Landau-type Inequalities and $L^p$-bounded Solutions of Neutral Delay Systems

HANS GÜNZLER*

Mathematisches Seminar, Universität Kiel, Ludewig-Meyn-Str. 4, D 24098 Kiel, Germany

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In Section 1 relations between various forms of Landau inequalities $\|y^{(m)}\|^n \leq \lambda \|y\|^{n-m} \|y^{(m)}\|^m$ and Halperin–Pitt inequalities $\|y^{(m)}\| \leq \varepsilon \|y^{(n)}\| + S(\varepsilon) \|y\|$ are discussed, for arbitrary norms, intervals and Banach-space-valued $y$. In Section 2 such inequalities are derived for weighted $L^p$-norms, Stepanoff- and Orlicz-norms.

With this, Esclangon–Landau theorems for solutions $y$ of linear neutral delay difference-differential systems are obtained: If $y$ is bounded e.g. in a weighted $L^p$- or Stepanoff-norm, then so are the $y^{(m)}$. This holds also for some nonlinear functional differential equations.

Keywords: Landau inequalities; Esclangon–Landau theorem; $L^p$-bounded solutions; Neutral differential-difference systems

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0 INTRODUCTION AND NOTATIONS

To prove that bounded solutions of certain linear differential equations are quasiperiodic, Esclangon [6,7] needed and demonstrated that such bounded solutions have bounded derivatives. This result was later used by Bohr and Neugebauer [4] to get the almost periodicity of bounded solutions of $n$th order linear equations with constant coefficients and almost periodic right-hand side.

* E-mail: guenzler@math.uni-kiel.de.
Landau [21] extended Esclangon’s result on the boundedness of the derivatives of bounded solutions to linear differential equations with only bounded coefficients. In the following we will call such theorems “Esclangon–Landau–” or “EL-results”. They have played an important role in the discussion of the asymptotic behaviour of solutions of differential equations, see e.g. Basit and Zhikov [2], Levitan and Zhikov [22, p. 95 and 97], and the references in [3, p. 596]. In [3] EL-results were obtained for difference-differential equations and the sup-norm.

For his EL-results Landau showed, under some additional assumptions and with the sup-norm, for a compact interval,

$$\|y^{(m)}\|^n \leq \lambda_n \|y\|^{n-m} \|y^{(n)}\|^m, \quad 0 < m < n \tag{0.1}$$

([20, 1913 for $n = 2, 21, 1930$ p. 182, Hilfssatz 3]); a qualitative form can be found in Hardy and Littlewood [12, p. 422, Theorem 3])

We will call results of this type Landau inequalities; a thorough discussion of the many results in this direction can be found in Chapter 1 of Mitrinović, Pečarić and Fink [25], mostly for scalar-valued $y$ and unbounded intervals.

To get EL-results for $L^p$-bounded and Banach-space-valued solutions, one needs however (0.1) or related inequalities for bounded intervals and such norms, then not so much can be found in the literature.

In Section 1 we discuss first the relations between various forms of (0.1), especially the asymptotic form (for compact intervals approaching the boundary) needed later, for vector-valued $y$.

It turns out that a stronger version of (0.1) is a Nirenberg inequality [26, appendix],

$$\|y^{(m)}\| \leq \varepsilon^{n-m}\|y^{(n)}\| + K\varepsilon^{-m}\|y\|, \quad 0 < \varepsilon \leq \varepsilon_0 \tag{0.2}$$

a weaker variant is obtained by replacing $K\varepsilon^{-m}$ by an arbitrary function $S(\varepsilon)$ (Halperin and Pitt [11]). Again relations, also with asymptotic versions, are discussed, for general intervals and norms.

In Section 2 we obtain Nirenberg and then Landau inequalities for weighted $L^p$ or Stepanoff-norms, arbitrary intervals and vector-valued functions. From the explicit form of the constants $K$ and $\lambda$ there (usually not optimal) we can deduce the asymptotic forms needed. For Orlicz-norms we get at least Halperin–Pitt inequalities, for bounded intervals.
This is applied in Section 3 to linear delayed neutral difference-differential equations and systems, with bounded operator-valued coefficients: For weighted $L^p$-norms or weighted Stepanoff $S^p$-norms still an EL-result is true, $1 \leq p \leq \infty$, if the weight function does not oscillate too wildly, similarly for Orlicz norms (Corollary 3.2). These results seem new and non-trivial even for bounded intervals and scalar-valued solutions. With an asymptotic Landau inequality even some non-linear functional differential equations can be treated (Proposition 3.6).

In the following $X$ is a Banach space over $K = \mathbb{R}$ or $\mathbb{C}$. $J \subset \mathbb{R}$ is an interval with endpoints $\alpha$ and $\beta$, $-\infty \leq \alpha < \beta \leq \infty$. For $f : J \to X$ and $M \subset J$

\[ g = f M \quad \text{means} \quad g = f \quad \text{on} \quad M, \quad g = 0 \quad \text{else in} \quad J; \quad (0.3) \]

$|f|$ is defined by $|f|(x) := \|f(x)\|$, $x \in J$. $|I|$ is the length of the interval $I \subset \mathbb{R}$; "a.e." is with respect to Lebesgue measure on $\mathbb{R}$. Integrals are usually (Bochner-) Lebesgue integrals (Hille–Phillips [15]). A seminorm $\|\|$ is a norm without "$\|x\| = 0$ implies $x = 0$". For $n$ natural

\[ C^{(n)}(J, X) := \{ f \in C^{n-1}(X, J) : f^{(n-1)} \text{ locally absolutely continuous and } f^{(n-1)'} \text{ exists a.e. in } J \}; \quad (0.4) \]

then

\[ f^{(n)}(x) := f^{(n-1)'}(x) \text{ where it exists in } J, \text{ else := 0.} \quad (0.5) \]

The $L^p$-spaces are spaces of measurable functions, not equivalence classes.

1 LANDAU, NIREDNERG AND HALPERIN–PITT INEQUALITIES

With the notations of the introduction we assume in the following:

$V$ linear $\subset X'$ (pointwise operations), $\| \| : V \to [0, \infty)$ seminorm satisfying: if $y \in C^{(1)}(J, X)$, $I$ compact $\subset J$, $yI$ (1.1)

and $y' I \in V$ and $\|yI\| = 0$, then $\|y'I\| = 0$; $n$ integer $\geq 2$.

Here $V$ can be e.g. $L^p(J, X)$, with $\|\| = \|\|_p$. 
DEFINITION 1.1 We say that the strong Landau (or Kolmogorov) inequality $L_n = L_n^s(\lambda) = L_n^s(\lambda, \|\|)$ holds (for $V$) if $0 \leq \lambda < \infty$ and

$$\|y^{(m)}\|^n \leq \lambda \|y\|^{n-m} \|y^{(n)}\|^m \quad \text{for} \quad 0 < m < n \quad (1.2)$$

and all $y \in C^n(J, X)$ with $y^{(m)} \in V$, $0 \leq m \leq n$ (see (0.4), (0.5)).

The weak Landau inequality $L_n^w = L_n^w(\lambda, \tau, \|\|)$ holds if with $\lambda, \tau \in (0, \infty]$ one has (1.2) with $y$ as after (1.2) and with the additional conditions

$$\|y^{(n)}\| > 0 \quad \text{and} \quad (\|y\|/\|y^{(n)}\|)^{1/n} \leq \tau. \quad (1.3)$$

The Landau inequality $L_n^L = L_n^L(\lambda, \tau, \|\|)$ holds if $\lambda, \tau \in (0, \infty]$ and for $y$ as after (1.2) with $0 \leq \|y\| \leq a$, $0 \leq \|y^{(n)}\| \leq b$, $0 < b$, where $a, b \in [0, \infty)$, and with $(a/b)^{1/n} \leq \tau$, one has

$$\|y^{(m)}\|^n \leq \lambda a^{n-m}b^m, \quad 0 < m < n. \quad (1.4)$$

The asymptotic Landau inequality $L_n^a = L_n^a(\lambda, \|\|)$ holds, with $0 < \lambda < \infty$, if to each $y \in C^n(J, X)$ with $y^{(m)} \in V$ for $0 \leq m \leq n$ and all compact $I \subset J$ and for which furthermore $\|y^{(n)}\|$ is $\neq 0$ in $I$ there exists a compact interval $I(y) \subset J$ such that (see (0.3))

$$\|y^{(m)}I\|^n \leq \lambda \|yI\|^{n-m} \|y^{(n)}I\|^m, \quad 0 < m < n, \quad I(y) \subset I \subset J \quad (1.5)$$

(the $y^{(m)}$ need not be in $V$).

Landau inequalities have been introduced in [20, $n = 2$, 21, p. 182, Hilfssatz 3], where Landau showed that $L_n$ holds for compact $J$, $\|f\| = \sup_J |f|$, $X = \mathbb{R}$, with $\lambda_2 = 4$, $\lambda_n = 2^n - 2^x$, $\tau = (1/2)|J|$; this implies immediately a strong Landau inequality for unbounded $J$. Kolmogorov [16] determined the optimal $\lambda_n$ (even $\lambda_{n,m}$) in $L_n^s$ for $J = \mathbb{R} = X$, $\|f\| = \sup_{\mathbb{R}} |f|$, $V = \text{bounded functions}$. The asymptotic form $L_n^a$ can be traced back to Hardy and Littlewood [12, p. 422, Theorem 3], it was used in [3, Lemma 2.5] for general $X$ and $\|\|_\infty$.

Obvious relations, for fixed $J, V, \|\|, n, \lambda$, any $\tau > 0$:

$$L_n^s(\lambda) = L_n(\lambda, \infty) = L_n^w(\lambda, \infty) \Rightarrow L_n(\lambda, \tau) \Rightarrow L_n^w(\lambda, \tau). \quad (1.6)$$
Also

\[ L_n^a(\lambda) \Rightarrow L_n^s(\lambda) \quad \text{for } \|y^{(n)}\| > 0, \quad (1.7) \]

provided \( V \) satisfies

\[ f \in V, I \text{ compact } \subset J \Rightarrow fI \in V \text{ and } \|fI\| \to \|f\| \text{ as } I \to J. \quad (1.8) \]

Even with (1.8), \( L_n^s(\lambda) \Rightarrow L_n^a(\lambda + \varepsilon) \) follows only for \( y \) with \( y^{(m)} \in V \); see Remarks 2.17(a) and (c).

**Example 1.2** \( y(t) = t + \varepsilon \sin t \) shows that already \( L_2^i \) and therefore \( L_2^a \) are false for any bounded \( J \), any \( \lambda \), any \( V = L^p \) with \( \|\|_p, 1 \leq p \leq \infty \) (see, however, Proposition 2.16, but also Example 1.13).

So for bounded \( J \) for (1.2) additional conditions are necessary. We work here with Landau's condition (1.3); for other types of Landau inequalities in this situation see Gorny [9], Levitan and Zhikov [22, p. 95], Redheffer and Walter [27].

