Explicit Exponential Decay Bounds in Quasilinear Parabolic Problems

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This paper deals with classical solutions u(x, t) of some initial boundary value problems involving the quasilinear parabolic equation

\[ g(k(t)\|\nabla u\|^2)\Delta u + f(u) = u, \quad x \in \Omega, \quad t > 0, \]

where \( f, g, k \) are given functions. In the case of one space variable, i.e. when \( \Omega := (-L, L) \), we establish a maximum principle for the auxiliary function

\[ \Phi(x, t) := e^{2\alpha t} \left\{ \frac{1}{k(t)} \int_0^{k(t)u^2} g(s) \, ds + \alpha u^2 + 2 \int_0^t f(s) \, ds \right\}, \]

where \( \alpha \) is an arbitrary nonnegative parameter. In some cases this maximum principle may be used to derive explicit exponential decay bounds for \( |u| \) and \( |u_x| \). Some extensions in \( N \) space dimensions are indicated. This work may be considered as a continuation of previous works by Payne and Philippin (Mathematical Models and Methods in Applied Sciences, 5 (1995), 95–110; Decay bounds in quasilinear parabolic problems, In: Nonlinear Problems in Applied Mathematics, Ed. by T.S. Angell, L. Pamela, Cook, R.E., SIAM, 1997).

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1 INTRODUCTION

Using a maximum principle approach Payne and Philippin [8] derived pointwise decay bounds for solutions of some initial boundary value problems involving the parabolic differential equation $\Delta u + f(u) = u_t$, $x \in \Omega$, $t > 0$, where $\Omega$ is a bounded convex domain in $\mathbb{R}^N$. This paper deals with classical solutions $u(x, t)$ of some initial boundary value problems involving the quasilinear parabolic equation

$$g(k(t)|\nabla u|^2)\Delta u + f(u) = u_t, \quad x \in \Omega, \quad t > 0,$$

where $f, g, k$ are given functions. In Section 2 we consider the case of one space variable $x \in (-L, L)$. Under certain hypotheses we establish a maximum principle for the auxiliary function

$$\Phi(x, t) := e^{2\alpha t} \left\{ \frac{1}{k(t)} \int_0^{k(t)u_x^2} g(\xi) \, d\xi + \alpha u^2 + 2 \int_0^u f(s) \, ds \right\},$$

where $\alpha$ is an arbitrary nonnegative parameter.

When $f$ is zero and $k$ is an exponential function we compute in Section 3.1 a critical value $c_0$ that depends on the boundary conditions and on $g$ in such a way that for $0 \leq \alpha < c_0$, $\Phi$ takes its maximum value initially. This fact leads to explicit exponential decay bounds for $|u|$ and $|u_x|$.

When $f$ is not zero and $k = 1$, we show under certain assumptions that if the initial data $u(x, 0)$ is nonnegative and small enough in some sense that will be made precise in Section 3.2, the solution $u(x, t)$ cannot blow up in finite time. Depending on $f$ we then determine $\alpha_1 < c_0$ such that for $0 \leq \alpha < \alpha_1$, $\Phi$ takes its maximum value initially. This leads again to explicit exponential decay bounds for $u(\geq 0)$ and $|u_x|$.

In Section 4, the results of Sections 2 and 3.1 are extended in $\mathbb{R}^N$ in the case of the parabolic equation $g(k(t)|\nabla u|^2)\Delta u = u_t$, $x \in \Omega$, $t > 0$. We refer to [9] for a similar investigation involving the parabolic equation $\nabla(g(|\nabla u|^2)) = u_t$, $x \in \Omega$, $t > 0$.

For maximum principle results related to parabolic partial differential equations we refer to [10–14].
2 MAXIMUM PRINCIPLE FOR $\Phi(x, t)$

In this section we establish the following result.

**Theorem 1** Let $u(x, t)$ be the solution of the initial boundary value problem

$$g(ku_x^2)u_{xx} + f(u) = u_t, \quad x \in (-L, L), \ t \in (0, T),$$

$$u(\pm L, t) = 0, \ t \in [0, T],$$

$$u(x, 0) = h(x), \ x \in [-L, L],$$

with $h(\pm L) = 0$, $h \neq 0$. Let $\Phi(x, t)$ be defined on $u(x, t)$ by

$$\Phi(x, t) := e^{2\alpha t} \left\{ \frac{1}{k(t)} G(ku_x^2) + \alpha u^2 + 2F(u) \right\},$$

where $\alpha$ is an arbitrary nonnegative parameter and with

$$F(s) := \int_0^s f(\xi) \, d\xi,$$

$$G(\sigma) := \int_0^\sigma g(\xi) \, d\xi.$$

Assume that the given functions $g \in C^2$, $k \in C^1$ are strictly positive and satisfy the inequality

$$(2\alpha k - k') [\sigma g(\sigma) - G(\sigma)] \geq 0, \quad t \in (0, T), \ \sigma \geq 0,$$

and that the given function $f \in C^1$ satisfies the inequality

$$sf(s) - 2F(s) \geq 0, \quad s \in \mathbb{R}.$$

We then conclude that $\Phi$ takes its maximum value either at an interior critical point $(\bar{x}, \bar{t})$ of $u$, or initially. In other words we have

$$\Phi(x, t) \leq \max \left\{ \Phi(\bar{x}, \bar{t}), \ \max_{[-L, L]} \Phi(x, 0) \right\}$$

with $u_x(\bar{x}, \bar{t}) = 0$ (i), $\max_{[-L, L]} \Phi(x, 0) = 0$ (ii).
For the proof of Theorem 1 we compute

\[ \Phi_x = 2e^{2\alpha t}\{u_xu_{xxx}g(ku_x^2) + \alpha uu_x + f(u)u_x\} \]
\[ = 2e^{2\alpha t}u_x(u_t + \alpha u), \quad (12) \]

\[ \Phi_{xx} = 2e^{2\alpha t}\{2k u_x^2u_{xx}^2 + g u_{xx}^2 + g u_x u_{xxx} + \alpha u_x^2 + \alpha uu_{xx} + f'u_x^2 + fu_{xx}\} \]
\[ = 2e^{2\alpha t}\{g u_{xx}^2 + u_x u_{xt} + \alpha u_x^2 + \alpha uu_{xx} + fu_{xx}\}, \quad (13) \]

\[ \Phi_t = e^{2\alpha t}\left\{ \frac{k'}{k^2}\left[gk u_x^2 - G(ku_x^2)\right] + 2g u_x u_{xt} + 2\alpha uu_t + 2fu_t \right. \]
\[ + \left. 2\alpha \left[\frac{1}{k}G(ku_x^2) + \alpha u^2 + 2F(u)\right] \right\}. \quad (14) \]

