Research Article

Hyers-Ulam Stability of Differential Equation $y'' + 2xy' - 2ny = 0$

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1. Introduction

Assume that $X$ and $Y$ are a topological vector space and a normed space, respectively, and that $I$ is an open subset of $X$. If for any function $f : I \to Y$ satisfying the differential inequality

$$
\left\| a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) \right\| \leq \varepsilon 
$$

(1.1)

for all $x \in I$ and for some $\varepsilon \geq 0$, there exists a solution $f_0 : I \to Y$ of the differential equation

$$
a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) = 0 
$$

(1.2)

such that $\|f(x) - f_0(x)\| \leq K(\varepsilon)$ for any $x \in I$, where $K(\varepsilon)$ is an expression of $\varepsilon$ only, then we say that the above differential equation satisfies the Hyers-Ulam stability (or the local Hyers-Ulam stability if the domain $I$ is not the whole space $X$). We may apply this terminology for other differential equations. For more detailed definition of the Hyers-Ulam stability, refer to [1–6].

Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [7, 8]). Here, we will introduce a result of Alsina and Ger (see [9]): If a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality

$$
\left\| a_n(x)f^{(n)}(x) + a_{n-1}(x)f^{(n-1)}(x) + \cdots + a_1(x)f'(x) + a_0(x)f(x) + h(x) \right\| \leq \varepsilon
$$

(1.3)

for all $x \in I$ and for some $\varepsilon \geq 0$, there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation

$$
a_n(x)f^{(n)}(x) + a_{n-1}(x)f^{(n-1)}(x) + \cdots + a_1(x)f'(x) + a_0(x)f(x) + h(x) = 0 
$$

(1.4)

such that $\|f(x) - f_0(x)\| \leq K(\varepsilon)$ for any $x \in I$, where $K(\varepsilon)$ is an expression of $\varepsilon$ only, then we say that the above differential equation satisfies the Hyers-Ulam stability (or the local Hyers-Ulam stability if the domain $I$ is not the whole space $\mathbb{R}$). We may apply this terminology for other differential equations.
\[ |y'(x) - y(x)| \leq \varepsilon, \quad \text{where } I \text{ is an open subinterval of } \mathbb{R}, \text{ then there exists a solution } f_0 : I \to \mathbb{R} \]

of the differential equation \( y'(x) = y(x) \) such that \( |f(x) - f_0(x)| \leq 3\varepsilon \) for any \( x \in I \).

This result of Alsina and Ger has been generalized by Takahasi et al.: They proved in [10] that the Hyers-Ulam stability holds true for the Banach space-valued differential equation \( y'(x) = Ay(x) \) (see also [11]).

Using the conventional power series method, the author in [12] investigated the general solution of the inhomogeneous Legendre differential equation of the form

\[ (1 - x^2)y''(x) - 2xy'(x) + p(p + 1)y(x) = \sum_{m=0}^{\infty} a_m x^m \quad (1.3) \]

under some specific conditions, where \( p \) is a real number and the convergence radius of the power series is positive. Moreover, he applied this result to prove that every analytic function can be approximated in a neighborhood of 0 by the Legendre function with an error bound expressed by \( C(x^2/(1 - x^2)) \) (see [13–15]).

Let us consider the error function and the complementary error function defined by

\[ \text{erf } x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad \text{erfc } x = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt = 1 - \text{erf } x, \quad (1.4) \]

respectively. We recursively define the integrals of the error function as follows:

\[ i^{-1} \text{erfc } x = \frac{2}{\sqrt{\pi}} e^{-x^2}, \quad i^0 \text{erfc } x = \text{erfc } x, \quad i^m \text{erfc } x = \int_x^{\infty} i^{m-1} \text{erfc } t dt \quad (1.5) \]

for any \( m \in \mathbb{N}_0 \). Suppose that we are given a nonnegative integer \( n \), and we introduce a differential equation

\[ y''(x) + 2xy'(x) - 2ny(x) = 0, \quad (1.6) \]

whose general solution is given by

\[ y(x) = A i^n \text{erfc } x + B i^n \text{erfc } (-x) \quad (1.7) \]

(see [16, §7.2.2]).