Throughout each of the following lemmas, \( J, X, V \) and its seminorm \( \| \| \) are fixed and satisfy (1.1).

**Lemma 1.3** If \( L_2^a \) holds with \( \lambda_2 \leq 1 \), then \( L_n^a \) holds for \( n \geq 2 \), with \( \lambda_n = \lambda_2^{n/2} \).

**Proof** By induction one can show (see [18, p. 232, (2.14)]), for \( \lambda = \lambda_{n,m} \) in (1.5)

\[ \lambda_{n,m} \leq \lambda_{2,1}^{nm(n-m)/2}, \quad 0 < m < n, \quad 2 \leq n. \quad (1.9) \]

**Lemma 1.4** If \( L_2^i \) holds with \( \lambda \geq 1 \) and \( n \geq 2 \), then \( L_n^s \) holds with \( \lambda_n \) as in Lemma 1.3.

**Proof** As for Lemma 1.3.

**Lemma 1.5** If \( L_2(\lambda_2, \tau) \) holds with \( \tau > 0 \), \( \lambda_2 \geq 1 \), and \( n \geq 2 \), then \( L_n(\lambda_n, \tau) \) holds with \( \lambda_n = \lambda_2^{2^{n-1}} \).

**Proof** This has been shown by Landau [21, pp. 182–183], for \( \|\|_\infty \), compact \( J, X = \mathbb{R}, \tau = (1/2)|J|, \lambda_2 = 4 \). His ingenious proof works also in our more general situation, for the \( \chi \) on p. 182 e.g. one has to use \( \chi := \max(\lambda_2^{2^{n-2}}, \max\{\|y^{(m)}\|: 0 < m < n\}) \).
Question: Can one improve this to $\lambda_n = A \cdot \lambda_2^n$ as in Lemma 1.3? (Yes under the assumptions of Lemma 1.15 via $L_2 \Rightarrow N_2 \Rightarrow N_n \Rightarrow L_n$ of below.)

**Example 1.6** If $y_k(t) = t + k^{-3} \sin kt$, $t \in J = [0, 1]$, $k \in \mathbb{N}$, with $\|y\|_p$, $1 \leq p \leq \infty$, and the nonlinear $V$ containing just the $y_k$ and their derivatives up to order 3, one can show that $L_3^v$ holds, but $L_2^v$ holds for no $\lambda < \infty$.

Do there exist such linear $V$ as in (1.1)?

**Lemma 1.7** Assume $V \subset L^1(J, X)$, $J$ bounded, $V$ containing all bounded continuous $f$, assume further the existence of $C_1, C_2 \in (0, \infty)$ with

$$C_1 \|f\|_1 \leq \|f\|_V \leq C_2 \sup_j |f|, \text{ } f \text{ bounded } \in V. \quad (1.10)$$

Then $L_2(\lambda, \tau)$ and $L_2^V(\lambda, \tau)$ are equivalent.

Examples are $L^p(J, X)$ or more general Orlicz-spaces with Lebesgue measure.

**Proof** With $y$ as before (1.4) with $\|y''\| < b$ and $0 < a$ define $w_{u,v}(t) := y(t) + xu \sin vt$ with $x \in X, \|x\| = 1$. Then $w_{u,v}^{(j)} \in V$, with $w_{r} := w_{r/(C_2)^{1/p}}$, one has $\|w_r - y\| \leq \varepsilon, \|w_r' - y'\| \leq \varepsilon$, for $0 < \varepsilon < b - \|y''\|$. With a continuity argument there is $s \in (1, \infty)$ with $\|w^s\| = b + \varepsilon(b/a)$; then $\|w_s\|/\|w^s\| \leq (\|y\| + \varepsilon)/(b + \varepsilon(b/a)) \leq a/b \leq \tau^2$. $L_2^v$ yields $\|w''\|^2 \leq (\|y''\|^2 - \varepsilon)^2 \leq \|w''_r\|^2 \leq (\|y''\|^2 + \varepsilon(b + \varepsilon(b/a)), \varepsilon \rightarrow 0$ gives $L_2$. This works for any $V$ containing $z$ with $\|z^{(j)}(v \cdot)\| \leq C_0, 0 < \delta_0 \leq \|z''(v \cdot)\|$ for $v \leq 1, j = 0, 1, 2$, then without (1.10).

**Corollary 1.8** $L_2^w$ implies $L_2^w$ for $n \geq 2$, with $\lambda_n$ of Lemma 1.5, provided $V, \|\| \text{ are as in Lemma 1.7.}$

Question: Direct proof of $L_2^w \Rightarrow L_2^w$, for more general $V$? Characterization of $V$ with $L_2^w \Rightarrow L_2^w$?

**Definition 1.9** We say that a Nirenberg inequality $N_n = N_n(K, \sigma) = N_n(K, \sigma, V, \|\|)$ holds iff with $K, \sigma \in [0, \infty]$ for all $y$ as after (1.2) one has

$$\|y^{(m)}\| \leq \varepsilon^{n-m}\|y^{(n)}\| + \frac{K}{\varepsilon^m}\|y\| \quad \text{for } 0 < \varepsilon \text{ real } \leq \sigma, \quad 0 < m < n. \quad (1.11)$$
A strong $N_n^3$ holds means $N_n(K, \infty, V, \| \|)$ is true.

**Lemma 1.10** If $N_2(K, \sigma)$ holds with $4K \geq 1$, then $N_n(K_n, \sigma)$ holds for $n \geq 2$ with

$$K_n = 2^{n-4}(4K)^{\frac{n}{2}}.$$  \hfill (1.12)

**Proof by induction** If $N_k$ holds for $2 \leq k \leq n$, one has, with $N_2$ and $y_m := \| y^{(m)} \|$, $y$ as after (1.2) for $n+1$,

$$y_{n-1} \leq \varepsilon y_n + K_n\varepsilon^{1-n}y_0, \quad 0 < \varepsilon \leq \sigma_n = \sigma,$$

$$y_n \leq \eta y_{n+1} + Ky_{n-1}/\eta \leq \eta y_{n+1} + \frac{K\varepsilon}{\eta} y_n + \frac{KK_n}{\varepsilon^{n-1}} y_0, \quad 0 \leq \eta \leq \sigma.$$

With $\varepsilon = \eta/(2K) \leq \sigma$ one gets, with $\delta := 2\eta$

$$y_n \leq 2\eta y_{n+1} + \frac{2KK_n}{\eta \varepsilon^{n-1}} y_0 = \delta y_{n+1} + \frac{(2K)^n K_n 2^n}{\varepsilon^n} y_0,$$  \hfill (1.13)

which is (1.11) for $n + 1$ and $m = n$, with even $0 < \delta \leq 2\sigma$.

Substituting this in (1.11), one gets for $0 < m < n$

$$y_m \leq \varepsilon^{n-m} \delta y_{n+1} + K_n((4K)^n \varepsilon^{n-m} \delta^{n} + \varepsilon^{-m})y_0$$

for $0 < \varepsilon \leq \sigma$ and $0 < \delta \leq 2\sigma$. So $\delta = \varepsilon$ is possible, yielding with $(4K)^n \geq 1$

$$y_m \leq \varepsilon^{(n+1)-m} y_{n+1} + K_n(2(4K)^n)\varepsilon^{-m} y_0, \quad 0 < \varepsilon \leq \sigma_{n+1} = \sigma;$$

with (1.13), this holds for $0 < m < n + 1$.

So

$$K_{n+1} = 2(4K)^n K_n = 2(4K)^n 2^{n-4}(4K)^{\frac{n}{2}} = 2^{n+1-4}(4K)^{\frac{n+1}{2}}.$$  \hfill (1.14)

In the above, the case $\sigma = \infty$ gives

**Lemma 1.11** If $N_2^3(K)$ holds with $4K \geq 1$, then $N_2^3(K_n)$ holds, with (1.12).

**Lemma 1.12** For each $n \geq 2$, $N_n^3$ and $L_n^3$ are equivalent, with

$$\lambda_n = (1 + e^{1/\varepsilon})^n K_{n-1}^{n-1} \text{ resp. } K_n = n^{-1/(n-1)} \lambda_n,$$  \hfill (1.14)

provided $K_n \geq 1$ resp. $\lambda_n \geq 1$. 
Remark For \( n = 2, \lambda_2 = 4K_2 \) by (1.15); (1.14) can be improved to \( K_n = (1 - (1/n))n^{-1/(n-1)}\lambda_n \), which is optimal.

Proof We show the equivalence even for each fixed \( m, 0 < m < n \). If, with \( y_j := \|y^{(j)}\| \), (1.11) holds for all \( \varepsilon > 0 \), the right side has its minimum for \( 0 = (n - m)e^{n - m - 1}y_n - mK\varepsilon^{-m-1}y_0 \) or \( y_n = 0 \Rightarrow y_m = 0 \Rightarrow (1.2) \) \( \varepsilon = (mK\varepsilon/((n - m)y_n))^{1/n} \). This \( \varepsilon \) gives (1.2), with

\[
\lambda_{n,m} = \left( \left( \frac{m}{n-m} \right)^{(n-m)/n} + \left( \frac{n-m}{m} \right)^{m/n} \right)^n K_{n,m}^{n-m}. \tag{1.15}
\]

With \( t^{1/t} \leq e^{1/e} \) for \( 1 \leq t < \infty \) and \( K_{n,m} = K_n \) this gives Part 1 of (1.14). Conversely, (1.2) for \( m \) implies for \( 0 < \varepsilon < \infty \)

\[
y_m^n \leq \lambda_{n,m} y_0^{n-m} y_m^m = \lambda_{n,m} \left( \frac{n}{m} \right)^{-1} \left( \frac{n}{m} \right)^{n-m} (\varepsilon^{-m} y_0)^{n-m} (\varepsilon^{n-m} y_n)^m \\
\leq \left( \lambda_{n,m} \left/ \left( \frac{n}{m} \right) \right. \right)^{(n-m)/n} (\varepsilon^{n-m} y_n + \varepsilon^{-m} y_0)^n.
\]

This gives (1.11) with

\[
K_{n,m} = \left( \lambda_{n,m} / \left( \frac{n}{m} \right) \right)^{1/(n-m)}. \tag{1.16}
\]

If \( \lambda_{n,m} = \lambda_n \geq 1 \), this gives Part 2 of (1.14.)