Combining (13) and (14) we obtain after some reduction

\[ g\Phi_{xx} - \Phi_t = e^{2\alpha t}\left\{ \frac{1}{k^2}(2\alpha k - k')\left[gk u_x^2 - G(ku_x^2)\right] + 2g^2u_{xx}^2 \right. \]
\[ - 2f^2 - 2\alpha uf - 2\alpha^2u^2 - 4\alpha F(u) \right\}. \quad (15) \]

From (12) we compute

\[ gu_{xx} = \frac{1}{2}u_x^{-1}\Phi_x e^{-2\alpha t} - (f + \alpha u). \quad (16) \]

Inserting (16) into (15) we obtain the parabolic differential inequality

\[ g\Phi_{xx} + u_x^{-2}c(x, t)\Phi_x - \Phi_t \]
\[ = e^{2\alpha t}\left\{ \frac{1}{k^2}(2\alpha k - k')\left[gk u_x^2 - G(ku_x^2)\right] + 2\alpha [uf - 2F(u)] \right\} \geq 0, \quad (17) \]

where the last inequality in (17) follows from (9) and (10). In (17), c(x, t) is regular throughout \((-L, L) \times (0, T)\). It then follows from Nirenberg’s maximum principle [6,10] that \( \Phi \) takes its maximum value (i) at a critical
point \((\bar{x}, \bar{t})\) of \(u\), or (ii) initially, or (iii) at a boundary point \((\hat{x}, \hat{t})\) with \(\hat{x} = \pm L, \hat{t} \in (0, T]\). The conclusion of Theorem 1 will follow if we can show that (iii) implies (i) or (ii). If fact (12) and the boundary conditions (4), imply \(\Phi_x(\pm L, t) = 0\). It then follows from Friedman’s maximum principle [3,10] that \(\Phi\) can take its maximum value at a boundary point \((\hat{x}, \hat{t})\) with \(\hat{x} = \pm L\) only if \(\Phi\) is identically constant in \((-L, L) \times (0, \hat{t})\), in which case the two possibilities (i) and (ii) hold in (11). This achieves the proof of Theorem 1.

It is worthy to note that \(\Phi\) can be constant only for some particular choice of the data in problem (3), (4), (5). In fact \(\Phi = \text{const.}\) implies equality in (17), i.e. also in (9) and in (10). This implies then that

\[
f(u) = \lambda u, \quad \lambda = \text{const.},
\]

and that

\[
either k(t) = e^{2\alpha t} \quad \text{or} \quad g = \text{const.}
\]

Moreover \(\Phi = \text{const.}\) implies \(\Phi_x = u_x(u_t + \alpha u) \equiv 0\), from which we conclude

\[
either u_x \equiv 0 \quad \text{or} \quad u_t + \alpha u \equiv 0.
\]

The first equation in (20), together with (3) and (18), leads to

\[
u(x, t) = e^{\lambda t},
\]

which is impossible in view of (4). The second equation in (20) leads to

\[
u(x, t) = h(x)e^{-\alpha t},
\]

which solves (3) only if we have

\[
g(h^2)h'' + (\lambda + \alpha)h = 0, \quad x \in (-L, L).
\]

To conclude this section we note that Theorem 1 remains true even if we replace (4) by any one of the following pairs of boundary conditions:

\[
u(-L, t) = u_{x}(L, t) = 0, \quad (4)_2
\]

\[
u_{x}(-L, t) = u(L, t) = 0, \quad (4)_3
\]

\[
u_{x}(-L, t) = u_{x}(L, t) = 0. \quad (4)_4
\]
Moreover if both inequalities (9) and (10) in Theorem 1 are reversed we then conclude that $\Phi$ takes its minimum value either at a critical point of $u$, or initially.

3 ELIMINATION OF THE FIRST POSSIBILITY (i) IN (11)

We note that the realization of (ii) in (11) leads to lower exponential decay bounds for both $|u|$ and $|u_x|$. In this section we impose restrictions on the parameter $\alpha \geq 0$ so that the first possibility (i) in (11) cannot occur. We shall investigate two particular cases.

3.1 First Case: $f(u) = 0$, $k(t) = e^{2\mu t}$, $\mu \leq \alpha$

We consider the parabolic problem (3), (4)$_k$, $k = 1, 2, 3$ or 4, and (5) with the particular choices $f(u) = 0$, $k(t) = e^{2\mu t}$, $\mu = \text{const.} \leq \alpha$. We assume (9) so that the conclusion (11) of Theorem 1 holds. Note that if $\mu = \alpha$, assumption (9) is satisfied for any arbitrary function $g > 0$. If $\mu < \alpha$, (9) is satisfied if and only if $g' \geq 0$.

Suppose that we have the first possibility (i) in (11), i.e.

$$
\Phi(x, t) \leq \alpha u^2(x, \bar{t}) e^{2\alpha \bar{t}} = \alpha u_0^2 e^{2\alpha \bar{t}},
$$

(27)

with

$$
u_0^2 := \max_{(-L,L)} u^2(x, \bar{t}).
$$

(28)

We can rewrite (27) evaluated at $t = \bar{t}$ as

$$
e^{-2\mu \bar{t}} G(e^{2\mu \bar{t}} u_0^2) \leq \alpha (u_0^2 - u^2(x, \bar{t})).
$$

(29)

Making use of the mean value theorem we may bound the left hand side of (29) as follows:

$$
u_0^2(x, \bar{t}) g_{\min} \leq e^{-2\mu \bar{t}} G(e^{2\mu \bar{t}} u_0^2),
$$