In Section 2 of this paper, using power series method, we will investigate the general solution of the inhomogeneous differential equation:

\[ y''(x) + 2xy'(x) - 2ny(x) = \sum_{m=0}^{\infty} a_m x^m, \quad (1.8) \]

where the radius of convergence of the power series \( \sum_{m=0}^{\infty} a_m x^m \) is \( \rho > 0 \), whose value is in general permitted to have infinity. Moreover, using the idea from [12–14], we will prove the Hyers-Ulam stability of the differential equation (1.6) in a class of special analytic functions (see the class \( C_K \) in Section 3).

In this paper, \( \mathbb{N}_0 \) denotes the set of all nonnegative integers.
2. General Solution of \((1.8)\)

In the following theorem, we solve the inhomogeneous differential equation \((1.8)\).

**Theorem 2.1.** Assume that \(n\) is a nonnegative integer, the radius of convergence of the power series \(\sum_{m=0}^{\infty} a_m x^m\) is \(\rho > 0\), and that there exists a real number \(\mu \geq 0\) with

\[
|a_{2m}| \leq \mu^2 m(2m + 2)(2m + 1)|a_{2m+2}| \quad (\text{if } n \text{ is odd}),
\]

\[
|a_{2m+1}| \leq \mu^2 m(2m + 3)(2m + 2)|\beta_{2m+3}| \quad (\text{if } n \text{ is even})
\]

for all sufficiently large integers \(m\), where

\[
a_{2m} = \frac{2^{m-1}}{(2m)!} \sum_{k=0}^{m-2} \frac{(2k)!}{2^k} a_{2k} \prod_{i=k+1}^{m-1} (n - 2i),
\]

\[
\beta_{2m+1} = \frac{2^{m-1}}{(2m + 1)!} \sum_{k=0}^{m-2} \frac{(2k + 1)!}{2^k} a_{2k+1} \prod_{i=k+1}^{m-1} [n - (2i + 1)]
\]

for any \(m \in \{2, 3, \ldots\}\). Let us define \(\rho_0 = \min\{\rho, 1/\mu\}\) and \(1/0 = \infty\). Every solution \(y : (-\rho_0, \rho_0) \to \mathbb{C}\) of the inhomogeneous differential equation \((1.8)\) can be represented by

\[
y(x) = y_h(x) + \sum_{m=2}^{\infty} \frac{a_{m-2}}{m(m-1)} x^m + \sum_{m=2}^{\infty} a_{2m} x^{2m} + \sum_{m=2}^{\infty} \beta_{2m+1} x^{2m+1},
\]

(2.3)

where \(y_h(x)\) is a solution of the homogeneous differential equation \((1.6)\).

**Proof.** Assume that a function \(y : (-\rho_0, \rho_0) \to \mathbb{C}\) is given by (2.3). We first prove that the function \(y_p(x)\), defined by \(y(x) - y_h(x)\), satisfies the inhomogeneous differential equation \((1.8)\). Since

\[
y_p'(x) = \sum_{m=2}^{\infty} \frac{a_{m-2}}{m-1} x^{m-1} + \sum_{m=2}^{\infty} 2ma_{2m} x^{2m-1} + \sum_{m=2}^{\infty} (2m + 1)\beta_{2m+1} x^{2m},
\]

\[
y_p''(x) = \sum_{m=0}^{\infty} a_m x^m + \sum_{m=1}^{\infty} (2m + 2)(2m + 1)\alpha_{2m+2} x^{2m}
\]

\[
+ \sum_{m=1}^{\infty} (2m + 3)(2m + 2)\beta_{2m+3} x^{2m+1},
\]

(2.4)
we have

\[ y'_p(x) + 2xy'_p(x) - 2ny_p(x) = \sum_{m=0}^{\infty} a_m x^m + \sum_{m=1}^{\infty} (2m + 2) (2m + 1) \alpha_{2m+2} x^{2m+1} \]

\[ + \sum_{m=1}^{\infty} (2m + 3)(2m + 2) \beta_{2m+3} x^{2m+1} + \sum_{m=2}^{\infty} \frac{2a_{m-2}}{m-1} x^m \]

\[ + \sum_{m=2}^{\infty} 4m \alpha_{2m} x^{2m} + \sum_{m=2}^{\infty} 2(2m + 1) \beta_{2m+1} x^{2m+1} \]

