Research Article

A Note on Generalized $|A|_k$-Summability Factors for Infinite Series

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1. Introduction

A weighted mean matrix, denoted by $(N, p_n)$, is a lower triangular matrix with entries $p_k / P_n$, where $\{p_k\}$ is a nonnegative sequence with $p_0 > 0$, and $P_n := \sum_{k=0}^{n} p_k$.

Mishra and Srivastava [1] obtained sufficient conditions on a sequence $\{p_k\}$ and a sequence $\{\lambda_n\}$ for the series $\sum a_n p_n \lambda_n / np_n$ to be absolutely summable by the weighted mean matrix $(N, p_n)$.

Recently Savaş and Rhoades [2] established the corresponding result for a nonnegative triangle, using the correct definition of absolute summability of order $k \geq 1$.

Let $A$ be an infinite lower triangular matrix. We may associate with $A$ two lower triangular matrices $\bar{A}$ and $\hat{A}$, whose entries are defined by

$$\bar{a}_{nk} = \sum_{i=k}^{n} a_{ni}, \quad \hat{a}_{nk} = \bar{a}_{nk} - \bar{a}_{n-1,k},$$

respectively. The motivation for these definitions will become clear as we proceed.

Let $A$ be an infinite matrix. The series $\sum a_k$ is said to be absolutely summable by $A$, of order $k \geq 1$, written as $|A|_k$, if

$$\sum_{k=0}^{\infty} n^{k-1} |\Delta t_{n-1}|^k < \infty,$$
where $\Delta$ is the forward difference operator and $t_n$ denotes the $n$th term of the matrix transform of the sequence $\{s_n\}$, where $s_n := \sum_{k=1}^{n} a_k$.

Thus
\[
\begin{align*}
t_n &= \sum_{k=1}^{n} a_{nk} s_k = \sum_{k=1}^{n} a_{nk} \sum_{\nu=1}^{k} a_\nu = \sum_{\nu=1}^{n} a_\nu \sum_{k=\nu}^{n} a_{nk} = \sum_{\nu=1}^{n} \bar{a}_\nu a_\nu, \\
t_n - t_{n-1} &= \sum_{\nu=1}^{n} \bar{a}_\nu a_\nu - \sum_{\nu=1}^{n-1} \bar{a}_{\nu-1,\nu} a_\nu = \sum_{\nu=1}^{n} \bar{a}_\nu a_\nu,
\end{align*}
\]

since $\bar{a}_{n-1,n} = 0$.

A sequence $\{\lambda_n\}$ is said to be of bounded variation ($bv$) if $\sum_n |\Delta \lambda_n| < \infty$. Let $bv_0 = bv \cap c_0$, where $c_0$ denotes the set of all null sequences.

A positive sequence $\{b_n\}$ is said to be an almost increasing sequence if there exist an increasing sequence $\{c_n\}$ and positive constants $A$ and $B$ such that $Ac_n \leq b_n \leq Bc_n$ (see [3]). Obviously, every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say $b_n = e^{(1)^n} n$.

A positive sequence $\gamma := \{\gamma_n\}$ is said to be a quasi $\beta$-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that
\[
Kn^\beta \gamma_n \geq m^\beta \gamma_m
\] (1.4)
holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is quasi $\beta$-power increasing sequence for any nonnegative $\beta$, but the converse need not be true as can be seen by taking an example, say $\gamma_n = n^\beta$ for $\beta > 0$ (see [4]). If (1.4) stays with $\beta = 0$ then $\gamma$ is simply called a quasi-increasing sequence. It is clear that if $\{\gamma_n\}$ is quasi $\beta$-power increasing then $\{n^\beta \gamma_n\}$ is quasi-increasing.

A positive sequence $\gamma = \{\gamma_n\}$ is said to be a quasi-$f$-power increasing sequence, if there exists a constant $K = K(\gamma, f) \geq 1$ such that $Kf_n \gamma_n \geq f_m \gamma_m$ holds for all $n \geq m \geq 1$, where $f := \{f_n\} = \{n^\mu (\log n)^\mu\}$, $\mu > 0$, $0 < \beta < 1$ was considered instead of $n^\beta$ (see [5, 6]).

Given any sequence $\{x_n\}$, the notation $x_n = O(1)$ means $x_n = O(1)$ and $1/x_n = O(1)$.

Quite recently, Savas and Rhoades [2] proved the following theorem for $|A|_{k^\gamma}$-summability factors of infinite series.