(30)
where $g_{\text{min}} > 0$ is the minimum value of $g$. From (29) and (30) we obtain the inequality

$$u^2(x, \bar{t}) g_{\text{min}} \leq \alpha (u^2_\text{M} - u^2(x, \bar{t})),$$

that may be rewritten as

$$\frac{|u_x(x, \bar{t})|}{\sqrt{u^2_\text{M} - u^2(x, \bar{t})}} \leq \sqrt{\frac{\alpha}{g_{\text{min}}}}. \quad (32)$$

Integrating (32) from the critical point $\bar{x}$ to the nearest zero $\bar{x} \in [-L, L]$ of $u(x, \bar{t})$, we obtain

$$\alpha \geq \alpha_0 := \frac{\pi^2 g_{\text{min}}}{4|\bar{x} - \bar{x}|^2}. \quad (33)$$

Since $|\bar{x} - \bar{x}|$ is unknown, we need an upper bound for this quantity in (33). Obviously we may use

$$|\bar{x} - \bar{x}| \leq \Lambda := \begin{cases} L, & \text{if we have (4)1}, \\ 2L, & \text{if we have (4)k, } k = 2, 3 \text{ or } 4, \end{cases} \quad (34)$$

if we assume the existence of $\bar{x} \in [-L, L]$ when we have the boundary conditions (4)4. This will be the case e.g. if the initial data $h(x)$ have zero mean value, i.e. if we have

$$\int_{-L}^{L} h(x) \, dx = 0. \quad (35)$$

In fact with the auxiliary function $\rho(\sigma)$ defined by

$$\rho(\sigma) := \frac{1}{2} \sigma^{-1/2} \int_{0}^{\sigma} g(\xi)\xi^{-1/2} \, d\xi, \quad (36)$$

we have

$$\frac{d}{dt} \int_{-L}^{L} u(x, t) \, dx = \int_{-L}^{L} u_t(x, t) \, dx = \int_{-L}^{L} g(e^{2\mu}u_x^2)u_{xx} \, dx
\begin{align*}
\quad & = \int_{-L}^{L} (\rho(e^{2\mu}u_x^2)u_x)_x \, dx \quad (\rho(e^{2\mu}u_x^2)u_x|_{-L}^{L} = 0. \quad (37)
\end{align*}$$
It then follows from (35) and (37) that
\[
\int_{-L}^{L} u(x, t) \, dx = \int_{-L}^{L} h(x) \, dx = 0, \quad \forall t \in [0, T], \tag{38}
\]
i.e. the zero mean value property of \(u(x, t)\) is inherited from the zero mean value property of the initial data if we have the boundary conditions (4). But this implies the existence of \(\bar{x} \in [-L, L]\) such that \(u(\bar{x}, t) = 0\).

We conclude from the above investigation that, for \(0 \leq \alpha \leq \alpha_0\), the first possibility (i) in (11) cannot hold, so that \(|u|\) and \(|u_x|\) must decay exponentially. This shows in fact that \(u(x, t)\) cannot blow up and will exist for all \(t > 0\). These results are summarized next.

**Theorem 2**  Let \(u(x, t)\) be the solution of the parabolic problem (3), (4), \(k = 1, 2, 3, \) or 4, and (5). If we have (4) and (5), we require that the initial data \(h(x)\) satisfy the further condition (35). Assume either

\[
\mu = \alpha < \alpha_0, \tag{39}
\]
or
\[
\mu < \alpha < \alpha_0 \quad \text{and} \quad g' \geq 0, \tag{40}
\]

with
\[
\alpha_0 := \frac{\pi^2 g_{\text{min}}}{4\Lambda^2}, \tag{41}
\]

where \(\Lambda\) is defined in (34). Then we may take \(T = \infty\) in (3) and (4). Moreover the function \(\Phi\) defined as
\[
\Phi(x, t) := e^{2\alpha t} \{ e^{-2\mu t} G(e^{2\mu t} u_x^2) + \alpha u^2 \}, \tag{42}
\]
takes its maximum value initially. The resulting inequality (11) with \(\alpha \to \alpha_0\) takes the form
\[
e^{-2\mu t} G(e^{2\mu t} u_x^2) + \alpha_0 u^2 \leq H^2 e^{-2\alpha_0 t}, \quad \forall \mu \leq \alpha_0, \tag{43}
\]
with

\[ H^2 := \max_{[-L,L]} \{ G(h') + \alpha_0 h^2 \} \quad (44) \]

We note that the quantities \( \alpha_0 \) and \( H^2 \) are explicitly computable in terms of the initial and boundary data.

A weaker but more practical version of (43) is

\[ g_{\min} u_x^2(x, t) + \alpha_0 u^2 \leq H^2 e^{-2\alpha_0 t}. \quad (45) \]

Integrating (45) over \((-L,L)\) we obtain

\[ g_{\min} \int_{-L}^{L} u_x^2(x, t) \, dx + \alpha_0 \int_{-L}^{L} u^2 \, dx \leq 2L H^2 e^{-2\alpha_0 t}. \quad (46) \]

Moreover depending on the boundary conditions (4), \( u(x,t) \) is admissible for the variational characterization of the first or second eigenvalue of a vibrating string of length 2L with fixed or free ends. We have actually

\[ g_{\min} \int_{-L}^{L} u_x^2(x, t) \, dx \geq \alpha_0 \int_{-L}^{L} u^2 \, dx, \quad (47) \]

valid in all cases considered in Theorem 2. From (46) and (47) we obtain the following decay bound for \( \int_{-L}^{L} u^2 \, dx \):

\[ \int_{-L}^{L} u^2(x, t) \, dx \leq L \alpha_0^{-1} H^2 e^{-2\alpha_0 t}. \quad (48) \]

We shall now derive a pointwise lower bound for \( |u(x,t)| \) that is proportional to the distance \( |x - \bar{x}| \) from \( x \) to the nearest zero \( \bar{x} \) of \( u(x,t) \). To this end we rewrite (45) as

\[ \frac{|u_x|}{\sqrt{(H^2/\alpha_0) e^{-2\alpha_0 t} - u^2}} \leq \frac{\alpha_0}{g_{\min}}. \quad (49) \]

