The $\lambda$-function in the Space $\mathcal{P}(l_2^2)$

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In this note, motivated by the question 1 in (Aron and Lohman, Pacific J. Math. 127 (1987), 209–231), we obtain an explicit formula for the $\lambda$-function in the real space $\mathcal{P}(l_2^2)$. From this we see that the $\lambda$-function is continuous and attained at each point of the unit ball of $\mathcal{P}(l_2^2)$, the space of real-valued continuous 2-homogeneous polynomials on $l_2^2$.

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Given a normed space $E$, $B_E$ denotes its closed unit ball, $\text{ext}(B_E)$ the set of extreme points of $B_E$, and $S_E$ the closed unit sphere of $E$. If $x \in B_E$, a triple $(e, y, \lambda)$ is said to be amenable to $x$ if $e \in \text{ext}(B_E)$, $y \in B_E$, $0 < \lambda \leq 1$, and $x = \lambda e + (1 - \lambda)y$. In this case, we define

$$\lambda(x) = \sup \{\lambda : (e, y, \lambda) \text{ is amenable to } x\}.$$ 

$E$ is said to have the $\lambda$-property if each $x \in B_E$ admits an amenable triple. If, in addition, $\inf \{\lambda : x \in B_E > 0\}$, then $E$ is said to have the uniform $\lambda$-property. For more details about $\lambda$-property and $\lambda$-functions in Banach spaces we refer to [1,2,4,5].

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Aron–Lohman [1] introduced the \( \lambda \)-function, and calculated explicitly the \( \lambda \)-function for the classical spaces \( C_x(T), l_1(X), l_\infty(X) \) and \( c(X) \). They showed that every finite dimensional normed space has the uniform \( \lambda \)-property.

Choi–Kim [3] obtained an explicit formula for the norm of the real space \( \mathcal{P}(2l_2^2) \): Let \( a, b, c \in \mathbb{R}, |a| \leq 1, |b| \leq 1 \) and \( |c| \leq 2 \). Suppose \( P(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}(2l_2^2) \) for the real Banach space \( l_2^2 \). Then

\[
\|P(x, y)\| = 1 \text{ if and only if } 4 - c^2 = 4(|a + b| - ab) \quad (*)
\]

Using (*) we also classified the extreme points of the unit ball of \( \mathcal{P}(2l_2^2) \):

For the real Banach space \( l_2^2 \),

\[
P(x, y) = ax^2 + by^2 + cxy \in \text{ext}(B_{\mathcal{P}(2l_2^2)})
\]

if and only if

\[
|a| = |b| = 1 \text{ or } 0 \leq |a| < 1, \ a = -b, \ 4a^2 = 4 - c^2 \quad (**)
\]

In this note, motivated by the question in [1], we obtain an explicit formula for the \( \lambda \)-function in the real space \( \mathcal{P}(2l_2^2) \) using (*) and (**). From this we see that the \( \lambda \)-function is continuous and attained at each point of the unit ball of \( \mathcal{P}(2l_2^2) \). Finally, we give an explicit formula for the norm and the \( \lambda \)-function in \( \mathcal{P}(2l_2^2) \).

**Lemma 1** Let \( P(x, y) = ax^2 + by^2 + cxy \) in \( \mathcal{P}(2l_2^2) \), \( \|P\| \leq 1 \). Then

\[
\lambda(ax^2 + by^2 + cxy) = \lambda(\text{sign}(ab) \min\{|a|, |b|\}x^2 + \max\{|a|, |b|\}y^2 + |c|xy).
\]

**Proof** It follows from the fact that the \( \lambda \)-function is invariant with respect to isometries.

**Theorem 2** Let \( P(x, y) = ax^2 + by^2 + cxy \) in \( \mathcal{P}(2l_2^2) \), \( \|P\| \leq 1 \). Then

\[
\lambda(ax^2 + by^2 + cxy) = \frac{1}{2} + \frac{1}{4} \left| a + b \right| - \sqrt{(a - b)^2 + c^2}.
\]