Example 1.13 \( N_n^2 \) trivially implies \( N_n \); the converse is in general false:

\[
J = [3, \infty), \quad X = \mathbb{R}, \quad \|f\| = \sup \{ |f(t)| / t : 3 \leq t \}, \\
V = \{ f \in C(J, \mathbb{R}) : \|f\| < \infty \}.
\]

\( y_\delta = t + \delta \sin t \) shows, that \( L_2^\delta \) is false for any \( \lambda \in [0, \infty) \) – though \( |J| = \infty \). One can show however that \( N_2 (14, 3) \) is true (Landau’s \( L_2 (4, (1/2)|J|) \) for compact \( I \) and \( \| \|_\infty \) gives \( L_2^\delta (4) \) for \( J, \| \|_\infty \), then \( N_2^\delta (1) \) by Lemma 1.12 and the above remark; apply this to \( f = y/t \). For \( |J| < \infty \) a simpler example follows, with \( \| \|_p \), from Example 1.2, Lemma 1.12 and Proposition 2.1.
LEMMA 1.14 \( N_n(K, \sigma) \) implies \( L_n(\lambda(\tau), \tau) \) for each real \( \tau > 0 \) and \( n \geq 2 \), with

\[
\lambda(\tau) = (K + \varrho^{n})^{\max(\varrho^{n}, \varrho^{n(n-1)})}, \quad \varrho := \frac{\tau}{\sigma}. \quad (1.17)
\]

**Proof** If \( y_m := \| y^{(m)} \| \leq \varepsilon^n m y_n + K_{n,m} \varepsilon^{-m} y_0 \) for \( 0 < \varepsilon \leq \sigma \), \( 0 \leq y_n \leq b \), \( y_0 \leq a < \infty, 0 < b \), then if \( \varepsilon := -(a/b)^{1/n}, \leq \sigma \), one gets

\[
y_m^n \leq \left( \left( \frac{\sigma}{\tau} \right)^{n-m} + K_{n,m} \left( \frac{\tau}{\sigma} \right)^{m} \right)^{n} a^{n-m} b^m \quad \text{if } \left( \frac{a}{b} \right)^{1/n} \leq \tau, \quad (1.18)
\]

which is \( L_n \), with (1.17).

LEMMA 1.15 If \( J, V, \| \| \) are as in Lemma 1.7 with (1.10), then for any \( \lambda, \tau, \sigma \in (0, \infty) \), \( L^w_2(\lambda, \tau) \) implies \( N_2(K, \sigma) \) with

\[
K = \max \left\{ \frac{\lambda}{4}, \frac{\sigma}{C_1} \frac{C_2}{C_1} (\tau^{-2} + 4/|J|^2) \right\}. \quad (1.19)
\]

**Proof** If, with \( y_m := \| y^{(m)} \|, \), \( y_0/y_2 \leq \tau^2 \) with \( y_2 > 0 \), then \( y_1^2 \leq \lambda y_0 y_2 \leq (\lambda/4)(\varepsilon y_2 + (1/\varepsilon)y_0)^2 \) implies even \( N_2(\lambda/4, \infty) \).

If \( 0 \leq y_2 < y_0 \tau^{-2} \), (2.8) of Section 2 and (1.10) give

\[
y_1 \leq c \left( y_2 + \frac{4}{|J|^2} y_0 \right), \quad c := C_2/C_1. \quad (1.20)
\]

So

\[
y_1 \leq c(y_0 \tau^{-2} + 4|J|^{-2} y_0) \leq \varepsilon y_2 + \sigma c(\tau^{-2} + 4|J|^{-2}) y_0/\sigma \leq \varepsilon y_2 + \frac{K}{\varepsilon} y_0
\]

if \( 0 < \varepsilon \leq \sigma \), with \( K = \sigma c(\tau^{-2} + 4|J|^{-2}) \).

**Remark** Lemmas 1.15 and 1.14 give a new proof of \( L^w_2(\lambda, \tau) \Rightarrow L_2(\tilde{\lambda}, \tau) \) of Lemma 1.7, but only with \( \tilde{\lambda} > \lambda \) in general, even for optimal \( \sigma \).

**Question:** Can one extend Lemma 1.15 to more general norms resp. to \( L^w_n \Rightarrow N_n, n \geq 2 \)? (For norms as in proposition 2.1, \( N_n \) always holds.)
DEFINITION 1.16 We say that a Halperin–Pitt inequality $H_n = H_n(S) = H_n(S, V, \| p \|)$ holds if with $S : (0, \infty) \to [0, \infty)$ one has for all $y$ as after (1.2)

$$\| y^{(m)} \| \leq \varepsilon \| y^{(n)} \| + S(\varepsilon) \| y \|, \quad 0 < m < n, \quad 0 < \varepsilon. \quad (1.21)$$

An asymptotic Halperin–Pitt inequality $H_n^a(S)$ holds if for each $y \in C^{(n)}(J, X)$ with $y^{(m)} I \in V$ for $0 \leq m \leq n$ and all compact intervals $I \subset J$ there is a compact $I(y) \subset J$ such that

$$\| y^{(m)} I \| \leq \varepsilon \| y^{(n)} I \| + S(\varepsilon) \| y I \|, \quad 0 < m < n, \quad 0 < \varepsilon,$n

$I(y) \subset \text{compact } I \subset J.$ \quad (1.22)$

The pointwise $H_n^a$ is defined as $H_n^a$, but with $S$ depending on $y$, similarly for $H_n^*$.  

Remark If (1.21) holds only for $0 < \varepsilon \leq \text{some } \sigma < \infty$, with $S(\varepsilon) := S(\sigma)$ for $\varepsilon > \sigma$ it holds for all $\varepsilon > 0$, we can assume $\sigma = \infty$, $H_n^a \equiv H_n$.

Such inequalities seem to have been considered first by Halperin and Pitt [11, Theorem 1, (2.1.2), Theorems 3 and 4] in their study of the closedness of ordinary differential operators and their adjoints in $L^p$.

Lemma 1.17 For any $n \geq 2$, $H_2$ implies $H_n$ with suitable $S$.

Proof Similar as for Lemma 1.10, with

$$S_{n+1}(\varepsilon) := S_n\left(\frac{\varepsilon}{2}\right) + 2S_2\left(\frac{\varepsilon}{2}\right) S_n\left(1/\left(2S_2\left(\frac{\varepsilon}{2}\right)\right)\right) \quad \text{if } 0 < \varepsilon \leq 1/2. \quad (1.23)$$

Also similarly as Lemma 1.3, with (1.23), one gets

Lemma 1.18 For $n \geq 2$, $H_2^a$ implies $H_n^a$ with suitable $S$.

Lemma 1.19 For $n \geq 2$, $H_2^a$ implies $H_n^*$.  

Collecting some of the above results, one has for $n \geq 2$

$$L_2^a \Rightarrow L_n^a \iff N_n^a \Rightarrow (\ast) L_n^s \iff N_n^s \Rightarrow N_n \Rightarrow L_n \Rightarrow L_n^w$$

$$N_n \Rightarrow H_n \equiv H_n^s \Rightarrow H_n^* \quad N_n^a \Rightarrow H_n^a \Rightarrow H_n^{*a} \Rightarrow (\ast) H_n^*, \quad (1.24)$$
where \( N^a_n \) is defined as \( L^a_n, H^a_n \) with \( \sigma = \infty \) and \( \|y^{(n)}I\| \neq 0 \); for (*) the assumption (1.8) is needed, and only (1.7) holds;
\[
L_2 \iff L^w_2 \iff N_2 \text{ if (1.10) holds.} \tag{1.25}
\]

Question: For what \( V, \|\| \) is \( L^w_n \Rightarrow H_n \) true, at least for \( n = 2 \)?

2 INEQUALITIES FOR WEIGHTED \( L^p \)-NORMS

In this section \( J, X \) are as in the introduction, \( w: J \rightarrow (0, \infty) \) is a Lebesgue measurable weight function with
\[
C_\delta := \sup \left\{ \frac{w(s)}{w(t)} : s, t \in J, |s - r| \leq \delta \right\}, \quad 0 < \delta \leq \infty, \tag{2.1}
\]
\[
\|f\|_{p,w} := \left( \int_J |f|^p w \, dt \right)^{1/p} \quad \text{resp.} \quad \mu_L - \sup_J |f| \tag{2.2}
\]
for Bochner–Lebesgue measurable \( f: J \rightarrow X, 1 \leq p \leq \infty, \mu_L \) Lebesgue measure; \( \|\|_p := \|\|_{p,1} \).

**Proposition 2.1** If \( 1 \leq p \leq \infty \) and \( w \) is a weight function with \( C_{\delta_0} < \infty \) for some \( 0 < \delta_0 \leq \infty \), then \( \|\|_{p,w} \) satisfies an asymptotic Nirenberg inequality, i.e. for any \( J, y \in C^2(J, X) \), \( I \) compact interval \( \subset J \) one has
\[
\|y'I\|_{p,w} \leq \varepsilon \|y''I\|_{p,w} + \frac{K}{\varepsilon} \|yI\|_{p,w} \quad \text{for } 0 < \varepsilon \leq \sigma \tag{2.3}
\]
with \( \sigma, K \) given by (2.10) resp. (2.13), (2.14).

Independent of \( p \) and \( |I| \geq 1 \) one can use
\[
K = 32C_{\delta_0}^2, \quad \sigma = (1/2) \min(\delta_0, |I|), \quad 1 \leq p \leq \infty. \tag{2.4}
\]

The case \( X = \mathbb{C}, p = 2, w \equiv 1 \) is due to Nirenberg [26, p. 671, (1)], also for functions of several variables; see also [25, p. 11 and p. 22] for \( p = 2 \), and [25, pp. 30–33 and p. 37], recalling (1.25).

**Remark 2.2** (a) In (2.3) the \( \|y''I\|_{p,w} \) can be \( \infty \) if \( p > 1 \) (see Corollaries 2.5/6). Also, \( \|y''I\| > 0 \) is not needed.

(b) For \( 0 \notin \bar{J} \) unbounded and \( \omega \) real, \( w = t^{\omega} \) or \( e^{\omega t} \) we have finite \( C_\delta \). Here \( C_\delta \rightarrow 1 \) has \( \delta \rightarrow 0 \), so \( K \rightarrow 2^{4 - 2/p} \) resp. 1 if \( n_I \rightarrow \infty \) resp. \( \delta_0 \rightarrow 0 \).
(c) In proposition 2.1 bounded $J$ or $\delta_0 = \infty$ are also admissible; but then $C_{\delta_0} < \infty$ implies $0 < \inf_J w \leq \sup_J w < \infty$, one can assume $w \equiv 1$. See example 2.3.

(d) For $p = \infty$, $w \equiv 1$ the $K = 1$ of (2.10) cannot be improved by Remark 2.9. See also [25, p. 11 and p. 22].

(e) For $p = \infty$ and $J = [\alpha, \beta)$ with $\beta \leq \infty$, proposition 2.1 can be extended to arbitrary decreasing $w : J \to (0, \infty)$ and $I = [\alpha, x)$, $\alpha + \delta \leq x \leq \beta$, with $\sigma = (1/2)C\delta\omega$, $K = (C\omega)^2$, $\omega := w(\alpha)/w(\alpha + \delta)$, $\delta \in (0, |J|)$, $C \geq 1$.

True also for $p < \infty$?

**Example 2.3** (2.3) becomes false for $J = [0, 1)$, $w = 1/(1 - t)$, $y = 1 - t + \eta \sin t$, $1 \leq p \leq \infty$: $C_\delta = \infty$. See Remark 2.2(e).

