Steffensen Pairs and Associated Inequalities

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Let \( x_1, \ldots, x_n \) be positive numbers and \( \alpha \geq 2. \) It is known that if \( \sum_{i=1}^{n} x_i \leq A, \sum_{i=1}^{n} x_i^\alpha \geq B^\alpha, \) then for any \( k \) such that \( k \geq (A/B)^{1/(\alpha - 1)} \), there are \( k \) numbers among \( x_1, \ldots, x_n \) whose sum is bigger than or equal to \( B. \) We express this statement saying that a pair of functions \( (x^\alpha, x^{1/(\alpha - 1)}) \) is a Steffensen pair. In this paper we show how to find many Steffensen pairs.

Keywords: Steffensen inequality; Steffensen pair; Convex function; Tchebycheff inequality

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1. INTRODUCTION

Classical Steffensen’s inequality \[2\] states:

**Theorem A** Let \( f \) and \( g \) be integrable functions from \([a, b]\) into \( \mathbb{R} \) such that \( f \) is decreasing, and for every \( x \in [a, b], 0 \leq g(x) \leq 1. \) Then

\[
\int_{b - \lambda}^{b} f(x) \, dx \leq \int_{a}^{b} f(x)g(x) \, dx \leq \int_{a}^{a + \lambda} f(x) \, dx,
\]

where \( \lambda = \int_{a}^{b} g(x) \, dx. \)
In [1], the following discrete analogue of Steffensen’s inequality was proved:

**THEOREM B** Let \((x_i)_{i=1}^n\) be a decreasing finite sequence of nonnegative real numbers, and let \((y_i)_{i=1}^n\) be a finite sequence of real numbers such that for every \(i\), \(0 \leq y_i \leq 1\). Let \(k_1, k_2 \in \{1, \ldots, n\}\) be such that \(k_2 \leq y_1 + \cdots + y_n \leq k_1\). Then

\[
\sum_{i=n-k_2+1}^{n} x_i \leq \sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{k_1} x_i.
\]

As an immediate consequence of Theorem B, the following proposition was proved in [1]:

**PROPOSITION A** Let \(x_1, \ldots, x_n\) be nonnegative real numbers such that the following two conditions are satisfied: (i) \(\sum_{i=1}^{n} x_i \leq A\), (ii) \(\sum_{i=1}^{n} x_i^2 \geq B^2\), where \(A\) and \(B\) are positive real numbers. Let \(k \in \{1, \ldots, n\}\) be such that \(k \geq A/B\). Then there are \(k\) numbers among \(x_1, \ldots, x_n\) whose sum is bigger than or equal to \(B\).

To prove Proposition A we can assume that \(B \geq x_1 \geq \cdots \geq x_n\). Set \(y_i = x_i/B\). Then \(\sum_{i=1}^{n} y_i = A/B \leq k\). By Theorem B,

\[
\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} \frac{x_i^2}{B} \geq B.
\]

Proposition A shows that under certain conditions, a relatively small portion of \(x_1, \ldots, x_n\) has a relatively large sum. For example, if \(\sum_{i=1}^{n} x_i \leq 300\) and \(\sum_{i=1}^{n} x_i^2 \geq 10000\), then there are three numbers among \(x_1, \ldots, x_n\), say \(x_j, x_k, x_m\), such that \(x_j + x_k + x_m \geq 100\), i.e. \(x_j + x_k + x_m \geq \frac{1}{3} \sum_{i=1}^{n} x_i\).

We will restate Proposition A using the following definition:

**DEFINITION** Let \(\varphi : [c, \infty) \rightarrow [0, \infty), c \geq 0, \text{ and } \tau : (0, \infty) \rightarrow (0, \infty)\) be two strictly increasing functions. We say that \((\varphi, \tau)\) is a Steffensen pair on \([c, \infty)\) if the following is satisfied:

If \(x_1, \ldots, x_n\) are real numbers such that \(x_i \geq c\) for all \(i\), \(A\) and \(B\) are positive real numbers, and (i) \(\sum_{i=1}^{n} x_i \leq A\), (ii) \(\sum_{i=1}^{n} \varphi(x_i) \geq \varphi(B)\), then for any \(k \in \{1, \ldots, n\}\) such that \(k \geq \tau(A/B)\), there are \(k\) numbers among \(x_1, \ldots, x_n\) whose sum is bigger than or equal to \(B\).
Now Proposition A can be reformulated as follows:

**Proposition A'** \((x^2, x)\) is a Steffensen pair on \([0, \infty)\).

The following more general result was proved in [1]:

**Proposition B** If \(\alpha \geq 2\), then \((x^\alpha, x^{1/(\alpha-1)})\) is a Steffensen pair on \([0, \infty)\).

The purpose of this paper is to find more examples of Steffensen pairs.