Therefore, the \( \lambda \)-function is continuous and attained at each point of \( B_{\mathcal{P}(2l_2^2)} \).
THE $\lambda$-FUNCTION IN THE SPACE $\mathcal{P}(l^2_2)$

**Proof**  By Lemma 1, may assume that $|a| \leq |b| = b$ and $c \geq 0$.

Case I  $\|P\| < 1$. First, (*) shows that

$$4 - c^2 > 4(a + b) - 4ab. \quad (1)$$

(A) Suppose that $P(x, y) = \lambda(x^2 + y^2) + (1 - \lambda)Q(x, y)$ for some $0 < \lambda \leq 1$ and $Q \in \mathcal{P}(l^2_2)$, $\|Q\| \leq 1$.

By Proposition 1.2(b) [1] we may assume $\|Q\| = 1$. Then

$$Q(x, y) = \left(\frac{a - \lambda}{1 - \lambda}\right)x^2 + \left(\frac{b - \lambda}{1 - \lambda}\right)y^2 + \left(\frac{c}{1 - \lambda}\right)xy.$$ and (*) shows that

$$\left|\frac{a - \lambda}{1 - \lambda}\right| \leq 1, \quad \left|\frac{b - \lambda}{1 - \lambda}\right| \leq 1, \quad \left|\frac{c}{1 - \lambda}\right| \leq 2$$

and

$$4 - \left(\frac{c}{1 - \lambda}\right)^2 = 4\left|\frac{a + b - 2\lambda}{1 - \lambda}\right| - 4\left(\frac{a - \lambda}{1 - \lambda}\right)\left(\frac{b - \lambda}{1 - \lambda}\right). \quad (2)$$

If $a + b - 2\lambda \geq 0$, then Eq. (2) is equivalent to $4 - c^2 = 4(a + b) - 4ab$, contrary to (1). Suppose $a + b - 2\lambda < 0$. Solving Eq. (2), we get

$$\lambda = \frac{1}{2} + \frac{1}{4} \left(|a + b| \pm \sqrt{(a - b)^2 + c^2}\right).$$

Since $\lambda \leq \min\{(1 + a)/2, (1 + b)/2\} = (1 + a)/2$, we have

$$\lambda = \frac{1}{2} + \frac{1}{4} \left(|a + b| - \sqrt{(a - b)^2 + c^2}\right).$$

It is easy to check that

$$\frac{a + b}{2} < \frac{1}{2} + \frac{1}{4} \left(|a + b| - \sqrt{(a - b)^2 + c^2}\right) \leq \frac{1 + a}{2}.$$

Hence

$$\sup\{\lambda: (x^2 + y^2, Q, \lambda) \text{ is amenable to } P\}$$

$$= \frac{1}{2} + \frac{1}{4} \left(|a + b| - \sqrt{(a - b)^2 + c^2}\right).$$
(B) Suppose that \( P(x, y) = \lambda(-x^2 - y^2) + (1 - \lambda)Q(x, y) \) for some \( 0 < \lambda \leq 1 \) and \( Q \in \mathcal{P}^{(2, 2)} \), \( \|Q\| = 1 \).

Then

\[
Q(x, y) = \left( \frac{a + \lambda}{1 - \lambda} \right) x^2 + \left( \frac{b + \lambda}{1 - \lambda} \right) y^2 + \left( \frac{c}{1 - \lambda} \right) xy
\]

and (*) shows that

\[
\left| \frac{a + \lambda}{1 - \lambda} \right| \leq 1, \quad \left| \frac{b + \lambda}{1 - \lambda} \right| \leq 1, \quad \left| \frac{c}{1 - \lambda} \right| \leq 2
\]

and

\[
4 - \left( \frac{c}{1 - \lambda} \right)^2 = 4 \left| \frac{a + b + 2\lambda}{1 - \lambda} \right| - 4 \left( \frac{a + \lambda}{1 - \lambda} \right) \left( \frac{b + \lambda}{1 - \lambda} \right) .
\]

Solving Eq. (3), we get

\[
\lambda = \frac{1}{2} - \frac{1}{4} \left( |a + b| + \sqrt{(a - b)^2 + c^2} \right).
\]

Note that

\[
\frac{1 - b}{2} < \frac{1}{2} - \frac{1}{4} \left( |a + b| + \sqrt{(a - b)^2 + c^2} \right).
\]

Since \( \lambda \leq \min\{((1 - a)/2, (1 - b)/2) = (1 - b)/2 \), \( P \) does not admit an amenable triple \((-x^2 - y^2, Q, \lambda)\).