**Theorem 1.1.** Let $A$ be a triangle with nonnegative entries satisfying

(i) $\bar{a}_{mn} = 1$, $n = 0, 1, \ldots$,
(ii) $a_{n-1,\nu} \geq a_{\nu\nu}$ for $n \geq \nu + 1$,
(iii) $n a_{nn} = O(1)$,
(iv) $\Delta(1/a_{nn}) = O(1)$, and
(v) $\sum_{\nu=0}^{n} a_{\nu\nu} [a_{\nu+1,\nu+1}] = O(a_{nn})$.

If $\{X_n\}$ is a positive nondecreasing sequence and the sequences $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy

(vi) $|\lambda_n| \leq \beta_n$,
(vii) $\lim \beta_n = 0$,
(viii) $|\lambda_n| X_n = O(1)$,
Theorem 2.1. Let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences satisfying conditions (vi) and (vii) of Theorem 1.1 and

$$\sum_{n=1}^{m} \lambda_n = o(m), \quad m \to \infty. \quad (2.1)$$

If $\{X_n\}$ is a quasi $f$-increasing sequence and condition (x) and

$$\sum_{n=1}^{\infty} nX_n(\beta, \mu) |\Delta \beta_n| < \infty \quad (2.2)$$

are satisfied, then the series $\sum_{n=1}^{\infty} a_n \lambda_n / na_m$ is summable $|A|_{k, k \geq 1}$, where $\{f_n\} := \{n^\mu (\log n)^\mu\}$, $\mu \geq 0$, $0 \leq \beta < 1$, and $X_n(\beta, \mu) := (n^\beta (\log n)^\mu)X_n$.

Theorem 2.1 includes the following theorem with the special case $\mu = 0$.

Theorem 2.2. Let $A$ satisfying conditions (i)–(v) and let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences satisfying conditions (vi), (vii), and (2.1). If $\{X_n\}$ is a quasi $\beta$-power increasing sequence for some $0 \leq \beta < 1$ and conditions (x) and

$$\sum_{n=1}^{\infty} nX_n(\beta) |\Delta \beta_n| < \infty \quad (2.3)$$

are satisfied, where $X_n(\beta) := (n^\beta X_n)$, then the series $\sum_{n=1}^{\infty} a_n \lambda_n / na_m$ is summable $|A|_{k, k \geq 1}$.

If we take that $\{X_n\}$ is an almost increasing sequence instead of a quasi $\beta$-power increasing sequence then our Theorem 2.2 reduces to [8, Theorem 1].
Remark 2.3. The crucial condition, \( \{\lambda_n\} \in bv_0 \), and condition (viii) do not appear among the conditions of Theorems 2.1 and 2.2. By Lemma 3.3, under the conditions on \( \{X_n\} \), \( \{\beta_n\} \), and \( \{\lambda_n\} \) as taken in the statement of Theorem 2.1, also in the statement of Theorem 2.2 with the special case \( \mu = 0 \), conditions \( \{\lambda_n\} \in bv_0 \) and (viii) hold.

3. Lemmas

We shall need the following lemmas for the proof of our main Theorem 2.1.

Lemma 3.1 (see [9]). Let \( \{\varphi_n\} \) be a sequence of real numbers and denote

\[
\Phi_n := \sum_{k=1}^{n} \varphi_k, \quad \Psi_n := \sum_{k=n}^{\infty} |\Delta \varphi_k|.
\]

If \( \Phi_n = o(n) \) then there exists a natural number \( N \) such that

\[
|\varphi_n| \leq 2 \Psi_n
\]

for all \( n \geq N \).

Lemma 3.2 (see [7]). If \( \{X_n\} \) is a quasi \( f \)-increasing sequence, where \( \{f_n\} = \{n^\beta (\log n)^\mu\} \), \( \mu \geq 0 \), \( 0 \leq \beta < 1 \), then conditions (2.1) of Theorem 2.1,

\[
\sum_{n=1}^{m} |\Delta \lambda_n| = o(m), \quad m \to \infty,
\]

\[
\sum_{n=1}^{\infty} nX_n(\beta, \mu) |\Delta \lambda_n| < \infty,
\]

where \( X_n(\beta, \mu) = (n^\beta (\log n)^\mu X_n) \), imply conditions (viii) and

\[
\lambda_n \to 0, \quad n \to \infty.
\]

Lemma 3.3 (see [7]). If \( \{X_n\} \) is a quasi \( f \)-increasing sequence, where \( \{f_n\} = \{n^\beta (\log n)^\mu\} \), \( \mu \geq 0 \), \( 0 \leq \beta < 1 \), then under conditions (vi), (vii), (2.1) and (2.2), conditions (viii) and (3.5) are satisfied.