\[ - \sum_{m=2}^{\infty} \frac{2n a_{m-2}}{m(m-1)} x^m - \sum_{m=2}^{\infty} 2n \alpha_{2m} x^{2m} - \sum_{m=2}^{\infty} 2n \beta_{2m+1} x^{2m+1} \]

\[ = \sum_{m=0}^{\infty} a_m x^m + 12n x^2 + \sum_{m=2}^{\infty} (2m + 2)(2m + 1) \alpha_{2m+2} x^{2m} \]

\[ + 20 \beta_5 x^3 + \sum_{m=2}^{\infty} (2m + 3)(2m + 2) \beta_{2m+3} x^{2m+1} \]

\[ + \sum_{m=2}^{\infty} \frac{2(m-n)}{m(m-1)} a_{m-2} x^m - \sum_{m=2}^{\infty} 2(n - 2m) \alpha_{2m} x^{2m} \]

\[ - \sum_{m=2}^{\infty} 2(n - (2m + 1)) \beta_{2m+1} x^{2m+1} \]

(2.5)

for all \( x \in (-\rho_0, \rho_0) \).

It is not difficult to see that

\[ (2m + 2)(2m + 1) \alpha_{2m+2} = 2(n - 2m) \alpha_{2m} + \frac{n - 2m}{m(m-1)} a_{2m-2}, \]

\[ (2m + 3)(2m + 2) \beta_{2m+3} = 2[n - (2m + 1)] \beta_{2m+1} + \frac{n - (2m + 1)}{(2m+1)m} a_{2m-1} \]

(2.6)

for any \( m \in \mathbb{N} \). Hence, we obtain

\[ y''_p(x) + 2xy'_p(x) - 2ny_p(x) = \sum_{m=0}^{\infty} a_m x^m + 12n x^2 + \sum_{m=2}^{\infty} \frac{n - 2m}{m(m-1)} a_{2m-2} x^{2m} + 20 \beta_5 x^3 \]

\[ + \sum_{m=2}^{\infty} \frac{n - (2m + 1)}{m(m+1)} a_{2m-1} x^{2m+1} + \sum_{m=2}^{\infty} \frac{2(m-n)}{m(m-1)} a_{m-2} x^m \]

(2.7)

which proves that \( y_p(x) \) is a particular solution of the inhomogeneous equation (1.8).
We now apply the ratio test to the power series expression of \( y_p(x) \). If \( n \) is an odd integer not less than 0, then \( \beta_{n+2} = \beta_{n+4} = \beta_{n+6} = \cdots = 0 \). Hence, the power series \( \sum_{m=2}^{\infty} \beta_{2m+1} x^{2m+1} \) is a polynomial. And it follows from the first conditions of (2.1) and (2.6) that

\[
\lim_{m \to \infty} \left| \frac{\alpha_{2m+2}}{\alpha_{2m}} \right| = \lim_{m \to \infty} \frac{n - 2m}{(m+1)(2m+1)} + \frac{n - 2m}{m(2m+2)(2m+1)(2m-1)} \alpha_{2m-2} \alpha_{2m} \\
= \lim_{m \to \infty} \frac{|n - 2m|}{m(2m+2)(2m+1)(2m-1)} \alpha_{2m-2} \alpha_{2m} \\
\leq \lim_{m \to \infty} \frac{|n - 2m|}{m(2m+2)(2m+1)(2m-1)} \mu^2 (m-1) 2m(2m-1) \\
= \mu^2.
\] (2.8)

If \( n \geq 0 \) is an even integer, then we have \( \alpha_{n+2} = \alpha_{n+4} = \alpha_{n+6} = \cdots = 0 \). Thus, for each even integer \( n \geq 0 \), the power series \( \sum_{m=2}^{\infty} \alpha_{2m} x^{2m} \) is a polynomial. By the second conditions in (2.1) and (2.6), we get

\[
\lim_{m \to \infty} \left| \frac{\beta_{2m+3}}{\beta_{2m+1}} \right| = \lim_{m \to \infty} \frac{n - (2m+1)}{(2m+3)(m+1)} + \frac{n - (2m+1)}{(2m+3)(2m+2)(2m+1)m} \beta_{2m+1} \\
= \lim_{m \to \infty} \frac{|n - (2m+1)|}{(2m+3)(2m+2)(2m+1)m} \beta_{2m+1} \\
\leq \lim_{m \to \infty} \frac{|n - (2m+1)|}{(2m+3)(2m+2)(2m+1)m} \mu^2 (m-1)(2m+1) 2m \\
= \mu^2.
\] (2.9)

Therefore, the power series expression of \( y_p(x) \) converges for all \( x \in (-\rho_0, \rho_0) \).