Integrating (49) for fixed \( t \) from \( x \) to \( \bar{x} \) we obtain

\[ \arcsin \left( \frac{\sqrt{\alpha_0 |u|}}{He^{-\alpha_0 t}} \right) = \int_0^{|u|} \frac{d\xi}{\sqrt{(H^2/\alpha_0) e^{-2\alpha_0 t} - \xi^2}} \leq \sqrt{\frac{\alpha_0}{g_{\min}}} |x - \bar{x}|, \quad (50) \]
or

\[ |u(x, t)| \leq \frac{H}{\sqrt{g_{\text{min}}}} |x - \bar{x}| e^{-\alpha_0 t}, \quad x \in (-L, L), \ t > 0. \quad (51) \]

Of course we can substitute \( \bar{x} \) by \( +L \) or \( -L \) if we have the boundary conditions \((4)_k, k = 1, 2, 3.\)

### 3.2 Second Case: \( f(u) \neq 0, k(t) = 1 \)

In this section we consider the parabolic problem \((3), (4)_k, k = 1, 2, 3 \) or \( 4, \) and \((5)\) with the particular choice \( k(t) = 1, f(u) \neq 0. \) It is well known that the solution \( u(x, t) \) of this problem may not exist for all time. In fact \( u(x, t) \) may blow up at some time \( t^* \) which may be finite or infinite \([1,3].\)

However if blow-up does occur at \( t = t^* \), then \( u(x, t) \) will exist on the time interval \((0, t^*).\)

We want to establish conditions involving the data sufficient to prevent blow-up of \( u(x, t) \) and even sufficient to guarantee its exponential decay. To this end we first establish the following comparison result.

**Lemma 1** Let \( u(x, t) \) be the solution of the parabolic problem \((3), (4)_k, k = 1, 2, 3, \) or \( 4, \) and \((5)\) with \( h(x) \geq 0 \) and \( k(t) = 1. \) Assume moreover the following conditions on \( f \) and \( g: \)

\[ sf'(s) \geq f(s) > 0, \quad s > 0, \ f(0) = 0, \quad (52) \]

\[ \mu := \frac{f(u_M)}{u_M} \leq \alpha_0 := \frac{\pi^2 g_{\text{min}}}{4 \Lambda^2}, \quad (53) \]

\[ g'(\sigma) \geq 0, \quad \sigma \geq 0, \quad (54) \]

where \( u_M^2 \) has been defined in \((28). \) We then have the following bounds for \( u(x, t) :\)

\[ 0 \leq u(x, t) \leq U \exp \left\{ \left( \frac{f(u_M)}{u_M} - \alpha_0 \right) t \right\}, \quad t \in (0, T), \quad (55) \]
with

$$U := \max_{(-L,L)} \sqrt{h^2 + \frac{1}{\alpha_0} G(h'^2)}. \quad (56)$$

We note that condition (52) implies condition (10) and the fact that the ratio $f(s)/s$ is a nondecreasing function of $s$.

The lower bound in (55) follows from Nirenberg's and Friedman's maximum principles [3,6,10]. To establish the upper bound in (55) we introduce an auxiliary function $v(x, t)$ defined as

$$u(x, t) = v(x, t)e^{\mu t}, \quad (57)$$

with $\mu := f(u_M)/u_M$. Inserting (57) into (3) we obtain

$$e^{\mu t} [g(e^{2\mu t} v_x^2)v_{xx} - v_t] = u \left( \mu - \frac{f(u)}{u} \right) \geq 0, \quad (58)$$

where the above inequality results from the definition of $\mu$ together with the monotonicity of $f(s)/s$. The auxiliary function $v(x, t)$ then satisfies

$$2g(e^{2\mu t} v_x^2)v_{xx} - v_t \geq 0, \quad x \in (-L, L), \ t \in (0, T), \quad (59)$$

$$v(x, 0) = h(x), \quad x \in (-L, L). \quad (60)$$

Moreover $v(x, t)$ satisfies the same boundary conditions (4) as $u(x, t)$. Let $w(x, t)$ satisfy

$$g(e^{2\mu t} w_x^2)w_{xx} - w_t = 0, \quad x \in (-L, L), \ t \in (0, T), \quad (61)$$

$$w(x, 0) = h(x), \quad x \in (-L, L), \quad (62)$$

with the same boundary conditions as $u$ and $v$. From (59) and (61) we have

$$g(e^{2\mu t} v_x^2)v_{xx} - g(e^{2\mu t} w_x^2)w_{xx} - (v - w)_t \geq 0. \quad (63)$$
Using the mean value theorem we may rewrite the first two terms in (63) as follows:

\begin{align*}
  g(e^{2\mu t}v_x^2)_{xx} - g(e^{2\mu t}w_x^2)w_{xx} \\
  = g(e^{2\mu t}v_x^2)(v - w)_{xx} + w_{xx}[g(e^{2\mu t}v_x^2) - g(e^{2\mu t}w_x^2)] \\
  = g(e^{2\mu t}v_x^2)(v - w)_{xx} + w_{xx}g'(\xi)e^{2\mu t}(v - w)(v + w),
\end{align*}

for some intermediate value \( \xi \). We conclude from (63) and (64) that the function \( \omega := v - w \) satisfies a parabolic inequality of the following form:

\begin{align*}
  g(e^{2\mu t}v_x^2)\omega_{xx} + C(x, t)\omega_x - \omega_t \geq 0, \quad x \in (-L, L), \ t \in (0, T),
\end{align*}

where \( C(x, t) \) is regular throughout \((-L, L) \times (0, T)\). Since \( \omega(x, 0) = 0 \) and since \( \omega \) satisfies zero Dirichlet or Neumann boundary conditions, it follows that

\begin{align*}
  \omega := v - w \leq 0, \quad x \in (-L, L), \ t \in (0, T).
\end{align*}

From (57) and (66) we obtain

\begin{align*}
  0 \leq u(x, t) \leq e^{\mu t}w(x, t).
\end{align*}

Finally since we assume (53) and (54) we may use (43) to bound \( w(x, t) \). Dropping the first term in (43) we obtain

\begin{align*}
  \alpha_0 w^2 \leq H^2 e^{-2\alpha_0 t},
\end{align*}

where \( H^2 \) is defined in (44). The desired inequality (55) follows now from (67) and (68).

Lemma 1 is the main tool in the derivation of the following result.

