{1/s} \]
\[ = C \pi^{1/s-1/2} r^{2/s-1} \sigma^2/s \]
\[ = C \left( \pi^{2-s} r^{-2s} \sigma^4 \right)^{1/2s}. \]

\[ (3.19) \]

4. The Global Hardy-Littlewood Inequality

Finally, we should notice that the local Hardy-Littlewood inequality can be extended into the global case in the John domain. A proper subdomain \( \Omega \subset \mathbb{R}^n \) is called a \( \delta \)-John domain, \( \delta > 0, \) if there exists a point \( x_0 \in \Omega \) which can be joined with any other point \( x \in \Omega \) by a continuous curve \( \gamma \subset \Omega \) so that

\[ d(\xi, \partial \Omega) \geq \delta |x - \xi| \quad (4.1) \]

for each \( \xi \in \gamma. \) Here \( d(\xi, \partial \Omega) \) is the Euclidean distance between \( \xi \) and \( \partial \Omega. \)

Using the properties of John domain and the well-known Covering Lemma, we can prove the following global two-weight Hardy-Littlewood inequality.

**Theorem 4.1.** Let \( u \in D'(\Omega, \Lambda_0) \) and \( v \in D'(\Omega, \Lambda^2) \) be conjugate \( A \)-harmonic tensors in a John domain \( \Omega. \) Assume that \( q \leq p, \) \( v - c \in L^1(\Omega, \Lambda^2), \) \( (\omega_1, \omega_2) \in A_r(\lambda, \Omega), \) and \( \omega_1 \in A_r(\Omega) \) for some \( r > 1 \) and \( \lambda > 0. \) If \( s \) is defined by \( s = npt / (ntq + t(q-p)), \) \( 0 < t < \infty, \) then there exists a constant \( C, \) independent of \( u \) and \( v, \) such that

\[ \left( \int_{\Omega} |u - u_{Q_0}|^s \omega_1^{\lambda/s} \, dx \right)^{1/s} \leq C \left( \int_{\Omega} |v - c|^{t/p} \omega_2^{qt/aqs} \, dx \right)^{q/p} \quad (4.2) \]

for any real number \( \alpha > 1. \) Here \( c \) is any form in \( W^{1,\text{q,loc}}(\Omega, \Lambda) \) with \( d^*c = 0 \) and \( Q_0 \subset \Omega \) is a fixed cube.

It is easy to see that our global results can also be used to study \( K \)-quasiregular mappings and \( p \)-harmonic functions in \( \mathbb{R}^n \) as we did in the local cases. Similar to the local case, some global versions of the two-weight inequalities will be obtained if we choose \( \lambda \) and \( \alpha \) to be some special values in Theorem 4.1. Considering the length of the paper, we do not list these similar results here.
References


