A Refinement of Various Mean Inequalities*

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A new refinement of the classical arithmetic mean and geometric mean inequality is given. Moreover, a new interpretation of the classical mean is given and this refinement theorem is generalized.

Keywords: Arithmetic mean and geometric mean inequality; Harmonic mean; Refinement

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1 INTRODUCTION


Our main purpose of this paper is to give a new refinement of the classical arithmetic mean and geometric mean inequality (Theorem 2.1).

* The original concept of this research was inspired by the discussion held during the second author’s visit to the Faculty of Engineering of Yamagata University.
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Furthermore we give a new interpretation of the classical mean and generalize this refinement theorem (Theorem 3.2).

## 2 A REFINEMENT OF THE CLASSICAL MEAN INEQUALITY

Let \( \mathbb{R}_+ \) denote the set of all positive real numbers and \( \mathbb{R}_+^n \) its \( n \)-product. Recall the arithmetic mean, geometric mean, and harmonic mean;

\[
A_n(x_1, \ldots, x_n) \equiv \frac{x_1 + \cdots + x_n}{n}, \\
G_n(x_1, \ldots, x_n) \equiv (x_1 \cdots x_n)^{1/n}, \\
H_n(x_1, \ldots, x_n) \equiv \frac{1}{(1/n)(1/x_1 + \cdots + 1/x_n)},
\]

where \( n \in \mathbb{N} \) and \( (x_1, \ldots, x_n) \in \mathbb{R}_+^n \). The order relation among these means is well-known;

\[
H_n(x_1, \ldots, x_n) \leq G_n(x_1, \ldots, x_n) \leq A_n(x_1, \ldots, x_n), \tag{1}
\]

and the equality holds if and only if \( x_1 = x_2 = \cdots = x_n \) (see for instance [2]).

Given any \( x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n \) and \( k \) with \( 1 \leq k \leq n \) we first take the geometric means of any \( k \) terms and then consider the arithmetic mean of these \( nC_k \) numbers. So we obtain

\[
u(A, G, x; k) \equiv \frac{1}{nC_k} \sum_{1 \leq i_1 < \cdots < i_k \leq n} (x_{i_1} \cdots x_{i_k})^{1/k}, \tag{2}
\]

and by the similar procedure

\[
u(G, A, x; k) \equiv \left\{ \prod_{1 \leq i_1 < \cdots < i_k \leq n} \frac{x_{i_1} + \cdots + x_{i_k}}{k} \right\}^{1/nC_k}. \tag{3}
\]

By the definitions (2) and (3), we have

\[
u(A, G, x; 1) = u(G, A, x; n) = A_n(x_1, \ldots, x_n), \\
u(A, G, x; n) = u(G, A, x; 1) = G_n(x_1, \ldots, x_n),
\]

so \( \nu(A, G, x; 1) \geq u(A, G, x; n) \) and \( u(G, A, x; 1) \leq u(G, A, x; n) \). We will prove that \( \nu(A, G, x; k) \) and \( \nu(G, A, x; k) \) monotonously lie between \( A_n \) and \( G_n \).
Theorem 2.1 Fix \( n \in \mathbb{N} \) and \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). The refinement \( u(A, G, x; k) \) is nonincreasing and \( u(G, A, x; k) \) is nondecreasing with respect to \( k \) (\( 1 \leq k \leq n \)), that is,

\[
A_n = u(A, G, x; 1) \geq u(A, G, x; 2) \geq \cdots \geq u(A, G, x; n-1) \geq u(A, G, x; n) = G_n, 
\]

(4)

\[
G_n = u(G, A, x; 1) \leq u(G, A, x; 2) \leq \cdots \leq u(G, A, x; n-1) \leq u(G, A, x; n) = A_n. 
\]

(5)

In the above inequalities one equality occurs only if \( x_1 = x_2 = \cdots = x_n \).