**Theorem** Let \(\varphi, \tau: [0, \infty) \rightarrow [0, \infty)\) where \(c \geq 0\), be increasing and convex. Assume that \(\varphi, \tau\) satisfy the following condition:

\[
\varphi(xy) \geq \varphi(x)g(y) \quad \text{for all } x \geq c, y \geq 1,
\]

where \(g[1, \infty) \rightarrow [0, \infty)\) is strictly increasing. Set \(\varphi(x) = x\psi(x), \tau(x) = g^{-1}(x)\), where \(g^{-1}\) is the inverse function for \(g\). Then \((\varphi, \tau)\) is a Steffensen pair on \([c, \infty)\).

**Example** Let \(\alpha \geq 2\), \(\psi(x) = x^{\alpha-1}\). Then \(\psi(xy) = \psi(x)\psi(y)\). Hence \(\varphi(x) = x^\alpha, \tau(x) = x^{1/(\alpha-1)}\), and we arrive at Proposition B.

**Theorem 2** Let \(f: [0, \infty) \rightarrow \mathbb{R}\) be a twice differentiable function on \([0, \infty)\) such that \(f'(x) \geq 0\) and \(f''(x) \geq 0\) for all \(x \geq 0\). Assume that \(f(0) = 0\). Then the functions \(\psi, \tau\) from \([1, \infty)\) into \([0, \infty)\) given by

\[
\psi = g = \exp \circ f \circ \ln
\]

satisfy the conditions of Theorem 1.

**Remark** There are many functions satisfying the conditions of Theorem 2. For example, if \(f(x) = \sum_{i=1}^\infty a_i x^i\) is the sum of a series converging on \([0, \infty)\) and if \(a_1 \geq 1, a_i \geq 0\) for \(i = 2, 3, \ldots\), then \(f(x)\) satisfies the conditions of Theorem 2.

**Proposition 1** If \(\alpha \geq 1\), then \((x \exp(x^\alpha - 1), (1 + \ln x)^{(1/\alpha)})\) is a Steffensen pair on \([1, \infty)\).

**Proposition 2** Let \(a\) and \(b\) be real numbers satisfying the conditions \(b > a > 1\) and \(\sqrt{ab} \geq e\). Set

\[
\varphi(x) = \begin{cases} 
(x^{1+\ln b} - x^{1+\ln a})/\ln x, & \text{if } x > 1, \\
\ln b - \ln a, & \text{if } x = 1, 
\end{cases}
\]

\[
\tau(x) = x^{1/\ln \sqrt{ab}}.
\]

Then \((\varphi, \tau)\) is a Steffensen pair on \([1, \infty)\).
Remark Since $\sqrt{ab} \geq e$, $x \geq x^{1/\ln \sqrt{ab}}$ for $x \geq 1$. Therefore it is possible to take $\tau(x) = x$ in Proposition 2.

2. PROOF OF THEOREMS 1, 2 AND PROPOSITIONS 1, 2

Theorem 1 can be deduced easily from Theorem 6.5 in [1]. However the proof of Theorem 6.5 in [1] uses the integration over a general measure space. Because of this reason we give here a direct and elementary proof of Theorem 1 (although it follows closely the ideas of the proof of Theorem 6.5 in [1]).

