Research Article
Note on $q$-Nasybullin’s Lemma Associated with the Modified $p$-Adic $q$-Euler Measure

Taekyun Kim, Young-Hee Kim, Lee-Chae Jang, Seog-Hoon Rim, and Byungje Lee
1 Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, South Korea
2 Department of Mathematics and Computer Science, KonKuk University, Chungju 380-701, South Korea
3 Department of Mathematics Education, Kyungpook National University, Taegu 702-701, South Korea
4 Department of Wireless Communications Engineering, Kwangwoon University, Seoul 139-701, South Korea

Correspondence should be addressed to Young-Hee Kim, yhkim@kw.ac.kr

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We derive the modified $p$-adic $q$-measures related to $q$-Nasybullin’s type lemma.

1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, the symbols $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of rational integers, the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The $p$-adic absolute value in $\mathbb{C}_p$ is normalized in such a way that $|p|_p = 1/p$ (see [1–17]). For $f \in \mathbb{N}$ with $f \equiv 1 \pmod{2}$, let $\overline{f} = [f, p]$ be the least common multiple of $f$ and $p$. We set

$\mathbb{Z}_{\overline{f}} = \lim_{n \to \infty} \frac{\mathbb{Z}}{\overline{f} \mathbb{Z}^{(n)}} \quad \text{for} \quad n \geq 0,$

$\mathbb{Z}_{\overline{f}}^* = \bigcup_{0 \leq a < \overline{f}} (a + \overline{f} \mathbb{Z}_p)^p = \bigcup_{0 \leq a < \overline{f}} \left\{ (a + \overline{f} \mathbb{Z}_p) \right\} = \bigcup_{0 \leq a < \overline{f}} \left\{ x \in \mathbb{Z}_{\overline{f}} \mid x \equiv a \pmod{\overline{f} \mathbb{Z}_p} \right\},$  \hspace{1cm} (1.1)

where $a \in \mathbb{Z}$ lies in $0 \leq a < \overline{f}$.
When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$ (see [1–6, 18–23]). As the definition of $q$-number, we use the following notations:

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}$$

(see [1–23]).

Let $\text{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in \text{UD}(\mathbb{Z}_p)$, the $p$-adic $q$-invariant integral on $\mathbb{Z}_p$ is defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1 + q}{1 + q^{N}} \sum_{x=0}^{pN-1} f(x)(-q)^x$$

(see [2, 3]).

The $q$-Euler numbers, $\varepsilon_{n,q}$, can be determined inductively by

$$\varepsilon_{0,q} = 1, \quad q(qe + 1)^n + \varepsilon_{n,q} = \begin{cases} [2]_q & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases}$$

with the usual convention of replacing $e^i$ by $e_{i,q}$ (see [11]). The modified $q$-Euler numbers $E_{n,q}$ of $\varepsilon_{n,q}$ are defined in [2] as follows:

$$E_{0,q} = \frac{[2]_q}{2}, \quad (qe + 1)^n + E_{n,q} = \begin{cases} [2]_q & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases}$$

with the usual convention of replacing $E^i$ by $E_{i,q}$. For any positive integer $N$,

$$\mu_q(a + fpN\mathbb{Z}_p) = \frac{(-q)^a}{[fpN]_q}$$

is known as a measure on $\mathbb{Z}_q$ (see [9]). In [2], the Witt’s type formulas for $E_{n,q}$ are given by

$$E_{n,q} = \int_{\mathbb{Z}_p} q^{-x} [x]_q^n d\mu_q(x) = [2]_q \frac{1}{(1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1 + q}.$$  

The modified $q$-Euler polynomials are also defined by

$$E_{n,q}(x) = \left([x]_q + q^x E\right)^n = \sum_{l=0}^{n} \binom{n}{l} E_{l,q} q^{lx} [x]_q^{n-l},$$
with the usual convention of replacing $E^n$ by $E_{n,q}$ (see [2]). Thus, we note that

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} q^{-t} [x + t]_q^n d\mu_q(t) = [2]_q \frac{1}{(1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{lx}}{1 + q^l}. \quad (1.9)$$

Recently Govil and Gupta [22] have introduced a new type of q-integrated Meyer-König-Zeller-Durrmeyer (q-MKZD) operators, obtained moments for these operators, and estimated the convergence of these integrated q-MKZD operators. In this paper, we consider the q-extension which is in a direction different than that of Govil and Gupta [22].