Proof For any \( k \) with \( 2 \leq k \leq n \), by the inequality (1)

\[
\frac{1}{k} \sum_{1 \leq i_1 < \cdots < i_k \leq n} (x_{i_1} \cdots x_{i_k})^{1/k} \leq \frac{1}{k} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \left\{ (x_{i_1} \cdots x_{i_k})^{1/(k-1)} + (x_{i_1} x_{i_2} \cdots x_{i_k})^{1/(k-1)} + \cdots + (x_{i_1} \cdots x_{i_{k-1}})^{1/(k-1)} \right\} 
\]

which implies

\[
u(A, G, x; k) = \frac{1}{nC_k} \sum_{1 \leq i_1 < \cdots < i_k \leq n} (x_{i_1} \cdots x_{i_k})^{1/k} \leq \frac{1}{nC_k} \frac{n-(k-1)}{k} \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq n} (x_{i_1} \cdots x_{i_{k-1}})^{1/(k-1)} = \frac{1}{nC_{k-1}} \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq n} (x_{i_1} \cdots x_{i_{k-1}})^{1/(k-1)} = u(A, G, x; k-1).
\]

Hence \( u(A, G, x; k) \) is nonincreasing, and (5) is proved similarly.
Next we consider the equality case. If \( x_1 = x_2 = \cdots = x_n \) then \( G_n = A_n \), so all values \( u(A, G, x; k) \) and \( u(A, G, x; k) \) are equal. Suppose that there exists \( k \) satisfying \( u(A, G, x; k) = u(A, G, x; k - 1) \). Then for any \( i_1, \ldots, i_k \) with \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \)

\[
    x_{i_2} \cdots x_{i_k} = x_{i_1} \cdots x_{i_{k-1}},
\]

which implies \( x_{i_1} = x_{i_2} = \cdots = x_{i_k} \). Hence \( x_1 = x_2 = \cdots = x_n \).

Using the geometric mean and harmonic mean we obtain

\[
    u(G, H, x; k) = \left\{ \prod_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{(1/k)(1/x_{i_1} + \cdots + 1/x_{i_k})} \right\}^{1/n C_k}, \tag{6}
\]

\[
    u(H, G, x; k) = \left\{ \frac{1}{n C_k} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{(x_{i_1} \cdots x_{i_k})^{1/k}} \right\}^{-1}. \tag{7}
\]

As Theorem 2.1 we can prove that \( u(G, H, x; k) \) is nonincreasing and \( u(H, G, x; k) \) is nondecreasing.

### 3 A REFINEMENT OF A GENERALIZED MEAN

In order to generalize the previous inequalities we will regard the mean as the sequence of positive functions. Let \( f_k \) be a positive function on \( \mathbb{R}_+^k \) \((k = 1, 2, 3, \ldots)\). The sequence of functions \( F = \{f_k\} \) is called mean if the following conditions (M-1)–(M-5) hold:

(M-1) \( f_1(a) = a \ (\forall a > 0), \)

(M-2) for any \( k \in \mathbb{N} \)

\[
    f_k(x_1, \ldots, x_k) \leq f_k(y_1, \ldots, y_k) \quad \text{if} \quad 0 < x_i \leq y_i \ (i = 1, \ldots, k),
\]

(M-3) for any \( k \in \mathbb{N} \) and permutation \( \sigma \) of \( k \) elements

\[
    f_k(x_1, \ldots, x_k) = f_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}),
\]

(M-4) for any \( k, l \in \mathbb{N} \) and \((x_1, \ldots, x_k) \in \mathbb{R}_+^k \)

\[
    f_k(x_1, \ldots, x_k) = f_{kl}(x_1, \ldots, x_1, \overbrace{x_2, \ldots, x_2}^l, \ldots, \overbrace{x_k, \ldots, x_k}^l),
\]

with

\[
    f_{kl}(x_1, \ldots, x_l) = f_k(x_1, \ldots, x_l).
\]
The sequences generated by arithmetic, geometric, and harmonic means, \( \{A_n\} \), \( \{G_n\} \), and \( \{H_n\} \), satisfy the above conditions (M-1)-(M-5). So the above mean is a generalization of well-known three means.