Lemma 3.4 (see [7]). Let \( \{X_n\} \) be a quasi \( f \)-increasing sequence, where \( \{f_n\} = \{n^\beta (\log n)^\mu\} \), \( \mu \geq 0 \), \( 0 \leq \beta < 1 \). If conditions (vi), (vii), and (2.2) are satisfied, then

\[
n\beta_n X_n = O(1),
\]

\[
\sum_{n=1}^{\infty} \beta_n X_n < \infty.
\]
4. Proof of Theorem 2.1

Let \( T_n \) denote the \( n \)th term of the A-transform of the partial sums of the series \( \sum_{m=1}^{\infty} (a_n \lambda_n)/(na_{nm}) \). Then, we have

\[
T_n = \sum_{v=1}^{n} \sum_{i=1}^{v} \frac{a_{n,i}}{a_{ii}} = \sum_{i=1}^{n} \frac{a_{n,i} \lambda_i}{a_{ii}} = \sum_{i=1}^{n} \frac{a_{n,i} \lambda_i}{a_{ii}}.
\]  

Thus,

\[
T_n - T_{n-1} = \sum_{i=1}^{n} \left( \frac{\lambda_i}{a_{ii}} \right) (s_i - s_{i-1}) + \sum_{i=1}^{n-1} \frac{\lambda_{i+1} s_i}{a_{ii}} + \sum_{i=1}^{n} \frac{\lambda_{i+1} s_i}{a_{ii}} - \sum_{i=1}^{n} \frac{\lambda_{i+1} s_i}{a_{ii}}.
\]  

It is easy to see that

\[
\frac{\bar{a}_{ni} \lambda_i}{ia_{ii}} - \frac{\bar{a}_{n,i+1} \lambda_{i+1}}{(i+1) a_{i+1,i+1}} = \Delta_i \left( \frac{\bar{a}_{ni}}{ia_{ii}} \right) \lambda_i + \frac{\bar{a}_{n,i+1}}{(i+1) a_{i+1,i+1}} \Delta(\lambda_i).
\]  

Also we may write

\[
\Delta_i \left( \frac{\bar{a}_{ni}}{ia_{ii}} \right) \lambda_i = \frac{\Delta_i (\bar{a}_{ni}) \lambda_i}{ia_{ii}} + a_{n,i+1} \lambda_i \left( \frac{1}{ia_{ii}} - \frac{1}{(i+1) a_{i+1,i+1}} \right).
\]  

Therefore, for \( n > 1 \),

\[
T_n - T_{n-1} = \sum_{i=1}^{n-1} \Delta_i (\bar{a}_{ni}) \lambda_i s_i + \sum_{i=1}^{n-1} \bar{a}_{n,i+1} \lambda_i \left( \frac{1}{ia_{ii}} - \frac{1}{(i+1) a_{i+1,i+1}} \right) s_i + \sum_{i=1}^{n-1} \frac{\bar{a}_{n,i+1}}{(i+1) a_{i+1,i+1}} \Delta_i (\lambda_i) s_i + \frac{\lambda_n}{n} s_n
\]

\[= T_{n1} + T_{n2} + T_{n3} + T_{nk}, \quad \text{say.}\]
To complete the proof of the theorem, it will be sufficient to show that

\[
\sum_{n=1}^{\infty} n^{k-1} |T_{n1}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4.
\] (4.6)

Using Hőlder’s inequality and condition (iii),

\[
I_1 = \sum_{n=1}^{m+1} n^{k-1} |T_{n1}|^k \leq \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{i=1}^{n-1} \left| \Delta_i(\tilde{a}_m) \lambda_i s_i \right| \right)^k
= O(1) \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{i=1}^{n-1} \left| \Delta_i(\tilde{a}_m) \lambda_i s_i \right| \right)^k
= O(1) \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{i=1}^{n-1} \left| \Delta_i(\tilde{a}_m) \right| |\lambda_i|^k |s_i|^k \right) \times \left( \sum_{i=1}^{n-1} \left| \Delta_i(\tilde{a}_m) \right| \right)^{k-1}.
\] (4.7)