**Example 2.4** For general norms Proposition 2.1 becomes false:

For any interval $J, X = \mathbb{R}$, $V = \text{piecewise continuous bounded functions}$: $J \to \mathbb{R}$ one can construct $f_n \in C^2(J, \mathbb{R})$ with compact support and $c_n \in (0, 2^{-n}]$ such that with $\|f\| := \sum_{n=1}^{\infty} c_n|f(r_n)|, r_n = \text{rationals} \in J$, one has (1.1), (1.8), $\|f_n\| \to 0$, $\|f'_n\| \to 0$, $\|f''_n\| \to 1$, and $(\|f_n\|/\|f''_n\|)^{1/2} \to 0$.

So even $L^2, H_2$, and therefore $L^2, L^2, L^2, L^2, N^2, N^2, N_2, H^2$ are here false, for any finite $\lambda, \tau, K, \sigma, S$. See Example 3.5.

**Proof of Proposition 2.1** With the fundamental theorem of calculus for vector-valued functions ([15, Theorem 3.8.6, p. 88]) one shows for $y \in C^2(J, X)$

$$y(u) = y(x) + (u - x)y'(x) + \int_x^u \int_x^t y''(s) \, ds \, dt, \quad (2.5)$$

$$u, x \in I := [b - a, b + a] \subset J.$$  

With $v \in I$ one gets

$$y(v) - y(u) = (v - u)y'(x) + \int_u^v \int_x^t y''(s) \, ds \, dt. \quad (2.6)$$

If $v = b + z, u = b - z$, $0 \leq z \leq a$, integration with respect to $z$ over $[0, a]$ yields

$$\int_b^{b+z} y \, dz - \int_{b-z}^{b} y \, dz = a^2 y'(x) + \int_{b-z}^{a} \int_{b-z}^{b+z} y''(s) \, ds \, dt \, dz,$$

$$\|y'(x)\| \leq \frac{4}{|I|^2} \int_I |y| \, ds + \int_I |y''| \, ds, \quad x \in I \text{ compact} \subset J \quad (2.7)$$

(see Brown and Hinton [5], with 9 instead of 4 and $X = \mathbb{R}$).
If \( v = b + a, u = b - a \) in (2.6), one gets, for \( x \in I \),

\[
2a\|y'(x)\| \leq 2\|y\|_{\infty} + \int_{b-a}^{b+a} |t - x| dt \|y''(x)\|_{\infty} \leq 2\|y\|_{\infty} + (a^2 + (x - b)^2)\|y''(x)\|_{\infty}
\]

or

\[
\|y'(x)\| \leq \frac{|I|}{2} \|y''(x)\|_{\infty} + \frac{2}{|I|} \|yI\|_{\infty}, \quad x \in I \subset J. \tag{2.8}
\]

**Case \( p = \infty \):** For compact intervals \( M, I \) with \( M \subset I \subset J \) and \( |I| \leq \delta_0 \), (2.8) gives, on \( M \)

\[
w|y'| \leq \frac{|M|}{2} \left( \sup_M y'' \right)_{\infty} + \frac{2}{|M|} \left( \sup_M y \right)_{\infty}
\]

\[
\leq C_{|M|} \left( \frac{|M|}{2} \|y''\|_{\infty,w} + \frac{2}{|M|} \|y\|_{\infty,w} \right)
\]

\[
\leq C_{|I|} \left( \frac{|M|}{2} \|y''\|_{\infty,w} + \frac{2}{|M|} \|y\|_{\infty,w} \right).
\]

Since this holds for any such \( M \subset I \), one gets, with \( \varepsilon = (1/2)C \cdot |M| \), and now any compact interval \( I \subset J \)

\[
\|y'I\|_{\infty,w} \leq \varepsilon \|y''I\|_{\infty,w} + \frac{K}{\varepsilon} \|yI\|_{\infty,w}, \quad 0 < \varepsilon \leq \sigma, \tag{2.9}
\]

\[
K = C^2, \quad \sigma = \frac{1}{2} C \delta, \quad \delta := \min(|I|, \delta_0), \quad C_\delta \leq C \text{ arbitrary } < \infty. \tag{2.10}
\]

**Case \( 1 \leq p < \infty \):** Since \((u + v)^p \leq 2^{p-1}(u^p + v^p)\) for \( u, v \geq 0 \), (2.7) implies on \( I \) with H"older

\[
w|y'|^p \leq 2^{p-1} \left( 4^p |I|^{-2p} \left( \int_I |y| ds \right)^p + \left( \int_I |y''| ds \right)^p \right) \cdot \sup_I w
\]

\[
\leq 2^{p-1} \left( 4^p |I|^{-2p} \int_I |y|^p \sup_I w ds + \int_I |y''|^p \sup_I w ds \right).
\]

\[
\cdot |I|^{p(1-1/p)},
\]
\[
\int_I |y'|^p w \, ds \leq 2^{p-1} C_{|I|} \left( 4^p |I|^{-p} \int_I |y'|^p w \, ds + |I|^p \int_I |y''|^p w \, ds \right) \tag{2.11}
\]
provided $|I| \leq \delta_0$; (2.11) holds also for $p = 1$.

If now $M$ is any compact interval in $I$, subdivide it into $n$ intervals $I_j$ of length $|M|/n < \delta_0$. Adding the inequalities (2.11) for these $I = I_j$, writing $I$ instead of $M$ and using $(u + v)^{1/p} \leq u^{1/p} + v^{1/p}$, one gets

\[
\|y'/I\|_{p,w} \leq 2^{1-1/p} C_{|I|/n} \frac{|I|}{n} \|y''I\|_{p,w} + 2^{3-1/p} C_{|I|/n} \frac{n}{|I|} \|yI\|_{p,w}. \tag{2.12}
\]

Define \[
\sigma = 2C^{1/p} \frac{|I|}{n_I}, \quad \text{with } n_I \in \mathbb{N}, \quad C_{|I|/n_I} \leq 2C < \infty \tag{2.13}
\]
($C_0 < \infty$ implies $C_0 < \infty$ for any $0 < \delta < \infty$, so everything above is defined).

Then if $0 < \varepsilon \leq \sigma$, there is $m \geq n_I$ with $|I|/(m + 1) < \varepsilon \leq n_I/m$, so $n = m + 1$ and $C_{|I|/n}$ replaced by $2C$ in (2.12) yields

\[
\|y'/I\|_{p,w} \leq \varepsilon \|y''I\|_{p,w} + \frac{K}{\varepsilon} \|yI\|_{p,w} \quad \text{if } 0 < \varepsilon < \sigma,
\]
with

\[
K = 16C^{2/p} \left( 1 + \frac{1}{n_I} \right), \quad 1 \leq p < \infty. \tag{2.14}
\]

If one chooses $n_I$ with $|I|/n_I \leq \delta_0$, one gets (2.4) from (2.13) and (2.14), resp. (2.10).

**Special case** $w = 1$: Then $\delta_0 = \infty$, $C_0 = 1$, $n_I = 1$; $\sigma = |I|$ and $K = 32$ are possible by (2.10), (2.13), (2.14) for $1 \leq p \leq \infty$, for $p = 1$ even $K = 8$, and $K = 1$ for $p = \infty$ (see Remark 2.2(d)).

Proposition 2.1 yields, with $2C \geq C_{\min(|I|,\delta_0)} \geq C_{\min(|I|,\delta_0)}$ (no continuity of $C_0$ in $\delta$ is needed)

**Corollary 2.5** In Proposition 2.1 one can omit the $I$ in (2.3), with $K, \sigma$ of (2.10) if $p = \infty$, resp. (2.4), and $|I|$ replaced by $|J|$ (also if $\delta_0$ or $|J| = \infty$).
Special case $|J| = \delta_0 = \infty$, i.e. $w \equiv 1 \equiv C_\delta$: Then $\sigma = \infty$, so even $N_2^w$ and with Lemmas 1.11, 1.12 all $N_2^{\alpha_n}$, $L_2^w$ are true, $n \geq 2$, $\|\| = \|\|_{p}$, $1 \leq p \leq \infty$, $V = L^p(J, X)$. For $J = \mathbb{R} = X$ optimal $\lambda = \lambda_{n,m}$ for $L_2^w$ have been determined by Kolmogorov [16] for $p = \infty$, they are upper bounds for the $\lambda_{n,m}$ by Stein [28, Theorem 2], for $1 \leq p < \infty$. However even for monotone decreasing $w$ the $L_2^w$ is in general false by Example 1.13.

**Corollary 2.6** For any interval $J$, $1 \leq p \leq \infty$, $w$ as in Proposition 2.1 or Remark 2.2(e), $n \geq 2$ and $y \in C^{(n)}(J, X)$, if $y$ and $y^{(n)}$ belong to $L_1^w := \{f$ Bochner–Lebesgue measurable: $J \to X, \|f\|_{p,w} < \infty\}$, then $y^{(m)} \in L_1^w$, $0 < m < n$.\n
**Proof** Corollary 2.5 and Lemma 1.10.

For $X = \mathbb{C}$ and $w \equiv 1$ this has been shown by Halperin and Pitt [11, Theorems 1 and 3], $p = \infty = |J|$ already by Hardy and Littlewood [12, p. 422, Theorem 3(a)], and Esclangon [7]. $J = \mathbb{R}$, $1 \leq p < \infty$, $X = \mathbb{C}$ and $w \equiv 1$ can also be found in Stein [28, Theorem 3].

There are two ways of getting Landau inequalities from Proposition 2.1: either $N_2 \Rightarrow N_n \Rightarrow L_n$, or $N_2 \Rightarrow L_2 \Rightarrow L_n$ (Lemmas 1.10, 1.14, 1.5).