(C) Suppose that \( P(x, y) = \lambda (lx^2 - ly^2 + 2\sqrt{1 - l^2}xy) + (1 - \lambda)Q(x, y) \) for some \( 0 < \lambda \leq 1 \), \(-1 \leq l \leq 1 \) and \( Q \in \mathcal{P}^{(2, 2)} \), \( \|Q\| = 1 \).

Then

\[
Q(x, y) = \left( \frac{a - \lambda l}{1 - \lambda} \right) x^2 + \left( \frac{b + \lambda l}{1 - \lambda} \right) y^2 + \left( \frac{c + 2\lambda \sqrt{1 - l^2}}{1 - \lambda} \right) xy
\]

and (*) shows that

\[
\left| \frac{a - \lambda l}{1 - \lambda} \right| \leq 1, \quad \left| \frac{b + \lambda l}{1 - \lambda} \right| \leq 1, \quad \left| \frac{c + 2\lambda \sqrt{1 - l^2}}{1 - \lambda} \right| \leq 2
\]
and

\[
\left( \frac{c \pm 2\lambda \sqrt{1-l^2}}{1-\lambda} \right)^2 = 4 \left( 1 - \frac{a - \lambda l}{1 - \lambda} \right) \left( 1 - \frac{b + \lambda l}{1 - \lambda} \right). \tag{4}
\]

Solving Eq. (4), we get

\[
\lambda = \frac{4(1 - a)(1 - b) - c^2}{4((b - a)l + 2 - a - b \pm c\sqrt{1-l^2})}.
\]

Computation shows that

\[
\max_{-1 \leq l \leq 1} \frac{4(1 - a)(1 - b) - c^2}{4((b - a)l + 2 - a - b \pm c\sqrt{1-l^2})} = \frac{4(1 - a)(1 - b) - c^2}{4\min_{-1 \leq l \leq 1} (b - a)l + 2 - a - b \pm c\sqrt{1-l^2}} \quad \text{(by (1))}
\]

\[
= \frac{4(1 - a)(1 - b) - c^2}{4\left(2 - a - b - \sqrt{(a - b)^2 + c^2}\right)} = \frac{1}{2} - \frac{1}{4} \left( |a + b| - \sqrt{(a - b)^2 + c^2} \right)
\]

at \( l = (a - b)/\sqrt{(a - b)^2 + c^2} \). Thus we have

\[
\lambda \leq \frac{1}{2} - \frac{1}{4} \left( |a + b| - \sqrt{(a - b)^2 + c^2} \right).
\]

Computation shows that \( P \) admits an amenable triple

\[
\left( \frac{a - b}{\sqrt{(a - b)^2 + c^2}}, \frac{b - a}{\sqrt{(a - b)^2 + c^2}}, \frac{2|c|}{\sqrt{(a - b)^2 + c^2}} \right) = xy, Q, \frac{1}{2} - \frac{1}{4} \left( |a + b| - \sqrt{(a - b)^2 + c^2} \right).
\]
Hence

\[
\sup\{\lambda: (lx^2 - ly^2 \pm 2\sqrt{1-t^2}xy, Q, \lambda) \text{ is amenable to } P, -1 \leq t \leq 1\} = \frac{1}{2} - \frac{1}{4} \left| a + b - \sqrt{(a - b)^2 + c^2} \right|
\]

By the cases (A)-(C), we have

\[
\lambda(ax^2 + by^2 + cxy) = \max \left\{ \frac{1}{2} \pm \frac{1}{4} \left| a + b - \sqrt{(a - b)^2 + c^2} \right| \right\}
\]

\[
= \frac{1}{2} + \frac{1}{4} \left| a + b - \sqrt{(a - b)^2 + c^2} \right|
\]

Case 2: \|P\| = 1. First, (\ast) shows that \(4 - c^2 = 4(a + b) - 4ab\). \hfill (5)

(A') Suppose that \(P(x, y) = f(x^2 + y^2) + (1 - \lambda)Q(x, y)\) for some \(0 < \lambda \leq 1\) and \(Q \in \mathcal{P}(2l_2^2), \|Q\| = 1\).