Since \((\lambda_n)\) is bounded by Lemma 3.3, using (ii), (iii), (vi), (x), and property (3.7) of Lemma 3.4,

\[
I_1 = O(1) \sum_{n=1}^{m+1} (n a_{mn})^{k-1} \sum_{i=1}^{n-1} |\lambda_i|^k |s_i|^k |\Delta_i(\tilde{a}_m)|
= O(1) \sum_{n=1}^{m+1} (n a_{mn})^{k-1} \left( \sum_{i=1}^{n-1} |\lambda_i|^{k-1} |\lambda_i| |\Delta_i(\tilde{a}_m)| |s_i|^k \right)
= O(1) \sum_{i=1}^{m} |\lambda_i| |s_i|^k \sum_{n=i+1}^{m+1} (n a_{mn})^{k-1} |\Delta_i(\tilde{a}_m)|
= O(1) \sum_{i=1}^{m} |\lambda_i| |s_i|^k a_{ii} = O(1) \sum_{i=1}^{m} |\lambda_i| |s_i|^k
= O(1) \left[ \sum_{i=1}^{m} |\lambda_i| \sum_{r=1}^{i} \frac{|s_r|^k}{r} - \sum_{i=0}^{m-1} |\lambda_{i+1}| \sum_{r=1}^{i} \frac{|s_r|^k}{r} \right]
= O(1) \sum_{i=1}^{m-1} \Delta(|\lambda_i|) \sum_{r=1}^{i} |s_r|^k + O(1)|\lambda_m| \sum_{i=1}^{m} \frac{|s_i|^k}{i}
= O(1) \sum_{i=1}^{m-1} \Delta(|\lambda_i|) X_i + O(1)|\lambda_m|X_m
= O(1) \sum_{i=1}^{m} \beta_i X_i + O(1)|\lambda_m|X_m = O(1).
Now

\[ I_2 = \sum_{n=1}^{m+1} n^{k-1} |T_n^2|^k = \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{i=1}^{n-1} \tilde{a}_{n,i+1} \lambda_i \Delta \left( \frac{1}{ia_{ii}} \right) s_i \right|^k \]

\[ = O(1) \sum_{n=1}^{m+1} n^{k-1} \left\{ \sum_{i=1}^{n-1} |\tilde{a}_{n,i+1}| |\lambda_i| \left| \Delta \left( \frac{1}{ia_{ii}} \right) \right| |s_i| \right\}^k. \]  \hspace{1cm} (4.9)

From [2],

\[ \Delta \left( \frac{1}{ia_{ii}} \right) = \frac{1}{i+1} \left[ \Delta \left( \frac{1}{a_{ii}} \right) + \frac{1}{ia_{ii}} \right]. \]  \hspace{1cm} (4.10)

Thus, using (iv) and (ii),

\[ \left| \Delta \left( \frac{1}{ia_{ii}} \right) \right| = \left| \frac{1}{i+1} \left[ \Delta \left( \frac{1}{a_{ii}} \right) + \frac{1}{ia_{ii}} \right] \right| \]

\[ = \frac{1}{i+1} [O(1) + O(1)]. \]  \hspace{1cm} (4.11)

Hence, using Hölder’s inequality, (v), (iii), and the fact that the \( \lambda_n \)'s are bounded,

\[ I_2 = O(1) \sum_{n=1}^{m+1} n^{k-1} \left\{ \sum_{i=1}^{n-1} |\tilde{a}_{n,i+1}| |\lambda_i| \frac{1}{i+1} |s_i| \right\}^k \]

\[ = O(1) \sum_{n=1}^{m+1} n^{k-1} \left\{ \sum_{i=1}^{n-1} |\tilde{a}_{n,i+1}||\lambda_i||s_i| \right\}^k \]

\[ = O(1) \sum_{n=1}^{m+1} n^{k-1} \left\{ \sum_{i=1}^{n-1} |\tilde{a}_{n,i+1}| |\lambda_i|^k |s_i|^k \right\}^{k-1} \]

\[ = O(1) \sum_{n=1}^{m+1} \sum_{i=1}^{m} (na_{nn})^{k-1} \sum_{i=1}^{m} |\tilde{a}_{n,i+1}| |\lambda_i|^k |s_i|^k \]

\[ = O(1) \sum_{i=1}^{m} |\lambda_i|^k |s_i|^k a_{ii} \sum_{n=1}^{m+1} (na_{nn})^{k-1} |\tilde{a}_{n,i+1}| \]

\[ = O(1) \sum_{i=1}^{m} |\lambda_i|^k |s_i|^k a_{ii} \sum_{n=1}^{m+1} |\tilde{a}_{n,i+1}| \]

\[ = O(1) \sum_{i=1}^{m} |\lambda_i|^k |s_i|^k a_{ii} \]
\[ I_3 = O(1) \sum_{i=1}^{m} |\lambda_i| |\lambda_i|^{k-1} |s_i|^{k} \frac{1}{i} \]
\[ = \sum_{i=1}^{m} |\lambda_i| |s_i|^{k} \frac{1}{i} = O(1), \]

(4.12)

as in the proof of \( I_1 \).