**Lemma 1** Assume that $\psi(c, \infty) \rightarrow [0, \infty)$, $c \geq 0$, is increasing and convex. Set $\varphi(x) = x \psi(x)$. Let $x_1, \ldots, x_r$ be positive real numbers such that $x_i \geq c$, $i = 1, \ldots, r$. Set $m = \min\{x_1, \ldots, x_r\}$. Then

$$\sum_{i=1}^{r} \varphi(x_i) - \psi(m) \sum_{i=1}^{r} x_i \leq \varphi\left(\sum_{i=1}^{r} x_i\right) - \psi(rm) \sum_{i=1}^{r} x_i.$$  

**Proof** Since $\psi(x)$ is convex, it is well known (and easy to prove) that if $x_1 < x_2$ and $\delta \geq 0$, then

$$\psi(x_2) - \psi(x_1) \leq \psi(x_2 + \delta) - \psi(x_1 + \delta).$$

Using this fact we obtain

$$\sum_{i=1}^{r} \varphi(x_i) - \psi(m) \sum_{i=1}^{r} x_i = \sum_{i=1}^{r} x_i [\psi(x_i) - \psi(m)]$$

$$\leq \sum_{i=1}^{r} x_i [\psi(x_i + (r - 1)m) - \psi(rm)]$$

$$\leq \sum_{i=1}^{r} x_i \left[\psi\left(\sum_{i=1}^{r} x_i\right) - \psi(rm)\right]$$

$$= \psi\left(\sum_{i=1}^{r} x_i\right) \sum_{i=1}^{r} x_i - \psi(rm) \sum_{i=1}^{r} x_i$$

$$= \varphi\left(\sum_{i=1}^{r} x_i\right) - \psi(rm) \sum_{i=1}^{r} x_i.$$
Proof of Theorem 1  Let \( x_1, \ldots, x_n \) be real numbers such that \( x_i \geq c \) for all \( i \). Without loss of generality we can assume that \( x_1 \geq \cdots \geq x_n \). Let \( A \) and \( B \) be positive real numbers, and (i) \( \sum_{i=1}^{n} x_i \leq A \), (ii) \( \sum_{i=1}^{n} \varphi(x_i) \geq \varphi(B) \). Assume that \( k \geq \tau(A/B) \). We will prove that \( x_1 + \cdots + x_k \geq B \).

The inequality \( k \geq \tau(A/B) \) implies that \( g(k) \geq A/B \). Hence
\[
A \psi(x_k) = \psi(x_k) \frac{A}{B} \leq \psi(x_k) g(k) B.
\]

Since \( \psi(xy) \geq \psi(x)g(y) \), we obtain
\[
A \psi(x_k) \leq \psi(kx_k) B.
\] (1)

Now we have
\[
\varphi(B) \leq \sum_{i=1}^{n} \varphi(x_i) = \sum_{i=1}^{k} \varphi(x_i) + \sum_{i=k+1}^{n} x_i \psi(x_i)
\]
\[
\leq \sum_{i=1}^{k} \varphi(x_i) + \psi(x_k) \sum_{i=k+1}^{n} x_i
\]
\[
= \sum_{i=1}^{k} \varphi(x_k) + \psi(x_k) \left( \sum_{i=1}^{n} x_i - \sum_{i=1}^{k} x_i \right)
\]
\[
\leq \sum_{i=1}^{k} \varphi(x_k) - \psi(x_k) \sum_{i=1}^{k} x_i + A \psi(x_k).
\]

Lemma 1 implies that
\[
\varphi(B) \leq \varphi \left( \sum_{i=1}^{k} x_i \right) - \psi(kx_k) \sum_{i=1}^{k} x_i + A \psi(x_k).
\]

By (1), we obtain that
\[
\varphi(B) - \varphi \left( \sum_{i=1}^{k} x_i \right) \leq -\psi(kx_k) \sum_{i=1}^{k} x_i + \psi(kx_k) B
\]
\[
= \psi(kx_k) \left( B - \sum_{i=1}^{k} x_i \right).
\]
Assume that the conclusion of the theorem is wrong, that is, assume that $B - \sum_{i=1}^{k} x_i > 0$. Then we have

$$\varphi(B) - \varphi\left(\sum_{i=1}^{k} x_i\right) \leq \psi\left(\sum_{i=1}^{k} x_i\right) \left(B - \sum_{i=1}^{k} x_i\right)$$

$$= B\psi\left(\sum_{i=1}^{k} x_i\right) - \varphi\left(\sum_{i=1}^{k} x_i\right).$$

This implies that $\varphi(B) \leq B\psi(\sum_{i=1}^{k} x_i)$. Hence $\psi(B) \leq \psi(\sum_{i=1}^{k} x_i)$. It follows that $B \leq \sum_{i=1}^{k} x_i$, which contradicts the above assumption.