The second method gives nicer formulas, we prefer the first, it gives in general better $\lambda_n$:

**Proposition 2.7** For $J$, $p$, $w$ as in Proposition 2.1, $n \geq 2$, and any $0 < \tau < \infty$ one has

$$\|y^{(m)}\|_{p,w}^{n} \leq \lambda_n(\tau)\|y\|_{p,w}^{n-m} b^m, \quad 0 < m < n$$

(2.15) for any $y \in C^{(n)}(J, X)$, $I$ compact $\subset J$, $0 \leq \|y^{(n)}\|_{p,w} \leq b$ with $0 < b < \infty$,

$$\left(\|y\|_{p,w}/b\right)^{1/n} \leq \tau,$$  \hspace{1cm} (2.16)

$$\lambda_n(\tau) = \left(2^{n-4}(4K)^{\alpha} + \left(\frac{\sigma}{\tau}\right)^n\right) \max \left(\left(\frac{\tau}{\sigma}\right)^n, \left(\frac{\tau}{\sigma}\right)^{n(n-1)}\right),$$ \hspace{1cm} (2.17)

$$K = K(p, w, I) = \begin{cases} 32C^2/p, & \text{if } 1 \leq p < \infty, \\ 4C^2, & \text{if } p = \infty, \end{cases}$$ \hspace{1cm} (2.18)

$$2C \geq C_\delta, \quad \delta := \min(|I|, \delta_0),$$

$$\sigma = \sigma(p, w, I) = \begin{cases} C^{1/p}\delta, & \text{if } 1 \leq p < \infty, \\ C\delta, & \text{if } p = \infty, \end{cases}$$ \hspace{1cm} (2.19)
**Proof**  Since with $V := L^p_w(I, X)$ of Corollary 2.6 any $y \in C^{(2)}(I, X)$ with $y^{(j)} \in V$ for $0 \leq j \leq 2$ can be extended to an $z \in C^{(2)}(J, X)$, Proposition 2.1 gives $N_n(K, \sigma)$ for this $V$ and $\|\cdot\|_{p, w}$ restricted to $I$, with $K, \sigma$ of (2.18), (2.19). So Lemma 1.10 gives $N_m$ then Lemma 1.14 the $L_n$, with (2.15), (2.16). Here (2.13), (2.14), for minimal $n_j$ with $|I|/n_j \leq \delta$ and $2C \geq C_\delta$ one gets $2|I|/n_j \geq \delta := \min(|I|, \delta_0)$ for $p < \infty$, i.e. (2.19).

**Corollary 2.8**  If $\delta_0 < \infty$ or $|J| < \infty$, Proposition 2.7 remains true if there everywhere $I$ is replaced by $J$.

If $|J| = \infty$ and $w \equiv 1$, (2.15) holds with $I = J$ and $y$ as there, but with $\\tau = \infty$ (i.e. without (2.16)) and

$$\lambda = \lambda_n(\infty) = n \left( \frac{n}{n-1} 2^{n-4} (4K)^{(3)} \right)^{n-1}, \quad K \text{ of (2.18)}, \quad C = \frac{1}{2}. \quad (2.20)$$

**Proof**  The first part follows as Corollary 2.5. For the second part one can take $\\tau = t/|J|$ with fixed $t \in (0, \infty)$; $I \to J$ gives then, for any $y$ as before (2.16) and $\delta_0 = \infty$, inequality (2.15) without $I$ (also if some terms are $\infty$, with $0 \cdot \infty := 0$); instead of $\lambda_n(\\tau)$ one gets, with suitable $K', \lambda = (K' + 1/s)^n$ max($s, s^n - 1$), $s := (2^{1/p} t)^n$ if $p < \infty$ resp. $(2t)^n$ if $p = \infty$. The minimum with respect to $s \in (0, \infty)$ gives (2.20).

**Remark 2.9**  (a) The variable in $\\tau$ in (2.16) gives less flexibility than might appear: $L_n(\lambda_0, \tau_0)$ already implies $L_n(\lambda_0 \cdot \max(1, (\\tau/\tau_0)^{n(n-1)}), \tau)$ for any $\\tau > 0$.

(b) In Landau's case $p = \infty$, $w \equiv 1$, $\\tau = (1/2)|I|$, $X = \mathbb{R}$, for $n = 2$ our (2.17)–(2.19) give $\lambda_2 = 4$, which is optimal by Landau [20, Satz 2]; for $n \geq 3$ our $\lambda_n$ are much smaller than the $\lambda_n = 2^{n^2}$ of Landau [21, Hilfssatz 3].

(c) Even for $n = 2$, $p = \infty$, $w \equiv 1$ Proposition 2.7 is more general than Landau's result: $y''(t)$ need not exist everywhere, and in (2.15) and $\|y''I\|_{\infty} \leq b$ only the $\mu_L - \sup$ is used.

(d) Corollary 2.8 says that $L_n(\lambda(\tau), \tau)$ is true for any $J, \tau$ and $\|\cdot\|_{p, w}$ with $\delta_0 < \infty$ or $|J| < \infty$, and $\lambda(\tau)$ independent of $|J| \geq \delta_0$; however $\lambda_n(\tau) \to \infty$ as $|J| \to 0$ as it should: Example 2.10.

(e) For $|J| = \infty$ and $w \equiv 1$, Corollary 2.8 gives even the strong $L_n(\lambda(\infty))$ with explicit $\lambda$ for $\|\cdot\|_p$, $1 \leq p \leq \infty$ (also "Special case" after Corollary 2.5); for $p = \infty$ one has $\lambda_2(\infty) = 4$, which is optimal by Matorin [24] for $J = [0, \infty)$; for $p = \infty$ and $J = \mathbb{R}$, $\lambda = 2$ is optimal by...
Hadamard and Kolmogorov, $\lambda = 1$ for $p = 2$ by Hardy–Littlewood–Polya [25, p. 5]. More can be found in [10, 18, p. 229, 25, pp. 2–7, 28, 30, p. 4 and p. 9].

For increasing $w$, $|J| = \infty$ and $1 \leq p < \infty$ a strong $L^s_\omega$ has been shown by Goldstein–Kwong–Zettl [8, p. 23, 25, p. 37 (84.3)]; for $w = t^\beta$ see [5, 18, p. 238 Theorem 4, 25, p. 51 no. 102]; for decreasing $w$ this is false by Example 1.13. See also Remarks 2.17(c) and (f), Corollaries 2.11, 2.19, Examples 1.2, 2.3, 2.10.

(f) Proposition 2.7 and the first part of Corollary 2.8 hold also for $p = \infty$ and $J, w$ as in Remark 2.2(e), with suitable $\lambda$.

(g) If a strong $L^s_\omega$ holds for $J$ and the seminorm $\| \|$ (as in (e)), then for non-negative integer $m$ the $L^s_n$ is also true (with the same $\lambda$) for the seminorm $\|f\|_{[m]} := \sum_{j=0}^n \|f_j\|$ (Hölder, $p = n/(n-k)$); special cases have been treated by Upton [30]. The same holds for the later asymptotic $L^a_n$ of Proposition 2.16.

Example 2.10 For no $n \geq 2, 1 \leq p < \infty$ and fixed $\lambda, \tau$ a $L_n(\lambda, \tau)$ holds for arbitrary $J, J = [0, e], y = t (\sin \eta \sin t)$ if $n = 2$.

For applications to differential equations, we need asymptotic Landau inequalities; under additional assumptions one gets one already from Proposition 2.7:

Corollary 2.11 For $J, p, w, n$ as in Proposition 2.1 or Remark 2.9(f), to each $y \in C^{(n)}(J, X)$ with $\|y\|_{p, w} < \infty$ and $y^{(n)} \neq 0$ there exist $\lambda(y) < \infty$ and a compact $I(y)$ such that

$$\|y^{(m)}\|_{p, w} \leq \lambda(y)\|y\|_{p, w}^{n-m}\|y^{(n)}\|_{p, w}^m, \quad I(y) \subset I \subset J, \quad 0 < m < n.$$  

(2.21)

If $|J| = \infty$ and $w \equiv 1$, then $\|y\|_p = O(|I|^n)$ suffices for (2.21); if even $\|y\|_p = o(|I|^n)$, then any $\lambda(y) > \lambda_n(\infty)$ of (2.20) is possible in (2.21), independent of $y$.

This follows from Proposition 2.7 with $\tau(y) = (\|y\|_{p, w}/b_0)^{1/n}$ with $b_0 = \|y^{(n)}I(y)\|_{p, w} > 0$ for some compact $I(y), 0 < \delta_0 \leq |I(y)|$.

If $1 < p < \infty$, one gets e.g.

$$\lambda(y) = \left(2^{n-4}(4K)^{\frac{n}{2}} + \frac{1}{s}\right)^n \cdot \max(s, s^{-1}), \quad \text{with } s \geq s(y) := \left(\frac{\tau(y)2^{1/p}}{\delta_0 C_{b_0}^{1/p}}\right)^n,$$
with $K$ of (2.18) with $C = C_{\delta_0}/2$. This works also in the case $\|yI\| = O(|I|^{n})$, $\delta_0 = \infty$. For the case $o(|I|^{n})$ one can argue as in the proof of the second part of Corollary 2.8.

The last statement of Corollary 2.11 follows for $p = \infty$ also (except for the explicit $\lambda_n(\infty)$) from results of Gorny [9, 25, p. 7], or Redheffer and Walter [27].

For $f \in L^p_{\text{loc}}(J, X)$ and $w$ as before (2.1) the weighted Stepanoff norm is defined by

$$\|f\|_{s^p_w} := \sup\{\|fI\|_{p,w}; |I| = 1, \text{ interval } I \subset J\}, \quad 1 \leq p < \infty. \quad (2.22)$$

This definition and the above results yield

**Corollary 2.12** If $|I| = \infty$ and $1 \leq p < \infty$, then Corollary 2.5, Corollary 2.6 and Proposition 2.7 (with $|I| = 1$ in (2.4), (2.10), (2.13), (2.18)) hold also for $\|f\|_{s^p_w}$ instead of $\|f\|_{p,w}$ (also in (2.16)).

A strong Landau inequality $L^*_n$ for Stepanoff-norms can be found in Upton [30], for $J = \mathbb{R}$, $X = \mathbb{C}$, $w \equiv 1$.

**Corollary 2.13** For $J = [\alpha, \infty)$ resp. $J = [-\infty, \infty)$, $w$ as in Proposition 2.1, to $y \in C^{(n)}(J, X)$ with $y^{(n)} \not= 0$ and Stepanoff-norm $\|y\|_{s^p_w} < \infty$, there exist $\lambda(y) < \infty$ and a compact interval $I(y) \subset J$ such that

$$\|y^{(m)}I\|_{s^p_w}^{n-m} \leq \lambda(y)\|yI\|_{s^p_w}^{n-m}\|y^{(n)}I\|_{s^p_w}^{m}, \quad I(y) \subset I \subset J, \quad 0 < m < n. \quad (2.23)$$

**Proof** By assumption there is $t_0$ with $b_0 := \|y^{(n)}I_{t_0}\|_{s^p_w} > 0$, $I_t := [t, t + 1]$, so $\|y^{(n)}I\|_{s^p_w} \geq b_0$ if $I \supset I_{t_0} =: I(y)$. Furthermore $\|yI_{t}\|_{p,w} \leq \|yI\|_{s^p_w} =: b_0 < \infty$ for any $t \in J$. With $b(t) := \max(\|y^{(n)}I_{t}\|_{p,w}, b_0)$ one has

$$\|yI_{t}\|_{p,w}/b(t)\|yI\|_{s^p_w} \leq b_0 < \infty$$

for any $t \in J$.

So Proposition 2.7 gives, with $\delta := \min(1, \delta_0)$, $\|y^{(m)}I_t\|_{p,w}^{n-m} \lambda_n(\tau_0)\|yI\|_{s^p_w}^{n-m} \max(\|y^{(n)}I\|_{s^p_w}^{n-m}, b(t))$ for any $t \in J$, $I_t \subset I$.

For $I(y) \subset I$ this yields (2.23), with $\lambda(y) = \lambda_n(\tau_0)$ only depending on $\tau_0, p, \delta$.