Then

\[
Q(x, y) = \left(\frac{a - \lambda}{1 - \lambda}\right)x^2 + \left(\frac{b - \lambda}{1 - \lambda}\right)y^2 + \left(\frac{c}{1 - \lambda}\right)xy
\]

and (\ast) shows that

\[
\left| \frac{a - \lambda}{1 - \lambda} \right| \leq 1, \quad \left| \frac{b - \lambda}{1 - \lambda} \right| \leq 1, \quad \left| \frac{c}{1 - \lambda} \right| \leq 2
\]

and

\[
4 - \left(\frac{c}{1 - \lambda}\right)^2 = 4 \left| \frac{a + b - 2\lambda}{1 - \lambda} \right| - 4 \left(\frac{a - \lambda}{1 - \lambda}\right) \left(\frac{b - \lambda}{1 - \lambda}\right).
\hfill (6)
\]

If \(a + b - 2\lambda \geq 0\), then Eq. (6) is equivalent to

\[
\lambda \leq \min \left\{ \frac{1 + a}{2}, \frac{1 + b}{2}, \frac{a + b}{2} \right\} = \frac{a + b}{2}.
\]
THE $\lambda$-FUNCTION IN THE SPACE $\mathcal{P}(^2l_2^2)$

If $a + b - 2\lambda < 0$, we have

$$\frac{a + b}{2} < \lambda \leq \min\left\{\frac{1 + a}{2}, \frac{1 + b}{2}\right\} = \frac{1 + a}{2}.$$ 

Solving Eq. (6), we get $\lambda = (a + b)/2$. Thus $P$ does not admit an amenable triple if $a + b - 2\lambda < 0$. Hence

$$\sup\{\lambda: (x^2 + y^2, Q, \lambda) \text{ is amenable to } P\} = \frac{a + b}{2}.$$ 

(B') Suppose that $P(x, y) = \lambda(-x^2 - y^2) + (1 - \lambda)Q(x, y)$ for some $0 < \lambda \leq 1$ and $Q \in \mathcal{P}(^2l_2^2)$, $\|Q\| = 1$.

Then

$$Q(x, y) = \left(\frac{a + \lambda}{1 - \lambda}\right)x^2 + \left(\frac{b + \lambda}{1 - \lambda}\right)y^2 + \left(\frac{c}{1 - \lambda}\right)xy$$

and (*) shows that

$$\left|\frac{a + \lambda}{1 - \lambda}\right| \leq 1, \quad \left|\frac{b + \lambda}{1 - \lambda}\right| \leq 1, \quad \left|\frac{c}{1 - \lambda}\right| \leq 2$$

and

$$4 - \left(\frac{c}{1 - \lambda}\right)^2 = 4\left|\frac{a + b + 2\lambda}{1 - \lambda}\right| - 4\left(\frac{a + \lambda}{1 - \lambda}\right)\left(\frac{b + \lambda}{1 - \lambda}\right).$$

Solving Eq. (7), we get

$$\lambda = 1 - \frac{a + b}{2}.$$ 

Hence

$$\sup\{\lambda: (-x^2 - y^2, Q, \lambda) \text{ is amenable to } P\} = \min\left\{1 - \frac{a + b}{2}, \frac{1 - b}{2}\right\}.$$ 