Moreover, the convergence region of the power series for \( y_p(x) \) is the same as those of power series for \( y_p'(x) \) and \( y_p''(x) \). In this paper, the convergence region will denote the maximum open set where the relevant power series converges. Hence, the power series expression for \( y_p''(x) + 2xy_p'(x) - 2ny_p(x) \) has the same convergence region as that of \( y_p(x) \). This implies that \( y_p(x) \) is well defined on \( (-\rho_0, \rho_0) \) and so does for \( y(x) \) in (2.3) because \( y_h(x) \) converges for all \( x \in \mathbb{R} \) under our hypotheses.

Since every solution to (1.8) can be expressed as a sum of a solution \( y_h(x) \) of the homogeneous equation and a particular solution \( y_p(x) \) of the inhomogeneous equation, every solution of (1.8) is certainly in the form of (2.3).

**Remark 2.2.** We might have thought that the conditions presented in (2.1) were too strong. However, we can show that some familiar sequences \( \{a_m\} \) satisfy the conditions in (2.1). For example, let \( n = 0 \) and \( a_0 = a_1 = 0, a_2m = a_{2m+1} = 1/(m-1)! \) for all \( m \in \mathbb{N} \) and choose an arbitrary \( \mu > 0 \). Then, by some manipulations, we can show that the coefficients sequence...
{a_m}$ satisfies the second condition of (2.1) for all sufficiently large integers $m$ as we see in the following:

$$\mu^2 m(2m + 3)(2m + 2)|\beta_{2m+3}| = \frac{\mu^2}{(m-1)!} \left| 1 + \sum_{k=1}^{m-1} (-1)^k k \right| \geq \frac{1}{(m-1)!} = |a_{2m+1}|.$$  

(2.10)

### 3. Hyers-Ulam Stability of (1.6)

In this section, let $n$ be a nonnegative integer and let $\rho$ be a constant with $0 < \rho \leq \infty$. For a given $K \geq 0$, let us denote by $\mathcal{C}_K$ the set of all functions $y : (-\rho, \rho) \to \mathbb{C}$ with the properties (a) and (b):

(a) $y(x)$ is represented by a power series $\sum_{m=0}^{\infty} b_m x^m$ whose radius of convergence is at least $\rho$;

(b) it holds true that $\sum_{m=0}^{\infty} |a_m x^m| \leq K \sum_{m=0}^{\infty} |a_m x^m|$ for all $x \in (-\rho, \rho)$, where $a_m = (m + 2)(m + 1)b_{m+2} + 2(m - n)b_m$ for each $m \in \mathbb{N}_0$.

It should be remarked that the power series $\sum_{m=0}^{\infty} b_m x^m$ in (b) has the same radius of convergence as that of $\sum_{m=0}^{\infty} a_m x^m$ given in (a).

In the following theorem, we prove that if an analytic function satisfies some given conditions, then it can be approximated by a combination of integrals of the error function (see the last part of Section 1 or [16, §7.2.2]).

**Theorem 3.1.** Let $n$ be a nonnegative integer. For given constants $K$ and $\rho$ with $K \geq 0$ and $0 < \rho \leq \infty$, suppose that $y : (-\rho, \rho) \to \mathbb{C}$ is a function which belongs to $\mathcal{C}_K$. Assume that there exist constants $\mu, \nu \geq 0$ satisfying

$$\nu^2 |a_{2m+2}| \leq |a_{2m}| \leq \mu^2 m(2m + 2)(2m + 1)|a_{2m+2}|, \quad (3.1)$$

$$\nu^2 |\beta_{2m+3}| \leq |a_{2m+1}| \leq \mu^2 m(2m + 3)(2m + 2)|\beta_{2m+3}| \quad (3.2)$$