**Lemma 2.14** If $1 \leq p \leq \infty$, $w : J \to (0, \infty)$ with $w(s) \geq w(t)$ if $s \leq t$, $J = [\alpha, \beta)$, $x \in C^{(2)}(J, X)$, $y_j(x) := \|y(t)^{[\alpha, \beta]}\|_{p,w}$ for $x \in J$, and $y_2 \not= 0$, one has for $\beta = \infty$

$$\lim_{x \to \infty} y_0(x)/y_2(x) \leq \begin{cases} (2p)^{-1/p}, & \text{if } 1 < p < \infty, \\ 1/2, & \text{if } p = 1, \\ 1/2, & \text{if } p = \infty. \end{cases} \quad (2.24)$$

If $\beta < \infty$, at least $\lim_{x \to \beta} y_0(x)/y_2(x) \leq \chi = \chi(p, |J|, w, \rho) < \infty$. 

Proof Since $w$ and $y_2$ are monotone, the limits $w(\infty) \geq 0$ and $0 < y_2(\infty) \leq \infty$ are defined. We prove only the case $w \equiv 1$ needed below.

If $p < \infty$, to $\varepsilon > 0$ there is $C = 2((1 + \varepsilon)1/p - 1)^{-p}$ with

$$
(u + v)^p \leq (1 + \varepsilon)u^p + Cv^p, \quad u, v \in [0, \infty).
$$

(2.25)

This and (2.5) with $x = \alpha$ yields, with $A := \|y(\alpha)\|, B := \|y'(\alpha)\|$

$$
\|y(x)\|^p \leq C(A + (x - \alpha)B)^p + (1 + \varepsilon)(x - \alpha)^p \left(\int_{\alpha}^{x} |y''| \, dt \right)^p 
\leq C(A + (x - \alpha)B)^p + (1 + \varepsilon)(x - \alpha)^{p+p/q} \int_{\alpha}^{x} |y''|^p \, dt.
$$

(2.26)

Integrating, one gets for $x \in J$ and $1 \leq p < \infty$

$$
y_0(x) \leq C^{1/p}(A + (x - \alpha)B)(x - \alpha)^{1/p} + \left(\frac{1 + \varepsilon}{2p} \right)^{1/p} (x - \alpha)^2 y_2(x).
$$

(2.27)

Since $\varepsilon$ is arbitrary, one gets (2.24) for $p < \infty$.

$p = \infty$ has been shown in [3, (2.13)].

If $\beta < \infty$, (2.27) resp. (2.26) gives at least $\lim y_0/y_2 < \infty, 1 \leq p \leq \infty$.

Remark 2.15 (a) At least for $p = 1$ and $\infty$ the constants "1/2" in (2.24) cannot be improved; see also (2.13) in [3].

(b) Lemma 2.14 becomes false for increasing $w$, any $p$ (see Remark 2.17(e)).

(c) For $J = \mathbb{R}$ one can show that (2.24) still is true, with $\alpha = 0$ and $y_j(x) := \|y^{(j)}(-x, x)\|_{p,w}$.

(d) For Stepanoff-norms one has $\lim y_0(x)/(x^2 y_2(x)) \leq 1/2$ for $1 \leq p < \infty, w$ as in Lemma 2.14, with $I_x := [\alpha, x]$ resp. $[-x, x]$ and $y_j(x) := \|y^{(j)}I_x\|_{s^w}$.

Proposition 2.16 If $J = [\alpha, \beta], n \geq 2, 1 \leq p \leq \infty, y \in C^{(n)}(J, X)$ with $y^{(n)} \neq 0, y_j := \|y^{(j)}[\alpha, x]\|_p, \varepsilon > 0$, there exist $x_{\varepsilon,y} \in J$ with

$$
y_m(x)^n \leq (\lambda + \varepsilon)y_0(x)^{n-m}y_2(x)^m, \quad x_{\varepsilon,y} \leq x < \beta, \quad 0 < m < n.
$$

(2.28)

Here for $\beta = \infty$

$$
\lambda = \lambda_n(p) = (\lambda_2(p))^{n/8},
$$

(2.29)
\[ \lambda_2(\infty) = \frac{9}{2}, \quad \lambda_2(p) = \left( 32 \left( \frac{1}{8p} \right)^{1/(2p)} + \left( \frac{p}{2} \right)^{1/(2p)} \right)^2 \]

for \(1 < p < \infty\),

\[ \lambda_2(1) = \frac{(64 + 32\lambda)(1 + \lambda)}{\lambda} \]

\[ A(y) := 2 + 2\|y'(\alpha)\|/\lambda_2(\infty). \]

For \(\beta < \infty\) one has (2.28) only with some \(\lambda = \lambda(n, p, \|I\|, y(\alpha), y'(\alpha), y(x); 1 < p < \infty). \]

The proof follows for \(n = 2\) from Proposition 2.7 with \(w = 1, C = 1/2, \delta_0 = \infty, I = [\alpha, \infty), \sigma = (1/2)^{1/p}|I|\) resp. \((1/2)|I|, \beta = y_2(\alpha), \tau = \sqrt{\chi(p) + \eta}/I|\) by the assumptions, \(y_2(\infty)\) is defined \(\in (0, \infty)\); for \(I < \infty, \tau = \sqrt{\chi + \eta} \infty\) independent of \(I, |I| \geq x, y = 0 \) gives \(\lambda < \infty. \)

Lemma 1.3 gives the general case.

**Remark 2.17**

(a) Proposition 2.16 says that for unbounded \(I\) and \(V = L^n(I, X)\) with \(1 < p \leq \infty\) an asymptotic Landau inequality \(L^n_\lambda\) is true. For \(p = \infty\) this generalizes Lemma 2.5 of [3]. For \(p = 1\) or bounded \(I\) only a "pointwise \(L^n_\lambda\)" holds, the \(\lambda\) depends on \(y\); in all these cases exist \(y \in C^\infty\) with arbitrarily large \(\lambda\). These examples and Corollary 2.8 show also that \(L^n_{\lambda} \Rightarrow L^n_{\lambda}\) is false for \(p = 1, |I| = \infty, n = 2. \)

(b) (2.30) yields \(\lambda_2(1+) = 144\frac{1}{2} < \lambda_2(p) < \lambda_2(\infty)\) for \(1 < p < \infty. \)

(c) Corollary 2.11 gives more general (pointwise) asymptotic Landau inequalities if \(y\) is bounded in some way; for example, if \(y_0(x) = o(x^2)\) in Proposition 2.16, then \(\lambda = \lambda_0(\infty)\) of (2.20) is possible in (2.28), which is in general better than the \(\lambda\) of (2.29). See Remarks 2.9 and after (1.8), and (1.7).

(d) Proposition 2.16 and the remarks hold also for \(I = \mathbb{R}\) with e.g. \(y_j(x) := \|y^{(j)}[-x, x]\|_p, \) and the same \(\lambda_n(p)\) if \(1 < p \leq \infty. \) This, (a), (1.7) and Lemma 1.4 give again \(L^n_\lambda\) (for \(1 < p \leq \infty\)) of Remark 2.9(e).

(e) In all four cases \(I\) bounded or unbounded and \(w\) decreasing or increasing, there exist \(w\) and \(y\) showing that for no \(\lambda < \infty\) and no \(p\) an asymptotic Landau inequality \(L^n_\lambda\) is true for \(V = \mathbb{R}\).

(f) The examples (e) show also that for no \(\lambda\) and \(p\) a strong Landau inequality \(L^n_\lambda\) holds for general \(\|\|p, w\|\) except in the case \(|I| = \infty\) and \(w\) increasing (the \(y^{(j)} \) of (e) are \(\in L^n_w; \) except: [8], Remark 2.9(e)).
For the following Halperin–Pitt inequality we assume:

\[ J \text{ interval } \subset \mathbb{R}, \ V, K \text{-vectorspace } \subset X', \text{ with monotone seminorm } \| \|, \text{ i.e. } \|f\| \leq \|g\| \text{ if } f, g \in V \text{ with } |f| \leq |g|, \text{ and with } 1I \in V \text{ for } I \text{ compact interval } \subset J, \]

\[ \|1I\| \rightarrow 0 \text{ as } |I| \rightarrow 0, \quad (2.31) \]

\[ C_1 \int_I |f| \, dx \leq \|fI\| \text{ if } fI \in V \cap L^1, \ 0 < C_1 \text{ independent of } f, I. \quad (2.32) \]

**Proposition 2.18** For \( J, V \), \( \| \| \) as above and \( 0 < r < \infty \) there exists \( S : (0, \infty) \to (0, \infty) \) such that for any compact interval \( I \subset J \) with \( |I| \geq r \) and \( y \in C^{(2)}(J, X) \) with \( y(I) \tilde{I} \in V, \ 0 \leq j \leq 2 \) (see (0.3)), \( \tilde{I} \) compact \( \subset J \), one has

\[ \|y'I\| \leq \varepsilon \|y''I\| + S(\varepsilon) \|yI\|, \ 0 < \varepsilon < \infty, \ |I| \geq r. \quad (2.33) \]

**Proof** To \( \varepsilon > 0 \) choose \( \delta_\varepsilon \) with \( \|1M\| \leq C_1 \varepsilon \) if \( |M| \leq \delta_\varepsilon \), \( M \) compact interval \( \subset J \), then \( n \) minimal \( \in \mathbb{N} \) with \( |I|/n < \delta := \min(\delta_\varepsilon, r) \), and compact intervals \( I_j \) with \( |I_j| = |I|/n, I = \bigcup I_j \). With (2.7) one gets for \( y \in C^{(2)}(J, X) \) with \( y(I) \tilde{I} \in V \)

\[ \|y'I\| \leq \left\| \sum y'I_j \right\| \leq \sum \|y'I_j\| \leq \sum \|y'I_j\|_\infty \|1I_j\| \leq \sum \|y'I_j\|_\infty C_1 \varepsilon \]

\[ \leq C_1 \varepsilon \sum \left( \int_{I_j} |y''| \, dx + 4(|I|/n)^{-2} \int_{I_j} |y| \, dx \right) \]

\[ \leq C_1 \varepsilon \left( \int_J |y''| \, dx + 4(\delta/2)^{-2} \int_J |y| \, dx \right) \leq \varepsilon \|y''I\| + S(\varepsilon) \|yI\|, \]

with

\[ S(\varepsilon) := \frac{16 \varepsilon}{(\min(\delta_\varepsilon, r))^2}, \text{ with } \|1M\| \leq C_1 \varepsilon \text{ if } |M| \leq \delta_\varepsilon, \quad (2.34) \]

\( M \) compact interval \( \subset J \).

**Special case** \( V = L^1 \): Then one gets Proposition 2.1, \( p = 1, w \equiv 1 \), with \( K = 16 \), \( \sigma = r = |I| \).