We first remark that by the condition (M-4) with \( k = 1 \)

\[
f_k(a, \ldots, a) = f_1(a) = a \quad (\forall l \in \mathbb{N}, \forall a \in \mathbb{R}_+). \tag{8}
\]

Consider another condition (M-6);

(M-6) for any \( k, l \in \mathbb{N} \) and \( (x_{11}, \ldots, x_{1l}, \ldots, x_{kl}) \in \mathbb{R}_{+}^{kl} \)

\[
f_k(f_l(x_{11}, \ldots, x_{1l}), \ldots, f_l(x_{kl}, \ldots, x_{kl})) = f_k(x_{11}, \ldots, x_{1l}, \ldots, x_{kl}). \tag{9}
\]

We will show that (M-4) and (M-5) are equivalent to (M-6) under the condition (M-3) and (8) above.

**Proposition 3.1** Let \( F = \{ f_k \} \) be a sequence of positive functions. If \( F = \{ f_k \} \) is a mean then \( F \) satisfies (M-6). Conversely, if \( F = \{ f_k \} \) satisfies the conditions (8), (M-3), and (M-6) then (M-4) and (M-5) are valid.

**Proof** If \( F \) is a mean then

\[
f_{kl}(x_{11}, \ldots, x_{1l}, x_{21}, \ldots, x_{2l}, \ldots, x_{kl})
\]

\[
= f_k(f_l(x_{11}, \ldots, x_{1l}), \ldots, f_l(x_{kl}, \ldots, x_{kl}))
\]

by (M-5)

\[
= f_k(f_l(x_{11}, \ldots, x_{1l}), \ldots, f_l(x_{kl}, \ldots, x_{kl}))
\]

by (M-3), (M-5)

\[
= f_k(f_l(x_{11}, \ldots, x_{1l}), \ldots, f_l(x_{kl}, \ldots, x_{kl})), \quad \text{by (M-4)}
\]
so (M-6) holds. Conversely suppose that \( \mathcal{F} \) satisfies (8), (M-3), and (M-6). Put \( x_{ij} = x_i \) \((j = 1, \ldots, l)\) in (M-6) then (M-4) holds by (8). For any \( k \in \mathbb{N} \) and \( l \) with \( 1 \leq l \leq k \)

\[
\begin{align*}
&f_k \left( f_l(x_1, \ldots, x_l), \ldots, f_l(x_1, \ldots, x_l), x_{l+1}, \ldots, x_k \right) \\
&= f_k \left( f_l(x_1, \ldots, x_l), \ldots, f_l(x_1, \ldots, x_l), f_l(x_{l+1}, \ldots, x_{l+1}), \ldots, f_l(x_k, \ldots, x_k) \right) \quad \text{by (8)} \\
&= f_{kl}(x_1, \ldots, x_l, x_{l+1}, \ldots, x_k) \\
&\quad \ldots, x_{k}, \ldots, x_k) \quad \text{by (M-6)} \\
&= f_k(x_1, \ldots, x_k), \quad \text{by (M-3)} \\
&\quad \text{by (M-4)}
\end{align*}
\]

which implies (M-5).

The order relation of two means \( \mathcal{F} = \{f_k\} \) and \( \mathcal{G} = \{g_k\} \) is defined in each coordinate, that is, \( \mathcal{F} \leq \mathcal{G} \) if

\[
f_k(x_1, \ldots, x_k) \leq g_k(x_1, \ldots, x_k) \quad (\forall k \in \mathbb{N}, \forall (x_1, \ldots, x_k) \in \mathbb{R}^k_+).
\]