Let $K$ be a field over $\mathbb{Q}_p$. Then we call a function $\mu$ a $K$-measure on $\mathbb{Z}_f^*$ if $\mu$ is finitely additive function defined on open-closed subsets in $\mathbb{Z}_f^*$ whose values are in the field $K$. Any open-closed subset in $\mathbb{Z}_f^*$ is a disjoint union of some finite intervals $I_{a,n} = a + p^n \mathbb{Z}_p$ in $\mathbb{Z}_f^*$, where $a \in \mathbb{Z}$ is prime to $f$, and therefore a $K$-measure $\mu$ is determined by its values on all intervals in $\mathbb{Z}_f^*$. Let $Q(I)$ denote the set of all rational numbers, whose denominator is a divisor of $fp^n$ for some $n \geq 0$. In Section 2, we derive the modified $p$-adic $q$-measures related to $q$-Nasybullin’s type lemma.

### 2. The Modified $p$-Adic $q$-Measure

Let $T$ be a $K$-valued function defined on $Q(I)$ with the following property.

There exist two constants $A, B \in K$ such that

$$\sum_{k=0}^{p-1} T \left( \left[ \frac{x + k}{p} \right]_q \right)(-1)^k = AT \left( [x]_q \right) + BT \left( [px]_{q^{1/p}} \right), \quad (2.1)$$

$$T \left( [x + 1]_q \right) = T \left( [x]_q \right),$$

for any number $x \in Q(I)$. Suppose that $\rho$ is a root of the equation $y^2 = Ay + Bp$. Then we define

$$\mu(I_{a,n}) = \rho^{-n}(-1)^a T \left( \left[ \frac{a}{p^n f} \right]_{q^{n+1}} \right) + B\rho^{-(n+1)}(-1)^a T \left( \left[ \frac{a}{p^{n+1} f} \right]_{q^{n+1}} \right), \quad (2.2)$$
for any interval $I_{a,n}$. From (2.2), we note that

$$\sum_{k=0}^{p-1} \mu(I_{a+p^nf_k,n+1})$$

$$= p^{-n} \sum_{k=0}^{p-1} \left( \frac{a + p^nf_k}{p^{n+1}f} \right) q_{n+1} (-1)^{a+k} + B \rho^{-n} \sum_{k=0}^{p-1} \left( \frac{a + p^nf_k}{p^{n+1}f} \right) q_{n+1} (-1)^k$$

$$= p^{-n} (-1)^a \sum_{k=0}^{p-1} \left( \frac{k+a/p^{n+1}f}{p} \right) q_{n+1} (-1)^k + B \rho^{-n} (-1)^a \sum_{k=0}^{p-1} \left( \frac{a}{p^{n+1}f} + k \right) q_{n+1} (-1)^k$$

$$= p^{-n} (-1)^a AT \left( \frac{a}{p^{n+1}f} q_{n+1} \right) + B \rho^{-n} (-1)^a pT \left( \frac{a}{p^{n+1}f} q_{n+1} \right)$$

$$+ B \rho^{-n} (-1)^a T \left( \frac{a}{p^{n+1}f} q_{n+1} \right)$$

$$= p^{-n} (-1)^a (\rho A + Bp) T \left( \frac{a}{p^{n+1}f} q_{n+1} \right) + B \rho^{-n} (-1)^a T \left( \frac{a}{p^{n+1}f} q_{n+1} \right)$$

$$= \mu(I_{a,n}).$$

Thus, we have

$$\mu(I_{a,n}) = \sum_{b \equiv a \pmod{p^{n+1}}} \mu(I_{b,n+1}).$$

Therefore we obtain the following theorem.