**Theorem 3** Let \( u(x, t) \) be the solution of problem (3), (4), \( k = 1, 2, 3 \) or \( 4 \), and (5) with \( h(x) \geq 0 \), and \( k(t) = 1 \). Assume (52)–(54). Moreover assume that the data in problem (3)–(5) are small enough in the following sense:

\begin{align*}
  \frac{f(U)}{U} < \alpha_0 := \frac{\pi^2 g_{\text{min}}}{4\Lambda^2},
\end{align*}

where \( \Lambda \) is as in (44).
where $U$ is defined in (56). Then $u(x, t)$ exists for all time $t > 0$ (i.e. we may take $T = \infty$ in problem (3)–(5)). Moreover we have

$$\max_{(-L,L)} \frac{f(u(x, t))}{u(x, t)} < \alpha_0, \quad \forall t > 0. \quad (70)$$

For the proof of Theorem 3 we assume that (70) is not valid and show that this invalidity is self-contradictory. From the definition of $U$ we have

$$U \geq \max_{(-L,L)} h(x). \quad (71)$$

Since $f(s)/s$ is nondecreasing, (71) and (69) imply

$$\frac{f(h)}{h} \leq \frac{f(U)}{U} < \alpha_0. \quad (72)$$

If (70) is violated, there exists in view of (72) a first time $t = \tau$ for which we have

$$\max_{(-L,L)} \frac{f(u)}{u} = \alpha_0. \quad (73)$$

With $f(u_M)/u_M \leq \max_{(-L,L)} (f(u)/u)$, we obtain

$$\frac{f(u_M)}{u_M} \leq \alpha_0. \quad (74)$$

From (55) and (74) we obtain

$$u(x, t) \leq U, \quad x \in (-L, L), \quad 0 \leq t \leq \tau, \quad (75)$$

and we conclude that

$$\max_{(-L,L)} \frac{f(u(x, \tau))}{u(x, \tau)} \leq \frac{f(U)}{U} < \alpha_0, \quad (76)$$

so that (70) cannot actually be violated in a finite time $\tau$. This establishes (70) with $\tau = \infty$. 

We are now prepared to establish the following result:

**Theorem 4** Let \( u(x, t) \) be the solution of the parabolic problem (3), (4), (5) with \( k(x) \geq 0, k(t) = 1 \). Assume (52)–(54). Moreover assume that the data in problem (3)–(5) are small enough in the sense that there exists a constant \( \alpha_1 > 0 \) such that the inequality

\[
\frac{f(U)}{U} < \alpha_0 - \alpha_1
\]

is satisfied, where \( U \) is defined in (56). Then we conclude that the first possibility (i) in (11) cannot hold for \( \alpha, 0 \leq \alpha \leq \alpha_1 \). We are then led to the following decay bound for \( u^2 \) and \( u_x^2 \):

\[
G(u_x^2) + \alpha_1 u^2 + 2F(u) \leq \mathcal{H}^2 e^{-2\alpha_1 t}, \quad x \in (-L, L), \quad t > 0
\]

(valid for all time \( t > 0 \)) with

\[
\mathcal{H}^2 := \max_{(-L, L)} \{ G(h^2) + \alpha_1 h^2 + 2F(h) \}.
\]

Before proving Theorem 4 we show that the realization of (i) in (11) with \( \alpha := \alpha_1 \) implies the inequality

\[
\frac{f(u_M)}{u_M} \geq \alpha_0 - \alpha_1.
\]

In fact the realization of (i) in (11) with \( \alpha := \alpha_1 \) implies the inequality

\[
\{ G(u_x^2(x, t)) + \alpha_1 u^2 + 2F(u) \} e^{2\alpha_1 t} \leq [\alpha_1 u_M^2 + 2F(u_M)] e^{2\alpha_1 t},
\]

where \( u_M^2 \) is defined in (28). Evaluated at \( t = \bar{t} \), we obtain

\[
G(u_x^2(x, \bar{t})) \leq \alpha_1 [u_M^2 - u^2(x, \bar{t})] + 2[F(u_M) - F(u(x, \bar{t}))].
\]

Using the generalized mean value theorem and the monotonicity of \( f(s)/s \) we may rewrite the last term in (82) as follows:

\[
F(u_M) - F(u(x, \bar{t})) = \frac{F(u_M) - F(u(x, \bar{t}))}{u_M^2 - u^2(x, \bar{t})} [u_M^2 - u^2(x, \bar{t})]
\]

\[
= \frac{f(\xi)}{2\xi} [u_M^2 - u^2(x, \bar{t})] \leq \frac{1}{2} \frac{f(u_M)}{u_M} [u_M^2 - u^2(x, \bar{t})],
\]

(83)
where \( \xi \) is some intermediate value of \( u \). Moreover the left hand side of (82) may be bounded as follows:

\[
g_{\min} u_{\xi}^2(x, \bar{t}) \leq G(u_{\xi}^2(x, \bar{t})), \tag{84}
\]

with \( g_{\min} = g(0) \). From (82)–(84) we obtain the inequality

\[
g_{\min} u_{\xi}^2(x, \bar{t}) \leq \left( \alpha_1 + \frac{f(u_M)}{u_M} \right) [u_M^2 - u^2(x, \bar{t})], \tag{85}
\]

that may be rewritten as

\[
\frac{|u_{\xi}(x, \bar{t})|}{\sqrt{u_M^2 - u^2(x, \bar{t})}} \leq \sqrt{g_{\min}^{-1} \left( \alpha_1 + \frac{f(u_M)}{u_M} \right)}. \tag{86}
\]

Integrating (86) from the critical point \( \bar{x} \) to the nearest end \( \bar{x} = \pm L \) of the interval \((-L, L)\) with \( u(\bar{x}, \bar{t}) = 0 \), we obtain (80).