for all $m \in \mathbb{N}$. (See the definitions of $a_{2m}$ and $\beta_{2m+1}$ given in Theorem 2.1. Indeed, it is sufficient for the second inequalities in (3.1) and (3.2) to hold true for all sufficiently large integers $m$.) Let us define $\rho_0 = \min\{|\rho, 1/\mu|$, where $1/0 = \infty$. If the function $y$ satisfies the differential inequality

$$|y''(x) + 2xy'(x) - 2ny(x)| \leq \varepsilon \quad (3.3)$$
for all $x \in (-\rho_0, \rho_0)$ and for some $\varepsilon \geq 0$, then there exists a solution $y_h : \mathbb{R} \to \mathbb{C}$ of the differential equation (1.6) such that

$$|y(x) - y_h(x)| \leq \left(\frac{1}{2} + \frac{1}{\nu^2}\right)K\varepsilon x^2$$

for any $x \in (-\rho, \rho)$.

Proof. Since $y \in \mathcal{C}_K$, it follows from (a) and (b) that

$$y''(x) + 2xy'(x) - 2ny(x) = \sum_{m=0}^{\infty} [(m+2)(m+1)b_{m+2} + 2(m-n)b_m]x^m$$

for all $x \in (-\rho, \rho)$. It follows from the last equality and (3.3) that

$$\left|\sum_{m=0}^{\infty} a_m x^m\right| \leq \varepsilon$$

for any $x \in (-\rho, \rho)$. This inequality, together with (b), yields that

$$\sum_{m=0}^{\infty} |a_m x^m| \leq K \varepsilon$$

for each $x \in (-\rho, \rho)$.

By Abel’s formula (see [17, Theorem 6.30]), we have

$$\sum_{m=0}^{p} \left|\frac{a_m x^m}{(m+2)(m+1)}\right| \leq \frac{1}{(p+2)(p+1)} \sum_{m=0}^{p} \left|\frac{a_m x^m}{p+2}\right|$$

for $x \neq 0$. Hence,

$$\frac{1}{(p+2)(p+1)} \sum_{m=0}^{p} \left|\frac{a_m x^m}{p+2}\right| = \frac{1}{(p+2)(p+1)} \sum_{m=0}^{p} \left|\frac{a_m x^m}{p+1}\right|$$

for $x \neq 0$. Therefore,

$$\sum_{m=0}^{p} \left|\frac{a_m x^m}{(m+2)(m+1)}\right| \leq \frac{1}{(p+2)(p+1)} \sum_{m=0}^{p} \left|\frac{a_m x^m}{p+2}\right|$$

for $x \neq 0$. Hence,

$$\sum_{m=0}^{p} \left|\frac{a_m x^m}{(m+2)(m+1)}\right| \leq \frac{1}{(p+2)(p+1)} \sum_{m=0}^{p} \left|\frac{a_m x^m}{p+2}\right|$$

for $x \neq 0$. Therefore,

$$\sum_{m=0}^{p} \left|\frac{a_m x^m}{(m+2)(m+1)}\right| \leq \frac{1}{(p+2)(p+1)} \sum_{m=0}^{p} \left|\frac{a_m x^m}{p+2}\right|$$

for $x \neq 0$. Therefore,

$$\sum_{m=0}^{p} \left|\frac{a_m x^m}{(m+2)(m+1)}\right| \leq \frac{1}{(p+2)(p+1)} \sum_{m=0}^{p} \left|\frac{a_m x^m}{p+2}\right|$$

for $x \neq 0$. Therefore,

$$\sum_{m=0}^{p} \left|\frac{a_m x^m}{(m+2)(m+1)}\right| \leq \frac{1}{(p+2)(p+1)} \sum_{m=0}^{p} \left|\frac{a_m x^m}{p+2}\right|$$

for $x \neq 0$. Therefore,

$$\sum_{m=0}^{p} \left|\frac{a_m x^m}{(m+2)(m+1)}\right| \leq \frac{1}{(p+2)(p+1)} \sum_{m=0}^{p} \left|\frac{a_m x^m}{p+2}\right|$$

for $x \neq 0$. Therefore,

$$\sum_{m=0}^{p} \left|\frac{a_m x^m}{(m+2)(m+1)}\right| \leq \frac{1}{(p+2)(p+1)} \sum_{m=0}^{p} \left|\frac{a_m x^m}{p+2}\right|$$