**Theorem 2.1.** For $f \in \mathbb{N}$ with $f \equiv 1 \pmod{2}$ and $\bar{f} = [p, f]$, let $T$ be a $K$-valued function defined on $Q^{(f)}$ with the following properties.

There exist two constants $A, B \in K$ such that

$$\sum_{k=0}^{p-1} T \left( \frac{x+k}{p} \right) q_f (-1)^k = AT \left( [x]_q \right) + BT \left( [p x] q^{1/p} \right),$$

$$T \left( [x+1]_q \right) = T \left( [x]_q \right),$$

(2.5)
for any \( x \in Q^1 \). Suppose that \( \rho \) is a root of the equation \( y^2 = Ay + Bp \). Then there exists a \( K(\rho) \)-measure \( \mu \) on \( \mathbb{Z}_p^* \) such that

\[
\mu(I_{a,n}) = \rho^{-n}(1)^{a}T\left(\left[\frac{a}{p^n}\right]_{q^m}\right) + Bp^{-m(1+1)}(1)^{a}T\left(\left[\frac{a}{p^{n-1}q}\right]_{q^m}\right),
\]

for any interval \( I_{a,n} \).

From (1.9), we note that

\[
E_{n,q}(x) = \left[p\right]_q^m \sum_{a=0}^{p-1} (-1)^a E_{n,q}\left(\frac{x + a}{p}\right).
\]

Let \( E_{m,q}(x) \) be the \( m \)th \( q \)-Euler polynomials and let \( P_m([x]_q) \) be the \( m \)th \( q \)-Euler functions, that is, for \( 0 \leq x < 1 \),

\[
P_m([x]_q) = E_{m,q}(x).
\]

Note that \( \lim_{q \to 1} P_m([x]_q) = P_m(x) \) is the Euler function. By (2.7), we see that

\[
\frac{[2]_q}{[2]_{q^m}} \sum_{a=0}^{p-1} (-1)^a P_m\left(\left[\frac{x + i}{p}\right]_{q^m}\right) = P_m([x]_q).
\]

Thus, the \( q \)-Euler function \( P_m([x]_q) \) satisfies the properties of Theorem 2.1 with constants

\[
A = \left[p\right]_q^{-m} \frac{[2]_{q^m}}{[2]_q}, \quad B = 0.
\]

Then \( \rho \neq 0 \) is equal to \( \left[p\right]_q^{-m}(2)_{q^m} \), as \( \rho^2 = Ap + Bp \) reduces simply to \( \rho^2 = \left[p\right]_{q^m}^{-m}(2)_{q^m} \cdot \rho \). Therefore, we obtain the following theorem.

**Theorem 2.2.** For \( m \in \mathbb{Z}_+ \), let the function \( \mu_m = \mu_{m,q} \) be defined on \( I_{a,n} \) as follows:

\[
\mu_m(I_{a,n}) = \left[fp^m\right]_q^m \left[\frac{a}{p^n}\right]_{q^m}(1)^{a}P_m\left(\left[\frac{a}{p^n}\right]_{q^m}\right).
\]

Then \( \mu_m \) is a \( Q_p(q) \)-measure on \( \mathbb{Z}_p^* \)
For \( f \in \mathbb{N} \) with \( f \equiv 1 \pmod{2} \) and \( \overline{f} = [f, p] \), let \( \chi \) be a primitive Dirichlet character modulo \( \overline{f} \). Then the generalized \( q \)-Euler numbers are defined as follows:

\[
E_{n, \chi, q} = [\overline{f}]_q^n \sum_{a=0}^{\overline{f}-1} \chi(a)(-1)^a E_{n, q, \overline{f}} \left( \frac{a}{\overline{f}} \right). \tag{2.12}
\]