**Corollary 2.19** Proposition 2.18 holds for \( \| \|_{p, w}, 1 \leq p < \infty \) and the weight function \( w \) satisfying \( \inf_J w > 0 \) and \( w \) integrable over \( J \), with \( |J| < \infty \).
Proposition 2.18 can be applied to Orlicz-norms (see [17,23,31]): A
\[ \Phi : [0, \infty) \to [0, \infty] \]
will be called an Orlicz-function (OF) iff \( \Phi(0) = 0 \),
\( \Phi \neq 0 \), \( \Phi \neq \infty \) on \( (0, \infty) \), and \( \Phi \) is convex. Then for a measure space
\( (Y, \Omega, \mu) \), \( L^\Phi(\mu, X) := \{ f: Y \to X \mid f \text{ Bochner} - \mu - \text{measurable}, \int \Phi(t|f|) \, d\mu \leq 1 \} \) for some \( t \in (0, \infty) \), Orlicz–Luxenburg norm
\( \|f\|_\Phi := \inf\{s > 0 : \int \Phi(|f|/s) \, d\mu \leq 1 \} \).

For OF \( \Phi \), \( L^\Phi \) is a \( \mathbb{K} \)-vectorspace and \( \|\cdot\|_\Psi \) a monotone seminorm on
\( L^\Phi \), with \( \|f\|_\Phi = 0 \) iff \( f = 0 \) \( \mu \)-a.e. For any OF \( \Phi \), \( \Psi(t) := \sup\{st - \Phi(t) : 0 \leq s < \infty\} \) defines a "conjugate" OF such that for \( f \in L^\Phi(\mu, X), g \in L^\Psi(\mu, \mathbb{K}) \) one has \( fg \in L^1(\mu, X) \) (usual \( L^1 \)) and
\( \int_Y |fg| \, d\mu \leq 2\|f\|_\Phi \|g\|_\Psi \).

For \( Y = \) interval \( J \subset \mathbb{R}, \Omega = \) Lebesgue measurable sets \( \subset J \) and
\( \mu = \) Lebesgue measure \( \mu_L \) we write \( L^\Phi = L^\Phi(J, X) := L^\Phi(\mu_L | J, X) \), then
\( 0 < \|1J\|_\Psi \leq \infty \) if \( 0 < |J| < \infty \), so (2.32) holds with \( C_1 = 1/(2\|1J\|_\Psi) \).
(2.31) is true if \( \Phi(t) < \infty \) for \( 0 < t < \infty \):

**Corollary 2.20** If \( \Phi \) is an OF with \( \Phi(t) < \infty \) for \( 0 < t < \infty \) and \( |J| < \infty \),
then Proposition 2.18 is true for \( \|\cdot\|_\Psi, V = L^\Phi(J, X) \).

Question: Is such an asymptotic Halperin–Pitt inequality also true for
\( |J| = \infty \)? By Ha Huy Bang [10] at least a strong Landau inequality holds
for \( J = \mathbb{R}, X = \mathbb{C} \) and Orlicz-norms.

### 3 ESCLANGON–LANDAU THEOREMS FOR NEUTRAL SYSTEMS

In the following, we consider neutral delay differential-difference systems
\[ \sum_{k=0}^{n} \sum_{j=1}^{m} a_{jk}(t) y^{(k)}(t - t_j) = f(t); \quad (3.1) \]
here \( n \geq 1, m \geq 1, J = [\alpha, \beta) \) with \( -\infty < \alpha < \beta \leq \infty \), \( t_1 = 0 < t_j \leq \tau < \infty \)
for \( 1 < j \leq m, J' := [\alpha - \tau, \beta), f: J \to X \) Banach space over \( \mathbb{K} \), \( a_{jk}: J \to L(X) := \{ \text{continuous linear operators} : X \to X \} \) with operator norm. \( t_j \)
are not more general.

\( y \) is called a solution of (3.1) on \( J \) if \( y \in C^{(n)}(J', X) \) and (3.1) holds a.e.
on \( J \), with \( y^{(n)} \) of (0.5).

Systems of such equations are included: \( a_{jk} = r \times r \)-matrix \( (a_{jk, uv}) \),
y = column vector \( (y_1, \ldots, y_r) \).
Furthermore we assume that \( V \) is a \( \mathbb{K} \)-linear space \( \subset \mathcal{X}^J \) with monotone seminorm \( \| \cdot \| \) satisfying
\[
\| f, g \in V, \quad |f| = |g| \text{ a.e. implies } \| f \| = \| g \|, \tag{3.2}
\]
there is \( D_r < \infty \) with \( \| g_{i, t} \| \leq D_r \| g_{i, t} \| \) for \( 0 \leq t \leq \tau \), \( I \) compact interval with \( I, I, \subset \mathcal{J} \) and \( g : \mathcal{J}' \rightarrow \mathcal{X} \) with \( g_{i, |J} \in V \) and \( g_{i, |J} \in V \),
where \( g_s(s) := g(s - t), I_s := \{ s - t : s \in I \} \).

**THEOREM 3.1** Assume \( m, n, t, J, X, V, \| \| \) as above with (3.2), (3.3); assume further that the coefficients \( a_{ij} \) in (3.1) are bounded on \( J \), with
\[
D_r \cdot \sum_{j=2}^{m} \sup_{J} \| a_m \| < 1, \quad a_1n = 1. \tag{3.4}
\]
Assume finally that a pointwise asymptotic Halperin–Pitt inequality \( H_{\alpha}^{\| \cdot \|} \) holds for \( V, \| \| \) (Definition 1.16). If then \( y \) is a solution of (3.1) on \( J \), with \( fI, y_{i, t} \mid J \in V \) for all compact intervals \( I \subset J \), \( 0 \leq k \leq n, 1 \leq j \leq m \), such that \( \| f_{I, x} \| \) and \( \| y_{I, x} \| \) are \( O(\Psi(x)) \) for \( x \to \beta \) with some non-decreasing \( \Psi > 0 \) for \( I_x = [\alpha, x] \), then also \( \| y_{(k)}(I_x) \| = O(\Psi(x)), 0 < k < n \).

**Proof** With (3.2) and \( \| g \| \leq \| h \| = \| (\| h \|) \| \) if \( |g| \leq |h| \) on \( J, g, h \in V \) one gets for \( \alpha \leq x < \beta \), if \( \| a_{ij}(t) \| \leq A \) for \( t \in J \) (measurability of the \( a_{ij} \) is not needed)
\[
\| (y_{(n)}(I_x) \mid J) \| = \left\| f_{I, x} - \left( \sum_{j=2}^{m} a_m y_{ij}^{(n)} + \sum_{k=0}^{n-1} \sum_{j=1}^{m} a_{jk} y_{ij}^{(k)} \right) \right\| J_x \|
\]
\[
\leq K_1 \Psi(x) + \sum_{j=2}^{m} \sup_{J} |a_m| \cdot \| y_{ij}^{(n)}(I_x) \mid J \|
\]
\[
+ mnA \max\{ \| y_{ij}^{(k)}(I_x) \mid J \| : 1 \leq j \leq m, 1 \leq k \leq n \}. \tag{3.3}
\]
(3.3) and the monotonicity of \( \| \| \) give, for \( 1 \leq j \leq m, 0 \leq k \leq n \)
\[
\| (y_{ij}^{(k)}(I_x) \mid J) \| \leq \| (y_{ij}^{(k)}(\alpha, \alpha + \tau)) \mid J \| + D_r \| (y_{ij}^{(k)}(I_x) \mid J) \|.
\]
So if all \( \| (y_{ij}^{(k)}(\alpha, \alpha + \tau)) \mid J \| \leq B, < \infty \) by assumption, with suitable \( K_1 \) one gets for \( x \in J \)
\[
\| (y_{(n)}(I_x) \mid J) \| \leq K_1 \Psi(x) + \left( \sum_{j=2}^{m} \sup_{J} |a_m| \right) (B + D_r \| (y_{(n)}(I_x) \mid J) \|)
\]
\[
+ mnA (B + \max\{ \| (y_{ij}^{(k)}(I_x) \mid J) \| : 0 \leq k < n \}).
\]
By assumption $2\eta := 1 - D_\tau \sum \sup_j |a_m| > 0$, one has for $x \in J$

$$2\eta \|(y^{(n)} I_x) \|_J \leq m(n + 1) AB + K_1 \Psi(x) + mnA \max_{k<n} \|(y^{(k)} I_x) \|_J.$$  

(3.5)

$H_n^{(a)}$ and Lemma 1.19 imply $H_n^{(a)}$, so to the given $y$ there exist $c(y) \in J$ and $S'(0, \infty) \rightarrow [1, \infty)$ with

$$\|(y^{(k)} I_x) \|_J \leq \varepsilon \|(y^{(n)} I_x) \|_J + S(\varepsilon) \|(y I_x) \|_J,$$

$$c(y) \leq x < \beta, \quad 0 < k < n, \quad 0 < \varepsilon.$$

Choosing $\varepsilon = \eta/(mnA)$, (3.5) yields

$$\|(y^{(n)} I_x) \|_J \leq m(n + 1) AB + K_1 \Psi(x) + mnA \cdot S(\varepsilon) \|(y I_x) \|_J,$$

$$c(y) \leq x < \beta.$$

Since by assumption $\|(y I_x) \|_J = O(\Psi(x))$ and $\Psi$ is non-decreasing $\geq \Psi(\alpha) > 0$, one gets with suitable $K_2 \leq \infty$

$$\|(y^{(n)} I_x) \|_J \leq K_2 \Psi(x), \quad x \in J.$$

$H_n^{(a)}$ gives the same for $y^{(k)}$, $0 < k < n$.

**COROLLARY 3.2** Theorem 3.1 holds in the following cases, with $m, n, J, \ell, x, (3.1), (3.4)$ as there

(a) $\| \|_p, w$ of (2.2), $1 \leq p \leq \infty$, $w$ as in Proposition 2.1, and only $\|(y^{(n)} I_0) \|_J \in V = L^p_w(J, X)$; here $D_\tau = C_\tau$ of (2.1), $I_0 = [\alpha, \alpha + \min_{2 \leq j \ell_j}]$ resp. $\emptyset$.  
(b) $\| \|_p$ Stepanoff-norm $\| \|_p$ of (2.22), $1 \leq p \leq \infty$, $J = [\alpha, \infty)$, $w$ and $D_\tau$ as in (a). 
(c) $\| \|_a$ as in Proposition 2.18 with (3.3); special case: Orlicz-norm $\| \|_\Phi$ with Lebesgue measure as in Corollary 2.20, $|J| < \infty$. (Then $D_\tau = 1$ in (3.4).) 
(d) $\| \|_{\infty, w}, w$ decreasing, (3.4) with $D_\tau = 1$. 
(e) $J = \mathbb{R}$ and $m = 1$, with $I_x = [-x, x]$ (all $\ell_j = 0$, ordinary differential system (3.1), so $D_\tau = 1$ in (3.4)); especially for seminorms as in (a) and (b).
Corollary 3.3  With $m, n, t_{j}, J, X$, (3.1), (3.4) as in Theorem 3.1 and $y$ a solution of (3.1) on $J$ with $y^{(n)}|_{[\alpha - \tau, \alpha]} \in L^{p}$, if $f$ and $y | J \in V = L^{p}_{w}(J, X)$, then also $y^{(k)}|_{J} \in V$, $0 < k < n$, with $p, w$ as in Corollary 3.2(a). A corresponding result holds for $V = \{ g : J \to X \text{ Bochner-Lebesgue measurable} : \|g\|_{S^{p}_{w}} < \infty \}$ (as in Corollary 3.2(b)), resp. Orlicz-space $L^{\Phi}(J, X)$ with $|J| < \infty$ (Corollary 3.2(c)), resp. $J = \mathbb{R}$ etc. as in Corollary 3.2(e).