for $x \neq 0$. Therefore,

$$\sum_{m=0}^{p} \left|\frac{a_m x^m}{(m+2)(m+1)}\right| \leq \frac{1}{(p+2)(p+1)} \sum_{m=0}^{p} \left|\frac{a_m x^m}{p+2}\right|$$

for $x \neq 0$. Therefore,

$$\sum_{m=0}^{p} \left|\frac{a_m x^m}{(m+2)(m+1)}\right| \leq \frac{1}{(p+2)(p+1)} \sum_{m=0}^{p} \left|\frac{a_m x^m}{p+2}\right|$$

for $x \neq 0$. Therefore,

$$\sum_{m=0}^{p} \left|\frac{a_m x^m}{(m+2)(m+1)}\right| \leq \frac{1}{(p+2)(p+1)} \sum_{m=0}^{p} \left|\frac{a_m x^m}{p+2}\right|$$

for $x \neq 0$. Therefore,
for any \( x \in (-\rho_0, \rho_0) \) and \( p \in \mathbb{N} \), since

\[
\sum_{m=0}^{\infty} \frac{2}{(m+1)(m+2)(m+3)} = \sum_{m=0}^{\infty} \left\{ \frac{1}{m+1} - \frac{1}{m+2} + \frac{1}{m+3} - \frac{1}{m+2} \right\} = \frac{1}{2}
\]

(3.9)

Hence, it follows from (3.7) and (3.8) that

\[
\sum_{m=2}^{\infty} \left| \frac{a_{m-2}}{m(m-1)} x^m \right| = x^2 \sum_{m=2}^{\infty} |a_m| x^m \left( \frac{1}{m+2} + \frac{1}{m+1} \right) \leq K \varepsilon x^2
\]

(3.10)

for each \( x \in (-\rho_0, \rho_0) \).

Finally, it follows from Theorem 2.1, (3.1), (3.2), (3.7), and (3.10) that there exists a solution function \( y_h : \mathbb{R} \to \mathbb{C} \) of the homogeneous differential equation (1.6) such that

\[
|y(x) - y_h(x)| \leq \sum_{m=2}^{\infty} \left| \frac{a_{m-2}}{m(m-1)} x^m \right| + \sum_{m=2}^{\infty} |a_{m}| x^m + \sum_{m=2}^{\infty} |\beta_{m+1}| x^{m+1}
\]

\[
\leq K \varepsilon x^2 + \frac{1}{\varphi^2} \sum_{m=2}^{\infty} |a_{m-1}| x^m + \frac{1}{\varphi^2} \sum_{m=2}^{\infty} |\alpha_{m-1}| x^{m+1}
\]

(3.11)

\[
\leq \left( \frac{1}{2} + \frac{1}{\varphi^2} \right) K \varepsilon x^2
\]

for all \( x \in (-\rho_0, \rho_0) \).

If \( \rho \) is finite, then the local Hyers-Ulam stability of the differential equation (1.6) immediately follows from Theorem 3.1.

**Corollary 3.2.** Let \( n \) be a nonnegative integer. For given constants \( K \) and \( \rho \) with \( K \geq 0 \) and \( 0 < \rho < \infty \), suppose that \( y : (-\rho, \rho) \to \mathbb{C} \) is a function which belongs to \( \mathcal{C}_\rho \). Assume that there exist constants \( \mu, \nu \geq 0 \) satisfying the conditions in (3.1) and (3.2) for all \( m \in \mathbb{N} \). (It is sufficient for the second inequalities in (3.1) and (3.2) to hold true for all sufficiently large integers \( m \).) Let us define \( \rho_0 = \min \{ \rho, 1/\mu \} \) and \( 1/0 = \infty \). If the function \( y \) satisfies the differential inequality (3.3) for all \( x \in (-\rho_0, \rho_0) \) and for some \( \varepsilon \geq 0 \), then there exists a solution \( y_h : \mathbb{R} \to \mathbb{C} \) of the differential equation (1.6) such that

\[
|y(x) - y_h(x)| \leq \left( \frac{1}{2} + \frac{1}{\nu^2} \right) K \rho^2 \varepsilon
\]

(3.12)

for any \( x \in (-\rho_0, \rho_0) \).

We now deal with an asymptotic behavior of functions in \( \mathcal{C}_\rho \) under the additional conditions (3.1) and (3.2).