From (2.12) and (2.7), we can easily derive the following Witt’s formula:

\[
E_{n, \chi, q} = \int_{\mathbb{Z}/\overline{f}} [x]^n q^{-x} \chi(x) d\mu_q(x)
\]

\[
= \left[ d \right]_q^n [\overline{f}]_q^{\overline{f}-1} \sum_{a=0}^{\overline{f}-1} \chi(a)(-1)^a \int_{\mathbb{Z}/\overline{f}} \left[ \frac{a}{\overline{f}} + x \right] q^{-d} d\mu_q(x)
\]

\[
= \left[ \overline{f} \right]_q^n [\overline{f}]_q^{\overline{f}-1} \sum_{a=0}^{\overline{f}-1} \chi(a)(-1)^a E_{n, q, \overline{f}} \left( \frac{a}{\overline{f}} \right). \tag{2.13}
\]

We can compute a \( q \)-analog of the \( p \)-adic \( q \)-function by the following \( p \)-adic \( q \)-Mellin Mazur transform with respect to \( \mu_m \).

Let

\[
L(\mu_m, \chi) = \int_{\mathbb{Z}/\overline{f}} \chi(a) d\mu_m(a)
\]

\[
= \lim_{\rho \to \infty} \sum_{a \equiv (a) \pmod{\overline{f}} \atop a \in \mathbb{Z}, (a, p) = 1} \chi(a) \mu_m(I_{a, \rho}). \tag{2.14}
\]

Since the character \( \chi \) is constant on the interval \( I_{a,0} \),

\[
L(\mu_m, \chi) = \sum_{a \equiv (a) \pmod{\overline{f}} \atop (a, p) = 1} \chi(a) \mu_m(I_{a,0})
\]

\[
= \sum_{a \equiv (a) \pmod{\overline{f}} \atop (a, p) = 1} \chi(a) \left[ \frac{\overline{f}}{q} \right]_q^m \left[ \frac{a}{\overline{f}} \right] q^a \left( \frac{P_m \left( \frac{a}{\overline{f}} \right)}{q^a} \right) \tag{2.15}
\]

\[
= E_{m, \chi, q} - \chi(p) \left[ \frac{\overline{f}}{q} \right]_q^m \left[ p \right] q^m E_{m, \chi, q^p}.
\]
where $E_{m,\chi,q}$ are the $m$th generalized $q$-Euler numbers attached to $\chi$. For $m \in \mathbb{Z}_+$, we have

$$L(\mu_m, \chi w^{-m}) = E_{m,\chi w^{-m},q} - \chi w^{-m}(p) \frac{[2]_q}{[2]_q^p} [p]_q^m E_{m,\chi w^{-m},q}$$

$$= l_{p,q}(-m, \chi).$$

Assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$. Let $\omega$ be the Teichmüller character mod $p$. For $x \in \mathbb{Z}_p^\times$ we set $(x)_q = [x]_q / \omega(x)$. Note that $|(x)_q - 1|_p < p^{-1/(p-1)}$ and $(x)_q^s$ are defined by $\exp(s \log_p (x)_q)$ for $|s|_p \leq 1$. For $s \in \mathbb{Z}_p$, we define

$$l_{p,q}(s,x) = \int_{\mathbb{Z}_p^\times} (x)_q^s \chi(x) d\mu_q(x).$$

For (2.14), (2.16) and (2.17), we note that

$$l_{p,q}(-k, \chi w^k) = \int_{\mathbb{Z}_p^\times} [x]^k q \chi(x) d\mu_q(x) = \int_{\mathbb{Z}_p^\times} \chi(x) d\mu_k(x).$$

Since $|(x)_q - 1|_p < p^{-1/(p-1)}$ for $x \in \mathbb{Z}_p^\times$, we have $(x)_q^p \equiv 1 \pmod{p^n}$. Let $k \equiv k'(\pmod{p^n(p-1)})$. Then we have

$$l_{p,q}(-k, \chi w^k) \equiv l_{p,q}(-k', \chi w^k) \pmod{p^n}.$$  

Therefore, we obtain the following theorem.

**Theorem 2.3.** For $k \equiv k'(\pmod{p^n(p-1)})$, we have

$$L(\mu_k, \chi) \equiv L(\mu_{k'}, \chi) \pmod{p^n}.$$  

**References**