(C') Suppose that $P(x, y) = \lambda(lx^2 - ly^2 \pm 2\sqrt{1 - l^2}xy) + (1 - \lambda)Q(x, y)$ for some $0 < \lambda \leq 1$, $-1 \leq l \leq 1$ and $Q \in \mathcal{P}(^2l_2^2)$, $\|Q\| = 1$. 
Then

\[ Q(x, y) = \left( \frac{a - \lambda l}{1 - \lambda} \right) x^2 + \left( \frac{b + \lambda l}{1 - \lambda} \right) y^2 + \left( \frac{c + 2\lambda \sqrt{1 - l^2}}{1 - \lambda} \right) xy \]

and (*) shows that

\[
\left| \frac{a - \lambda l}{1 - \lambda} \right| \leq 1, \quad \left| \frac{b + \lambda l}{1 - \lambda} \right| \leq 1, \quad \left| \frac{c + 2\lambda \sqrt{1 - l^2}}{1 - \lambda} \right| \leq 2
\]

and

\[
\left( \frac{c + 2\lambda \sqrt{1 - l^2}}{1 - \lambda} \right)^2 = 4 \left( 1 - \frac{a - \lambda l}{1 - \lambda} \right) \left( 1 - \frac{b + \lambda l}{1 - \lambda} \right). \tag{8}
\]

Solving Eq. (8), we get

\[
l = \frac{a - b}{2 - a - b} \quad \text{and} \quad \lambda \leq \min \left\{ \frac{1 - a}{1 - l}, \frac{1 - b}{1 + l} \right\} = 1 - \frac{a + b}{2}.
\]

Computation shows that \( P \) admits an amenable triple

\[
\left( \frac{a - b}{2 - a - b} x^2 + \frac{b - a}{2 - a - b} y^2 + \frac{2|c|}{2 - a - b} xy, Q, 1 - \frac{a + b}{2} \right).
\]

Hence

\[
\sup\{ \lambda : (lx^2 - ly^2 \pm 2\sqrt{1 - l^2}xy, Q, \lambda) \text{ is amenable to } P, \ -1 \leq l \leq 1 \} = 1 - \frac{a + b}{2}.
\]

By the cases (A')–(C'), we have

\[
\lambda(ax^2 + by^2 + cxy) = \max\left\{ \frac{a + b}{2}, 1 - \frac{a + b}{2} \right\} = \frac{1}{2} + \frac{1}{4} |a + b| - \sqrt{(a - b)^2 + c^2} \quad \text{(by (5))}.
\]
By the cases 1 and 2, we have that
\[
\lambda(ax^2 + by^2 + cxy) = \frac{1}{2} + \frac{1}{4} \left| a + b \right| - \sqrt{(a - b)^2 + c^2}.
\]

The above argument shows that the \(\lambda\)-function is continuous and attained at each point of the unit ball of \(B_{\mathcal{P}(l_2^2)}\). This completes the proof.

Note that if \(E\) is a finite dimensional normed space, then \(x \in \text{ext}(B_E)\) if and only if \(\lambda(x) = 1\). From this fact and Theorem 2, we can reclassify the extreme points of the unit ball of \(\mathcal{P}(l_2^2)\).

We can give an explicit relation between the norm and the \(\lambda\)-function in \(\mathcal{P}(l_2^2)\).

**Theorem 3** Let \(P(x, y) = ax^2 + by^2 + cxy\) in \(\mathcal{P}(l_2^2)\), \(\|P\| \leq 1\). Then

\[
\|P\| + 2\lambda(P) = 1 + \max \left\{ |a + b|, \sqrt{(a - b)^2 + c^2} \right\}.
\]

**Proof** By Lemma 1, we may assume that \(|a| \leq |b| = b\) and \(c \geq 0\). From the proof of Lemma 2.1 [3] we get

\[
\|P\| = P \left( \frac{1}{2} - |a - b| \sqrt{2(a - b)^2 + c^2}, \right.
\]

\[
\left. \sqrt{\frac{1}{2} + |a - b| \sqrt{2(a - b)^2 + c^2}} \right),
\]

\[
= \left( |a + b| + \sqrt{(a - b)^2 + c^2} \right) / 2,
\]

which concludes the proof of the theorem combining Theorem 2.

**References**