**Corollary 3.3.** Let \( n \) be a nonnegative integer. For given constants \( K, \rho \), and \( \rho_1 \) with \( K \geq 0 \) and \( 0 < \rho_1 < \rho \leq \infty \), suppose that \( y : (-\rho, \rho) \to \mathbb{C} \) is a function belonging to \( \mathcal{C}_\rho \). Assume that there exist constants \( \mu \geq 0 \) and \( \nu > 0 \) satisfying the conditions in (3.1) and (3.2) for any \( m \in \mathbb{N} \). (It is sufficient
for the second inequalities in (3.1) and (3.2) to hold true for all sufficiently large integers $m$.) Then there exists a solution $y_h : \mathbb{R} \rightarrow \mathbb{C}$ of the differential equation (1.6) such that

$$|y(x) - y_h(x)| = O(x^2)$$  \hspace{1cm} (3.13)

as $x \rightarrow 0$.

Proof. Since $y \in C_K$, it follows from the first 4 lines of the proof of Theorem 3.1 that

$$y''(x) + 2xy'(x) - 2ny(x) = \sum_{m=0}^{\infty} a_m x^m$$  \hspace{1cm} (3.14)

for all $x \in (-\rho, \rho)$. As was remarked in the first part of Section 3, the radius of convergence of the power series $\sum_{m=0}^{\infty} a_m x^m$ is same as that of $\sum_{m=0}^{\infty} b_m x^m (= y(x))$, that is, it is at least $\rho$. Since $0 < \rho_1 < \rho$, if we set $\rho_0 = \min\{\rho_1, 1/\mu\}$, then there exists a constant $\delta > 0$ such that

$$|y''(x) + 2xy'(x) - 2ny(x)| = \left| \sum_{m=0}^{\infty} a_m x^m \right| \leq \delta$$  \hspace{1cm} (3.15)

for any $x \in (-\rho_0, \rho_0)$.

According to Theorem 3.1, there exists a solution $y_h : \mathbb{R} \rightarrow \mathbb{C}$ of the differential equation (1.6) satisfying

$$|y(x) - y_h(x)| \leq \left( \frac{1}{2} + \frac{1}{y^2} \right) K \delta x^2$$  \hspace{1cm} (3.16)

for any $x \in (-\rho_0, \rho_0)$. Hence, we have

$$|y(x) - y_h(x)| = O(x^2)$$  \hspace{1cm} (3.17)

as $x \rightarrow 0$. \hfill \Box

4. An Example

The conditions in (3.1) and (3.2) may seem too strong to construct some examples for the coefficients $a_m$’s. In this section, however, we will show that the sequence $\{a_m\}$ given in Remark 2.2 satisfies these conditions: let $n = 0$ and $a_0 = a_1 = 1$, $a_{2m} = a_{2m+1} = 1/(m-1)!$ for all $m \in \mathbb{N}$ and choose some constants $\mu > 0$ and $\nu = \sqrt{2}$. The second inequality in (3.2) has been verified in Remark 2.2.
The first inequality in (3.2) is also true for all \( m \in \mathbb{N} \) as we see in the following:

\[
\nu^2 |\beta_{2m+3}| = 2 \left| \frac{(-1)^m}{(2m+3)(2m+2)} \frac{1}{m!} \sum_{k=0}^{m-1} (-1)^k k! a_{2k+1} \right|
\leq \frac{2}{(2m+3)(2m+2)m} |a_{2m+1}| \left| 1 + \sum_{k=1}^{m-1} (-1)^k k \right|
\leq \frac{2}{(2m+3)(2m+2)m} |a_{2m+1}| \left[ \frac{m + 1}{2} \right] \tag{4.1}
\leq \frac{m + 1}{(2m+3)(2m+2)m} |a_{2m+1}|
\leq |a_{2m+1}|
\]

where \([x]\) denotes the largest integer not exceeding \( x \).

It is not difficult to show that

\[
\frac{1}{4k} \leq \frac{(2k + 2)!}{4^{k+1} (k+1)!} - \frac{(2k)!}{4^{k} k!(k-1)!} \leq \frac{2k - 1}{4k}, \tag{4.2}
\]

\[
\frac{1}{2} \leq \frac{(2k)!}{4^{k} k!(k-1)!} \leq k - \frac{1}{2} \tag{4.3}
\]

for all \( k \in \mathbb{N} \).