Consider two means \( \mathcal{F} = \{f_k\}, \mathcal{G} = \{g_k\} \) and fix \( n \in \mathbb{N} \). For any \( k \) with \( 1 \leq k \leq n \) and \( x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n \), as (2) and (3), we define

\[
u(\mathcal{F}, \mathcal{G}, x; k) \equiv f_{nk}(g_k(x_1, \ldots, x_k), \ldots, g_k(x_{n-k+1}, \ldots, x_n)),
\]

\[
u(\mathcal{G}, \mathcal{F}, x; k) \equiv g_{nk}(f_k(x_1, \ldots, x_k), \ldots, f_k(x_{n-k+1}, \ldots, x_n)). \quad (10)
\]

By the definition

\[
u(\mathcal{F}, \mathcal{G}, x; 1) = \nu(\mathcal{G}, \mathcal{F}, x; n) = f_n(x_1, \ldots, x_n),
\]

\[
u(\mathcal{F}, \mathcal{G}, x; n) = \nu(\mathcal{G}, \mathcal{F}, x; 1) = g_n(x_1, \ldots, x_n),
\]

so if \( \mathcal{F} \leq \mathcal{G} \) then

\[
u(\mathcal{G}, \mathcal{F}, x; 1) \geq \nu(\mathcal{G}, \mathcal{F}, x; n), \quad \nu(\mathcal{F}, \mathcal{G}, x; 1) \leq \nu(\mathcal{F}, \mathcal{G}, x; n).
\]

The following is a generalization of Theorem 2.1.
Theorem 3.2  Fix $n \in \mathbb{N}$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$. If $\mathcal{F} \leq \mathcal{G}$ then the refinement $u(\mathcal{G}, \mathcal{F}, x; k)$ is nonincreasing and $u(\mathcal{F}, \mathcal{G}, x; k)$ is nondecreasing with respect to $k$ ($1 \leq k \leq n$), that is,

$$u(\mathcal{G}, \mathcal{F}, x; 1) \geq u(\mathcal{G}, \mathcal{F}, x; 2) \geq \cdots \geq u(\mathcal{G}, \mathcal{F}, x; n - 1) \geq u(\mathcal{G}, \mathcal{F}, x; n),$$

(11)

$$u(\mathcal{F}, \mathcal{G}, x; 1) \leq u(\mathcal{F}, \mathcal{G}, x; 2) \leq \cdots \leq u(\mathcal{F}, \mathcal{G}, x; n - 1) \leq u(\mathcal{F}, \mathcal{G}, x; n).$$

(12)

Proof  Choose $k$ with $2 \leq k \leq n$. Since for any $(y_1, \ldots, y_k) \in \mathbb{R}^k_+$

$$f_k(f_{k-1}(y_1, \ldots, y_{k-1}), f_{k-1}(y_1, \ldots, y_{k-2}, y_k), \ldots, f_{k-1}(y_2, \ldots, y_k))$$

$$= f_{k(k-1)}(y_1, y_1, y_1, \ldots, y_k, \ldots, y_k, y_k) \quad \text{by (9)}$$

$$= f_{k(k-1)}(y_1, \ldots, y_k) \quad \text{by (M-3)}$$

$$= f_k(y_1, \ldots, y_k) \quad \text{by (M-4)},$$

we can deduce that

$$u(\mathcal{G}, \mathcal{F}, x; k)$$

$$= g_{\mathcal{C}_k} \left( f_k(x_1, \ldots, x_k), \ldots, f_k(x_{n-k+1}, \ldots, x_n) \right)$$

$$= g_{\mathcal{C}_k} \left( f_k \left( f_{k-1}(x_1, \ldots, x_{k-1}), f_{k-1}(x_1, \ldots, x_{k-2}, x_k), \ldots, \right. \right.$$

$$f_{k-1}(x_2, \ldots, x_k)) \right) \quad \text{by (9)}$$

According to the inequality $\mathcal{F} \leq \mathcal{G}$ and (M-2)

$$u(\mathcal{G}, \mathcal{F}, x; k)$$

$$\leq g_{\mathcal{C}_k} \left( g_k \left( f_{k-1}(x_1, \ldots, x_{k-1}), f_{k-1}(x_1, \ldots, x_{k-2}, x_k), \ldots, \right. \right.$$