For the proof of Theorem 4 we note that the assumption (77) implies (69), so that conclusion (70) of Theorem 3 holds. In particular we have

\[
\frac{f(u_M)}{u_M} \leq \alpha_0, \quad \forall t > 0, \tag{87}
\]

and (55) leads to

\[
u_M \leq U, \tag{88}
\]

from which we obtain using the monotonicity of \( f(s)/s \) and assumption (77)

\[
\frac{f(u_M)}{u_M} \leq \frac{f(U)}{U} < \alpha_0 - \alpha_1, \tag{89}
\]

in contradiction to (80), so that (i) in (11) cannot hold. The inequality (78) follows now from (ii) in (11). This achieves the proof of Theorem 4.
As an example, let \( u(x, t) \) be the solution of the following parabolic differential equation:

\[
  u_{xx} \sqrt{1 + u_x^2} + u^{1+\epsilon} = u_t, \quad x \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right), \quad t > 0,
\]

with the boundary conditions

\[
  u\left( -\frac{\pi}{2}, t \right) = u\left( \frac{\pi}{2}, t \right) = 0,
\]

and with the initial condition

\[
  u(x, 0) = a \cos x, \quad a = \text{const.} > 0.
\]

With \( \epsilon = \text{const.} \geq 0 \), the function \( f(s) := s^{1+\epsilon} \) satisfies (52). With \( g(\sigma) := (1 + \sigma)^{1/2} \), condition (54) is satisfied. Since \( g(s) \) is increasing we have \( g_{\text{min}} = 1 \). From (56) with \( \alpha_0 = 1 \) and \( h(x) = a \cos x \) we compute

\[
  U = \max_{(-\pi/2, \pi/2)} \left\{ a^2 \cos^2 x + \int_0^{a^2 \sin^2 x} \sqrt{1 + \xi} \, d\xi \right\}^{1/2} = \left\{ \frac{2}{3\varepsilon} \right\}^{1/2} \left\{ (1 + a^2)^{3/2} - 1 \right\}^{1/2}.
\]

From Theorem 3 we conclude that \( u(x, t) \) exists for all time \( t > 0 \) if (69) is satisfied, i.e. if we have \( 0 < a < \sqrt{(5/2)^{2/3} - 1} \approx 0.917 \). From Theorem 4 we have the decay estimate (78) with \( \alpha_1 := 1 - \{2/3\varepsilon\}^{1/2} > 0 \).

4 EXTENSION TO THE N-DIMENSIONAL CASE

The results of Sections 2 and 3.1 may be extended in case of \( N \) space variables \( x := (x_1, \ldots, x_N) \), \( N \geq 2 \). In this section we establish the following maximum principle analogous to Theorem 1.

THEOREM 5 Let \( \Omega \) be a bounded convex domain in \( \mathbb{R}^N \) with a \( C^{2+\epsilon} \) boundary \( \partial \Omega \). Let \( u(x, t) \) be the solution of the initial boundary value problem

\[
  g(k(t)\lVert \nabla u \rVert^2) \Delta u = u_t, \quad x \in \Omega, \quad t \in (0, T),
\]
where \( g \) and \( k \) are given positive functions, \( g \in C^2 \), \( k \in C^1 \). Let \( \Phi(x, t) \) be defined on \( u(x, t) \) by

\[
\Phi(x, t) := \left\{ \frac{1}{k(t)} G(k(t)|\nabla u|^2) + \alpha u^2 \right\} e^{2\alpha \beta t},
\]

with

\[
G(\sigma) := \int_0^\sigma g(\xi) \, d\xi.
\]

In (97), \( \alpha \) is an arbitrary nonnegative parameter, and \( \beta \) is a constant to be chosen in \((0, 1)\) as indicated below. We distinguish two cases.

If \( g'(\sigma) \geq 0 \), we assume

\[
2\alpha k - k' \geq 0,
\]

and we assume that two constants \( \lambda > 0 \) and \( \beta \in (0, 1) \) can be determined such that

\[
g(\sigma) - A(\lambda, N, \beta) \sigma g'(\sigma) \geq 0, \quad \sigma \geq 0,
\]

with

\[
A(\lambda, N, \beta) := \max \left\{ \lambda N, \frac{\lambda N + \lambda^{-1} - 2}{1 - \beta} \right\}.
\]

If \( g'(\sigma) \leq 0 \), we assume

\[
k'(t) \geq 0,
\]

and we assume that \( \beta \in (0, 1) \) can be determined such that

\[
\sigma g(\sigma) - \beta G(\sigma) \geq 0, \quad \sigma \geq 0,
\]

\[
g(\sigma) + B(N, \beta) \sigma g'(\sigma) \geq 0, \quad \sigma \geq 0,
\]
with

\[ B(N, \beta) := \max \left\{ N - 1, \frac{1}{1 - \beta} \right\}. \]  \hspace{1cm} (105)

We then conclude that \( \Phi(x, t) \) takes its maximum value either at an interior critical point \((\bar{x}, \bar{t})\) of \( u \), or initially. In other words we have

\[ \Phi(x, t) \leq \max \left\{ \Phi(\bar{x}, \bar{t}) \quad \text{with} \quad \nabla u(\bar{x}, \bar{t}) = 0 \quad \text{(i)}, \right. \]

\[ \left. \max_{\Omega} \Phi(x, 0) \quad \text{(ii)}. \right\} \hspace{1cm} (106) \]

We note the presence of a factor \( \beta \) in the decay exponent of \( \Phi(x, t) \). This factor makes Theorem 5 less sharp than Theorem corresponding to the one-dimensional case.

The existence of a classical solution of (94)–(96) will not be investigated in this paper. We refer to [1,5] for such existence results.

For the proof of Theorem 5 we proceed in two steps. We first construct a parabolic inequality of the following type:

\[ \mathcal{L} \Phi := g(k(t)|\nabla u|^2)\Delta \Phi + |\nabla u|^{-2}c(x, t) \cdot \nabla \Phi - \Phi_t \geq 0, \]  \hspace{1cm} (107)

where the vector field \( c(x, t) \) is regular throughout \( \Omega \times (0, T) \). Using the following notations: \( u, i := \partial u/\partial x_i, \ i = 1, \ldots, N \), \( u, _{ik} := \partial^2 u/\partial x_i \partial x_k, \ i, k = 1, \ldots, N \), \( u, i := \partial u/\partial t, u, iv = \sum_{i=1}^{N} u, i v, i = \nabla u \cdot \nabla v \), etc., we compute

\[ \Phi, t = \left\{ \frac{k'}{k^2} \left[ gk|\nabla u|^2 - G(k(t)|\nabla u|^2) \right] + 2\alpha uu_t + 2g u, ki u, k \\
+ 2\alpha \beta \left[ \frac{1}{k} G(k|\nabla u|^2) + \alpha u^2 \right] \right\} e^{2\alpha \beta t}, \]  \hspace{1cm} (108)

\[ \Phi, k = 2\{gu, ik u, i + \alpha uu, k\} e^{2\alpha \beta t}, \]  \hspace{1cm} (109)

\[ \Delta \Phi = 2\{2g'ku, ik u, ik u, \ell + gu, i(\Delta u)\}_i \\
+ gu, ik u, ik + \alpha|\nabla u|^2 + \alpha u \Delta u \} e^{2\alpha \beta t}. \]  \hspace{1cm} (110)