By using (4.2), we will now prove that

\[
\left| 1 + \sum_{k=1}^{m-1} \frac{(-1)^k}{4^k} \frac{(2k)!}{k!(k-1)!} \right| \to \infty \tag{4.4}
\]

as \( m \to \infty \): if \( m = 2\ell \) for some \( \ell \in \mathbb{N} \), then

\[
1 + \sum_{k=1}^{m-1} \frac{(-1)^k}{4^k} \frac{(2k)!}{k!(k-1)!} = \frac{1}{2} + \sum_{i=1}^{\ell-1} \left[ \frac{1}{4^{2i+1}} \frac{(4i)!}{(2i+1)!(2i)!(2i-1)!} - \frac{1}{4^{2i+1}} \frac{(4i+2)!}{(2i+1)!(2i)!(2i)!} \right]
\leq \frac{1}{2} + \sum_{i=1}^{\ell-1} \left( \frac{1}{8i} \right) \tag{4.5}
\]

\[
\to -\infty \quad \text{as} \quad m \to \infty.
\]
If $m = 2\ell + 1$ for some $\ell \in \mathbb{N}$, then

$$1 + \sum_{k=1}^{m-1} \frac{(-1)^k (2k)!}{4^k k!(k-1)!} = 1 + \sum_{i=1}^{\ell} \left[ \frac{1}{4^{2i}} \frac{(4i)!}{(2i)!(2i-1)!} - \frac{1}{4^{2i-1}} \frac{(4i-2)!}{(2i-1)!(2i-2)!} \right]$$

$$\geq 1 + \sum_{i=1}^{\ell} \frac{1}{8i - 4}$$

$$\to \infty \quad \text{as} \quad m \to \infty. \quad (4.6)$$

It then follows from (4.3) and (4.4) that

$$\mu^2 m(2m + 2)(2m + 1) |a_{2m+2}| = \mu^2 \frac{4^m m!}{2(2m-1)!} \left| \sum_{k=0}^{m-1} \frac{(-1)^k (2k)!}{4^k k!(k-1)!} a_{2k} \right|$$

$$= \mu^2 \frac{4^m m! (m-1)!}{(2m)!} |a_{2m}| \left| 1 + \sum_{k=1}^{m-1} \frac{(-1)^k (2k)!}{4^k k!(k-1)!} \right| \quad (4.7)$$

$$\geq \mu^2 \frac{1}{m} |a_{2m}| \left| 1 + \sum_{k=1}^{m-1} \frac{(-1)^k (2k)!}{4^k k!(k-1)!} \right|$$

for all sufficiently large integers $m$, which proves that the sequence $\{a_m\}$ satisfies the second inequality in (3.1).

Finally, we will show that the sequence $\{a_m\}$ satisfies the first inequality in (3.1). It follows from (4.3) that

$$\nu^2 |a_{2m+2}| = 2 \frac{4^m m!}{(2m + 2)!} \left| 1 + \sum_{k=1}^{m-1} \frac{(-1)^k (2k)!}{4^k k!(k-1)!} \right|$$

$$= 2 \frac{4^m m! (m-1)!}{(2m + 2)!} |a_{2m}| \left| 1 + \sum_{k=1}^{m-1} \frac{(-1)^k (2k)!}{4^k k!(k-1)!} \right|$$

$$\leq \frac{4}{(2m + 2)(2m + 1)} |a_{2m}| \left| 1 + \sum_{k=1}^{m-1} \frac{(-1)^k (2k)!}{4^k k!(k-1)!} \right|$$

$$\leq \frac{4}{(2m + 2)(2m + 1)} |a_{2m}| \left| 1 + \sum_{k=1}^{m-1} \frac{(2k)!}{4^k k!(k-1)!} \right|$$
\[
\begin{align*}
\leq & \frac{4}{(2m+2)(2m+1)} |a_{2m}| \left| 1 + \sum_{k=1}^{m-1} \left( k - \frac{1}{2} \right) \right| \\
= & \frac{2m^2 - 4m + 6}{(2m+2)(2m+1)} |a_{2m}| \\
< & |a_{2m}|
\end{align*}
\]
(4.8)

for each \( m \in \mathbb{N} \).

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**References**