$$\left. \left. f_{k-1}(x_2, \ldots, x_k) \right) \right) \quad \text{by (9)}$$
\[
\begin{align*}
&\frac{\sum_{i=1}^{k} f_{k-1}(x_1, \ldots, x_{k-1})}{\sum_{i=k+1}^{n} f_{k-1}(x_{n-k+1}, \ldots, x_{n})} \\
&= \sum_{i=k+1}^{n} \frac{f_{k-1}(x_{n-k+2}, \ldots, x_{n})}{f_{k-1}(x_{n-k+1}, \ldots, x_{n})}, \quad \text{by (M-3)}
\end{align*}
\]

\[
\begin{align*}
&\frac{\sum_{i=1}^{k} f_{k-1}(x_1, \ldots, x_{k-1})}{\sum_{i=k+1}^{n} f_{k-1}(x_{n-k+1}, \ldots, x_{n})} \\
&= \sum_{i=k+1}^{n} \frac{f_{k-1}(x_{n-k+2}, \ldots, x_{n})}{f_{k-1}(x_{n-k+1}, \ldots, x_{n})} \\
&= u(\mathcal{G}, \mathcal{F}, x; k-1).
\end{align*}
\]

Hence \( u(\mathcal{G}, \mathcal{F}, x; k) \) is nonincreasing and (12) is proved similarly.

**Remark** For any \( n \in \mathbb{N} \) and \( t \neq 0 \) consider the function \( M_n^t \) defined by

\[
M_n^t(x_1, \ldots, x_n) \equiv \left( \frac{x_1^t + \cdots + x_n^t}{n} \right)^{1/t} \quad (\forall (x_1, \ldots, x_n) \in \mathbb{R}_+^n).
\]

Because \( \lim_{t \to 0} M_n^t(x_1, \ldots, x_n) = (x_1 \cdots x_n)^{1/n} \), we define

\[
M_n^0(x_1, \ldots, x_n) \equiv (x_1 \cdots x_n)^{1/n} \quad (\forall (x_1, \ldots, x_n) \in \mathbb{R}_+^n).
\]

For a given \( n \) and \( (x_1, \ldots, x_n) \), \( M_n^t(x_1, \ldots, x_n) \) is nondecreasing with respect to \( t \). In particular, \( M_n^{-1} \), \( M_n^0 \), and \( M_n^1 \) are the harmonic, geometric, and arithmetic mean, respectively. So the functions \( M_n^t \) are interpolated in the harmonic, geometric, and arithmetic mean (see [2] for a detail of the function \( M_n^t \)). For a fixed \( t \), the sequence \( \{M_n^t\}_n \) satisfies the conditions (M-1)–(M-5). So \( \mathcal{M} = \{M_n^t\} \) is also a mean in our sense.

Fix \( n \in \mathbb{N}, x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n \) and choose \( k \) with \( 1 \leq k \leq n \). For any \( s, t \in \mathbb{R} \) let us consider

\[
\begin{align*}
&u(s, t, x; k) = u(\mathcal{M}^s, \mathcal{M}^t, x; k) \\
&= M_{s,\mathcal{C}_k}^s(M_k^t(x_1, \ldots, x_k), \ldots, M_k^t(x_{n-k+1}, \ldots, x_n))
\end{align*}
\]
If $s \leq t$ then $\mathcal{M}^s \leq \mathcal{M}^t$, so by Theorem 3.2 we can conclude that

\[
\begin{align*}
    u(t, s, x; 1) &\geq u(t, s, x; 2) \geq \cdots \geq u(t, s, x; n-1) \geq u(t, s, x; n), \\
    u(s, t, x; 1) &\leq u(s, t, x; 2) \leq \cdots \leq u(s, t, x; n-1) \leq u(s, t, x; n).
\end{align*}
\]

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