Moreover differentiating (94) we obtain

\[ gu, i(\Delta u)\}_i = -2kg'u, ik u, i u, k \Delta u + u, ik u, k. \]  \hspace{1cm} (111)
Combining (108), (110), and (111), we obtain after some reduction
\[
g\Delta \Phi - \Phi_t = \left\{ 4g' k [u,ik u,k u,\ell u,\ell - u,ik u,i u,k \Delta u] + 2g^2 u,ik u,ik \\
+ \frac{2\alpha}{k} [g k |\nabla u|^2 - \beta G(k |\nabla u|^2)] \\
- \frac{k'}{k^2} [g k |\nabla u|^2 - G(k |\nabla u|^2)] - 2\alpha^2 \beta u^2 \right\} e^{2\alpha \beta t}. \tag{112}
\]

In contrast to the one-dimensional case the quantity \( u,ik u,k u,\ell u,\ell - u,ik u,i u,k \Delta u \) is not identically zero. Depending on the sign of \( g' \), it seems convenient to substitute an upper bound or a lower bound for \( u,ik u,i u,k \).

If \( g' > 0 \), we use the arithmetic–geometric mean inequality in the following form:
\[
2u,ik u,i u,k \Delta u \leq \lambda |\nabla u|^2 (\Delta u)^2 + \lambda^{-1} |\nabla u|^{-2} (u,ik u,i u,k)^2 \\
\leq \lambda N |\nabla u|^2 u,ik u,ik + \lambda^{-1} |\nabla u|^{-2} (u,ik u,i u,k)^2, \tag{113}
\]

where \( \lambda \) is an arbitrary positive constant. Combining (112) and (113) we obtain
\[
g\Delta \Phi - \Phi_t \geq \left\{ 4g' k [u,ik u,k u,\ell u,\ell + 2g [g - N\lambda g' k |\nabla u|^2] u,ik u,ik \\
- 2\lambda^{-1} g g' k |\nabla u|^{-2} (u,ik u,i u,k)^2 + \frac{2\alpha}{k} [g k |\nabla u|^2 - \beta G(k |\nabla u|^2)] \\
- \frac{k'}{k^2} [g k |\nabla u|^2 - G(k |\nabla u|^2)] - 2\alpha^2 \beta u^2 \right\} e^{2\alpha \beta t}. \tag{114}
\]

Since \( g - N\lambda g' k |\nabla u|^2 \geq 0 \) by assumption (100) we may use the Cauchy–Schwarz inequality
\[
|\nabla u|^2 u,ik u,ik \geq u,ik u,k u,\ell u,\ell. \tag{115}
\]

We then obtain
\[
g\Delta \Phi - \Phi_t \geq \left\{ 2g |\nabla u|^2 [g + (2 - \lambda N) g' k |\nabla u|^2] u,ik u,k u,\ell u,\ell \\
- 2\lambda^{-1} g g' k |\nabla u|^{-2} (u,ik u,i u,k)^2 + \frac{2\alpha}{k} [g k |\nabla u|^2 - \beta G(k |\nabla u|^2)] \\
- \frac{k'}{k^2} [g k |\nabla u|^2 - G(k |\nabla u|^2)] - 2\alpha^2 \beta u^2 \right\} e^{2\alpha \beta t}. \tag{116}
\]
We now make use of (109) to represent $u_{,ik}u_{,i}$ as follows:
\[ g_{,ik}u_{,i} = -\alpha uu_{,k} + \cdots, \quad k = 1, \ldots, N, \]  
(117)
where dots stand for a term containing $\Phi,\ell$. From (117) we compute
\[ g^2u_{,ik}u_{,k}u_{,\ell}u_{,\ell} = \alpha^2|\nabla u|^2u^2 + \cdots, \]  
(118)
\[ g^2(u_{,ik}u_{,i}u_{,k})^2 = \alpha^2|\nabla u|^4u^2 + \cdots \]  
(119)
Inserting (118) and (119) into (116) we obtain after some reduction
\[ g\Delta\Phi - \Phi,\ell + \cdots \geq 2g^{-1}\alpha^2u^2[g + (2 - N\lambda - \lambda^{-1})g'k|\nabla u|^2] + 2\alpha \frac{gk|\nabla u|^2 - \beta G(k|\nabla u|^2)}{k'} \left( -\frac{k}{k^2}[gk|\nabla u|^2 - G(k|\nabla u|^2)] - 2\alpha^2\beta u^2 \right)e^{2\alpha\beta t}. \]  
(120)
Using (100) we obtain
\[ g^{-1}\alpha^2u^2[g + (2 - N\lambda - \lambda^{-1})g'k|\nabla u|^2] \geq \beta\alpha^2u^2. \]  
(121)
Combining (120) and (121) we are led to the desired inequality
\[ g\Delta\Phi - \Phi,\ell + \cdots \geq k^{-2}(2\alpha k - k')|gk|\nabla u|^2 - G(k|\nabla u|^2)|e^{2\alpha\beta t} \geq 0. \]  
(122)
If $g' \leq 0$, we use the inequality
\[ 2\Delta uu_{,ik}u_{,i}u_{,k} \geq -(N - 1)|\nabla u|^2u_{,ik}u_{,ik} + |\nabla u|^{-2}(u_{,ik}u_{,i}u_{,k})^2 \]  