Proof of Corollary 3.2(a)

(a) Since in Proposition 2.1 the $K$ and $\sigma$ are independent of $I \subset J$ with $|I| \geq \delta_{0} > 0$ (and of $y$), one has even an asymptotic $N^{a}_{2}$ and therefore $H^{a}_{2}$. For (3.3), (3.4) one can use $D_{r} = C_{r}$ defined by (2.1), $< \infty$ if some $C_{\delta_{0}} < \infty$. Since the $y^{(k)}$ are continuous for $0 \leq k < n$, automatically all these $(y^{(k)}_{j} I)|_{J} \in L^{p}_{w}$ or equivalently $\in L^{p}_{w}$ ($w$ and $1/w$ are locally bounded); for $p = 1$ also $(y^{(n)}_{j} I)|_{J} \in L^{p}_{w}$ by the definition (0.4) of $C^{(n)}(J, X)$.

(b) Follows as (a) with Corollary 2.12, case Corollary 2.5, for $\|\|_{S^{p}_{w}}$ and $I = I_{x}$, $\alpha + 1 \leq x < \infty$.

(c) Proposition 2.18 gives an asymptotic Halperin-Pitt inequality $H^{a}_{2}$ since in (2.33/34) the $S(\varepsilon)$ does not depend on $I (a y)$; this implies the pointwise $H^{a}_{2}$. $H^{a}_{2}$ holds especially for Orlicz-norms $\|\|_{\Phi}$ by Corollary 2.20, here the definition of $\|\|_{\Phi}$ gives equality in (3.3) with $D_{r} = 1$.

(d) Remark 2.2(e); (3.3) holds with $D_{r} = 1$ since $w$ is decreasing.

(e) Use the transformation $t \to -t, J = [0, \infty)$, in (a)–(d).

For Corollary 3.3, $(y^{(n)}|_{[\alpha, \alpha + k\rho]})|_{J} \in L^{p}_{w}$ follows by induction on $k$ with (3.1) if $\rho := \min \{t_{2}, \ldots, t_{m}\} > 0$; if $V = \text{Orlicz-space } L^{\Phi}$ and $m > 1$, the starting assumption needed is $(y^{(n)}_{j} I)|_{J} \in L^{\Phi}$ for $2 \leq j \leq m$.

Remark 3.4

(a) The special case $p = \infty, m = 1, w = 1, X = \mathbb{R}$ is the classical Esclangon–Landau theorem of [21, Satz 1]; in [3] this has been extended to $m > 1$ and general $X$.

(b) Without (3.4) Theorem 3.1 and the corollaries are in general false: Example 5.3 and Remarks 5.8 in [3].

(c) $a_{1n} = 1$ in (3.4) can be replaced by "$a_{1n}$ uniformly invertible", i.e. $\|a_{1n}(t)v\| \geq \eta_{0}\|v\|$ for $v \in X$, $t \in J$, with some $\eta_{0} > 0$, and $D_{r}/\eta_{0}$ in (3.4) instead of $D_{r}$. 


(d) Usually the condition \( "y^{(k)}_j|J \in V" \) for all \( j, k \), compact \( I \) can be weakened to \( (y^{(n)}_{t_j} [\alpha, \alpha + \rho]) | J \in V, \rho = \min\{t_2, \ldots, t_m\}, \) so for \( m = 1 \) (or \( V = L^p_\alpha \)) it can be omitted entirely: see Corollaries 3.2(a), 3.3.

(e) Corollaries 3.2 and 3.3 are even for bounded \( J \) (and \( m = 1 \), \( X = \mathbb{R} \)) non-trivial if \( p < \infty \). For \( p = \infty \) see Remark 2.9(b) of [3]; then essentially \( w \equiv 1 \) by Remark 2.2(c) here, except in Corollary 3.2(d).

(f) For \( p = \infty \) one can admit arbitrary variable \( t_j: J \to [0, \tau] \) in corollary 3.2(a) and (d), provided \( y^{(n)} \) is continuous.

(g) For decreasing \( w \) as in Corollary 3.2(d) and \( 1 \leq p < \infty \), one can get at least the boundedness of \( w(x) \cdot \int_0^x |y^{(k)}| \bigwedge p \, dt \), \( 0 < k \leq n \), if \( \|y|_x| J \|_p, w \) and \( \|f|_x| p, w \) are bounded: \( \Psi = w^{-1/p} \), \( \|f\|_p \) in Theorem 3.1.

(h) In Theorem 3.1 one can also use \( \|g\| = \int_1^\infty \|g\|_p \, d\mu(p), \mu \) Borel measure on \([1, \infty)\) with e.g. compact support.

(i) An analogue, where “bounded” is replaced by “uniformly continuous”, can be found in [3, Corollary 3.3(a), Theorem 4.1].

Example 3.5 By glueing together \( f_n \) as in example 2.4, to any \( -\infty < \alpha < \beta \leq \infty \) one can construct \( f \in C^2(\mathbb{R}, \mathbb{R}) \) and a monotone norm on the piecewise continuous bounded functions: \( J \to \mathbb{R} \) with (1.8), such that \( \|f\| \leq 1, \|f''\| \leq 1, \) but \( \|f'\| \to \infty \) as \( J \) compact \( \to J = [\alpha, \beta] \), and \( f \equiv 0 \) on some \([\alpha, \alpha + \epsilon] \).

Complementing example 2.4, this shows that here the pointwise asymptotic \( L_2^\alpha, N_2^\alpha \) and even \( S_2^\alpha \) are false. It shows further that already for the equation \( y'' = f \) Theorem 3.1 becomes false without \( S_2^\alpha \).

With an asymptotic Landau inequality one gets Esclangon–Landau results even for some non-linear functional differential equations/inequalities of Landau type [21, p. 179]:

In the following we assume \( m, n, t_j, J, X, V, \| \| \) with (3.2), (3.3) and \( a_{jn} \) with (3.4) as in Theorem 3.1, \( y \in C^{(n)}(J', X) \), \( y_{t_j}(t) := y(t - t_j) \), all \( (y^{(k)}_{t_j}| J \in V \) for compact \( I \subset J, 1 \leq j \leq m, 0 \leq k \leq n, \) with \( y = 0 \) on \( I \) if \( \|y| I \| J \| = 0 \). Then calculations similar as for Theorem 3.1 yield

**Proposition 3.6** If a.e. on \( J \)

\[
\sum_{j=1}^m a_{jn}(t) y^{(n)}_{t_j}(t) = F(t, \ldots, y^{(k)}_{t_j} | [\alpha, t], \ldots),
\]

if with finitely many constant real \( c_\gamma > 0, \gamma = (\gamma_{j,k}) \) multiindex with \( 1 \leq j \leq m, 0 \leq k \leq n - 1 \) and \( \gamma_{j,k} \in \mathbb{R} \) one has (with \( 0^0 := 1 \)) for all compact
intervals $I \subset J$

$$
\|F(\ldots, y^{(k)}_j, \ldots)|[\alpha, \beta] \| \leq \sum_{\gamma} c_\gamma \prod_{j,k} \|y^{(k)}_j|I|J\|^\gamma_j, 
$$

(3.7)

where $n - \sum_{j,k} k \gamma_{jk} \geq \varepsilon > 0$ if $c_\gamma \neq 0$, and if a pointwise asymptotic Landau inequality $L_n^{*\alpha}$ holds for $V$, $\| \|$, then as $x \to \beta$, $0 < k \leq n$, with

$$
\delta := \max\{\sum_{j,k} (n-k) \gamma_{jk} : c_\gamma \neq 0\}, I_x := [\alpha, x]
$$

$$
\|(y^{(k)}_I) |J| = O\left(\left(\|yI_x|J\|^{1+k((\delta/e) - (1/n))}\right)\right).
$$

(3.8)

This can be applied to systems (3.6) with $y$ column vector $(y_1, \ldots, y_r)$ with values in $X^r$, $a_{jn}$ matrix valued with components $a_{jn, ur}(t) \in L(X)$, with $a_{jn} \equiv$ unit matrix, $X = $ Banach algebra, $\gamma = (\gamma_{jku})$ with $\gamma_{jku}$ integers $\geq 0$, and the $v$th component of $F$ a polynomial with bounded $a_{\gamma,v}$ of the form

$$
F_v(t, \ldots, y^{(k)}_j |[\alpha, t], \ldots) = \sum_{\gamma} a_{\gamma,v}(t) \prod_{j,k,u} \left(y^{(k)}_{u,j}(t)\right)^{\gamma_{jku}}
$$

(3.9)

with $n - \sum_{j,k,u} k \gamma_{jku} \geq \varepsilon_0 > 0$ for all $\gamma$ with some $a_{\gamma,v} \neq 0$: With $\| \| = \| \|_\infty = \mu_L - \sup_{x \in X}$ instead of $X$ in Proposition 3.6, with Proposition 2.16 one gets an extension of Theorem 2.8 of [3].

Another application is $p = 1$, $\| \|_1, X = \mathbb{C}$ (or Banach algebra), $F$ as in (3.9) and the products in $\prod_{\gamma_{jku}}$ being convolution ($\alpha = 0$)

$$
(f \ast g)(t) := \int_0^t f(s)g(t-s) \, ds, \quad t \geq 0;
$$

then (3.7) follows with $\|(f \ast g)|I|_1 \leq \|f|I|_1 \cdot \|g|I|_1, L_n^{*\alpha}$ holds by Proposition 2.16.

Let as finally remark also additional terms/variables of the form

$$
\int_{t-h}^t g(s) \, ds \quad \text{or} \quad \sup_{[t-h, t]} g, \quad \text{with} \quad g = y^{(k)}_j \quad \text{or} \quad |y^{(k)}_j| \quad \text{and} \quad 0 < h \leq \tau,
$$

are possible, since

$$
\left\| \frac{1}{h} \int_{t-h}^t f \, ds \right\|_p \leq \|f\|_p, \quad f \in L^p(\mathbb{R}, X), \quad 1 \leq p \leq \infty, \quad 0 < h < \infty.
$$

(3.10)

(Almost) Periodic solutions of such equations have been considered in Bantsur and Trofimchuk [1] and the references there.
References