It follows from (3.6) that \( \beta_n = O(1/n) \) and hence that \( |\Delta \lambda_n| = O(1/n) \) by condition (vi).

Using (iii), Hölder’s inequality, and (v),

\[ I_3 = O(1) \sum_{n=1}^{m+1} n^{k-1} |T_{n3}|^k = O(1) \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{i=1}^{n-1} \frac{\tilde{a}_{n,i+1} (\Delta \lambda_i) s_i}{(i+1) a_{i+1,i+1}} \right)^k \]
\[ = O(1) \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{i=1}^{n-1} \frac{\tilde{a}_{n,i+1} \Delta \lambda_i |s_i|}{a_{i,i}^{k}} \right)^k \]
\[ = O(1) \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{i=1}^{n-1} \frac{\tilde{a}_{n,i+1} \Delta \lambda_i |s_i|}{a_{i,i}^{k}} \right)^{k-1} \sum_{i=1}^{n-1} a_{i,i} \tilde{a}_{n,i+1} |s_i| \]
\[ = O(1) \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{i=1}^{n-1} \frac{\tilde{a}_{n,i+1} |s_i|}{a_{i,i}^{k}} \right)^{k-1} \sum_{i=1}^{n-1} a_{i,i} \tilde{a}_{n,i+1} \]
\[ = O(1) \sum_{n=1}^{m+1} n \left( \frac{|\Delta \lambda_i|}{a_{i,i}} \right)^{k-1} |\Delta \lambda_i| |s_i|^{k} \]
\[ = O(1) \sum_{i=1}^{m} |\Delta \lambda_i| |s_i|^{k} = O(1) \sum_{i=1}^{m} |s_i|^{k} \beta_i. \]

(4.13)

Since \( |s_i|^k = i(T_i - T_{i-1}) \) by (x), we have

\[ I_3 = O(1) \sum_{i=1}^{m} i(T_i - T_{i-1}) \beta_i. \]

(4.14)
Using Abel’s transformation, (vi), (2.2), and properties (3.7) and (3.6) of Lemma 3.4,

$$I_3 = O(1) \sum_{i=1}^{m-1} T_i \Delta (i \hat{\beta}_i) + O(1) m T_n \hat{\beta}_n$$

$$= O(1) \sum_{i=1}^{m-1} i |\Delta \beta_i| X_i + O(1) \sum_{i=1}^{m-1} X_i \beta_i + O(1) m X_n \beta_n = O(1).$$

Using the boundedness of $\lambda_n$ and $(x)$,

$$I_4 = \sum_{n=1}^{m+1} n^{k-1} |T_n| n^k = \sum_{n=1}^{m+1} n^{k-1} \left| \frac{s_n \lambda_n}{n} \right|^k$$

$$= \sum_{n=1}^{m+1} \left| s_n \right|^k |\lambda_n|^k \frac{1}{n} = \sum_{n=1}^{m+1} \left| s_n \right|^k \frac{1}{n} |\lambda_n| |\lambda_n|^{k-1} = O(1),$$

as in the proof of $I_1$.

A weighted mean matrix, written $(\mathcal{N}, p_n)$, is a lower triangular matrix with entries $a_{mn} = p_0 / P_n$, where $\{p_n\}$ is a nonnegative sequence with $p_0 > 0$ and $P_n := \sum_{i=0}^{n} p_i \to \infty$, as $n \to \infty$.

**Corollary 4.1.** Let $\{p_n\}$ be a positive sequence satisfying

(i) $np_n \asymp O(P_n)$ and

(ii) $\Delta (P_n / p_n) = O(1)$.

and let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences satisfying conditions (vi), (vii), and (2.1). If $\{X_n\}$ is a quasi $f$-increasing sequence, where $\{f_n\} := \{n^\mu (\log n)^\nu\}$, $\mu \geq 0, 0 \leq \beta < 1$, and conditions (x) and (2.2) are satisfied, then the series $\sum_{n=1}^{\infty} (a_n P_n \lambda_n) / (np_n)$ is summable $|\mathcal{N}, p_n|_k$, $k \geq 1$.

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**References**