\[ + (N - 1)u_{,ik}u_{,k}u_{,\ell}u_{,\ell}, \]  
(123)
derived in [7]. Combining (112) and (123) we obtain
\[ g\Delta\Phi - \Phi,\ell \geq \left\{ 2(3 - N)gg'k u_{,ik}u_{,i}u_{,k}u_{,\ell} + 2g[g + (N - 1)g'k|\nabla u|^2]u_{,ik}u_{,ik} - 2gg'k|\nabla u|^{-2}(u_{,ik}u_{,i}u_{,k})^2 + \frac{2\alpha}{k}[gk|\nabla u|^2 - \beta G(k|\nabla u|^2)] \right. \]  
\[ - \left. \frac{k'}{k^2}[gk|\nabla u|^2 - G(k|\nabla u|^2)] - 2\alpha^2\beta u^2 \right\} e^{2\alpha\beta t} \]  
\[ \geq \left\{ 2(3 - N)gg'k u_{,ik}u_{,i}u_{,k}u_{,\ell} + 2g[g + (N - 1)g'k|\nabla u|^2]u_{,ik}u_{,ik} - 2gg'k|\nabla u|^{-2}(u_{,ik}u_{,i}u_{,k})^2 - 2\alpha^2\beta u^2 \right\} e^{2\alpha\beta t}, \]  
(124)
where the last inequality in (124) follows from assumptions (102) and (103). Now since \( g + (N - 1)g'k|\nabla u|^2 \geq 0 \) by assumption (104) we may use (115). Moreover inserting (118) and (119) we obtain after some reduction
\[
g \Delta \Phi - \Phi_t + \cdots \geq \{2\alpha^2 u^2 g^{-1}[g + g'k|\nabla u|^2] - 2\alpha^2 \beta u^2\} e^{2\alpha \beta t} \geq 0,
\]
where the last inequality follows from (104). The inequality (125) is again of the desired type.

It follows from Nirenberg's maximum principle [6,10] that \( \Phi \) takes its maximum value (i) at an interior critical point \((\bar{x}, \bar{t})\) of \( u \), or (ii) initially, or (iii) at a boundary point \((\bar{x}, \bar{t})\) with \( \bar{x} \in \partial \Omega \). The second step of the proof of Theorem 5 consists in showing that the later possibility (iii) cannot hold. To this end we compute the outward normal derivative of \( \Phi \) on \( \partial \Omega \). Using (94) rewritten in normal coordinates we obtain
\[
\frac{\partial \Phi}{\partial n} = 2e^{2\alpha \beta t} u_n u_{nn} g = -2(N - 1)e^{2\alpha \beta t} gK|\nabla u|^2 \leq 0 \quad \text{on} \ \partial \Omega,
\]
where \( K(\geq 0) \) is the average curvature of \( \partial \Omega \). Let \((\bar{x}, \bar{t})\) be a point at which \( \Phi \) takes its maximum value with \( \bar{x} \in \partial \Omega \). Friedman's boundary lemma [3,10] implies that \( \Phi \equiv \text{const.} \) in \( \Omega \times [0, \bar{t}] \), so that we must actually have \( \partial \Phi/\partial n = 0 \) on \( \partial \Omega \). Since we have \(|\nabla u|^2 > 0 \) on \( \partial \Omega \), we conclude then that the average curvature \( K \) vanishes identically on \( \partial \Omega \), which is clearly impossible. This achieves the proof of Theorem 5.

Now we want to select \( \alpha \geq 0 \) in such a way that the first possibility (i) in (106) cannot occur. To this end we proceed as in Section 3.1. In the particular case of \( k(t) = 1 \), this leads to the following result.

**Theorem 6** Let \( \Omega \) be a bounded convex domain in \( \mathbb{R}^N \) whose boundary is \( C^{2+\varepsilon} \). Let \( d \) be the radius of the greatest ball contained in \( \Omega \). Let \( u(x, t) \) be the solution of the parabolic problem (94)–(96) with \( k(t) \equiv 1 \). Assume that the hypotheses of Theorem 5 are satisfied. We then conclude that if
\[
0 \leq \alpha < \alpha_0 := \frac{\pi^2 g_{\min}}{4d^2},
\]
the first possibility (i) in (106) cannot occur. With \( \alpha \to \alpha_0 \) we are then led to the following decay bound for \( \Phi \): 
\[
G(|\nabla u|^2) + \alpha_0 u^2 \leq H^2 e^{-2\alpha_0 \beta t},
\]
with

\[ H^2 := \max_{\Omega} \{ G(\|\nabla h\|^2) + \alpha_0 h^2 \}. \]  

(129)

We note that in the context of Theorem 6, the quantity

\[ \psi := \|\nabla u\|^2 \]  

(130)

satisfies the parabolic inequality

\[ g \Delta \psi - \psi_t + \psi^{-1} \nabla \psi \cdot \bar{c} \geq 0, \]  

(131)

where the vector field \( \bar{c} \) is regular throughout \( \Omega \times (0, \infty) \). Moreover we have

\[ -2(N - 1)Ku_n^2 < 0 \text{ on } \partial \Omega. \]  

(132)

It then follows from (131) and (132) that \( \psi \) takes its maximum value initially. This shows that if \( g' \leq 0 \), we have

\[ \psi_{\min} = g(\psi_{\max}), \]  

(133)

with \( \psi_{\max} = \max_{\Omega} \|\nabla h\|^2 \).

As a first example consider \( g(\sigma) := (1 + \sigma)^{1/2} \). Since \( g'(\sigma) = \frac{1}{2}(1 + \sigma)^{-1/2} \geq 0 \), we have to determine the (greatest) \( \beta \in (0, 1) \) such that (100) is satisfied, i.e. such that \( A(\lambda, N, \beta) \leq 2 \), where \( A \) is defined in (101). This condition is satisfied only for \( N \leq 4 \). We are then led to \( \beta = 2 - \sqrt{N} > 0 \) if \( N = 2 \) or \( N = 3 \).

As a second example, consider \( g(\sigma) := (1 + \sigma)^{-\varepsilon}, \ 0 \leq \varepsilon \leq E := \min\{1/2, 1/(N - 1)\} \). Since \( g' = -\varepsilon(1 + \sigma)^{-1-\varepsilon} \leq 0 \), we have to determine the (greatest) \( \beta \in (0, 1) \) such that (103) and (104) are both satisfied. This will be the case for \( \beta = 1 - \varepsilon \).

We refer to [9] for similar results involving solutions of the parabolic differential equation

\[ (g(\|\nabla u\|^2)u_{ij})_{,j} = u_{,i}. \]  

(134)
References