**Proof of Theorem 2**

For $x > 1$,

$$\psi'(x) = \psi(x)f'(\ln x)\frac{1}{x} > 0,$$

$$\psi''(x) = \psi(x)f'(\ln x)\frac{1}{x^2} \left[f'(\ln x) - 1\right] + \psi(x)f''(\ln x)\frac{1}{x^2} \geq 0.$$

Therefore $\psi$ is increasing and convex. Let $y \geq 0$, be a fixed number. For $x \geq 0$, set

$$F(x) = f(x + y) - f(x) - f(y).$$

Then

$$F'(x) = f'(x + y) - f'(x) \geq 0, \quad F(0) = 0.$$ 

Hence $F(x) \geq 0$ for all $x \geq 0$. Thus

$$f(x + y) \geq f(x) + f(y)$$

for all $x, y \geq 0$. Therefore, for $x, y \geq 1$, we obtain

$$\psi(xy) = \exp(f(\ln xy)) = \exp(f(\ln x + \ln y))$$

$$\geq \exp[f(\ln x) + f(\ln y)]$$

$$= \exp(f(\ln x)) \cdot \exp(f(\ln y)) = \psi(x)\psi(y).$$

**Proof of Proposition 1**

For $\alpha \geq 1$, set $f(x) = e^{\alpha x} - 1$. Then $f(0) = 0$ and for all $x \geq 0$, $f'(x) \geq 1$, $f''(x) \geq 0$. Therefore by Theorem 2, functions $\psi$ and $g$ from $[1, \infty)$ into $[0, \infty)$ given by $\psi(x) = g(x) = \exp(e^{\alpha \ln x} - 1) = \exp(x^\alpha - 1)$ satisfy the conditions of Theorem 1. It follows by Theorem 1, that $\varphi(x) = x\psi(x) = x\exp(x^\alpha - 1)$ and $\tau(x) = g^{-1}(x) = (1 + \ln x)^{1/\alpha}$ is a Steffensen pair on $[1, \infty)$. 


Proof of Proposition 2  We prove this proposition using Theorem 1 and recent results from [3]. Let \( b > a > 1 \) and \( \sqrt{ab} \geq e \). Set

\[
h(x) = \begin{cases} 
\frac{(b^x - a^x)}{x}, & \text{if } x \neq 0, \\
\ln b - \ln a, & \text{if } x = 0.
\end{cases}
\]

By Proposition 3 in [3], \( h'(x) > 0 \).

Lemma 2  \( h''(x) \geq h'(x) \) for \( x \geq 0 \).

Proof  It is easy to see that

\[
h^{(n)}(x) = \int_a^b (\ln t)^n t^{x-1} \, dt.
\]  \hspace{1cm} (2)

We will use the following Tchebycheff inequality.

Let \( p, q : [a, b] \to \mathbb{R} \) be integrable increasing functions and let \( r : [a, b] \to [0, \infty) \) be an integrable function. Then

\[
\int_a^b r(t)p(t) \, dt \int_a^b r(t)q(t) \, dt \leq \int_a^b r(t) \, dt \int_a^b r(t)p(t)q(t) \, dt.
\]

Taking \( p(t) = q(t) = \ln t, r(t) = t^{x-1} \), we get

\[
\left( \int_a^b \ln t \cdot t^{x-1} \, dt \right)^2 \leq \int_a^b t^{x-1} \, dt \int_a^b (\ln t)^2 t^{x-1} \, dt.
\]

By (2), we obtain that for all \( x \),

\[
[h'(x)]^2 \leq h(x)h''(x). \tag{3}
\]

By Proposition 4 in [3], for every \( y \geq 0 \), \( F(x) = h(x + y)/h(x) \) is increasing as a function of \( x \). Therefore

\[
F'(x) = \frac{h'(x + y)h(x) - h(x + y)h'(x)}{[h(x)]^2} \geq 0.
\]

Hence

\[
h'(x + y)h(x) - h(x + y)h'(x) \geq 0 \tag{4}
\]

for all \( x \) and all \( y \geq 0 \).
Taking \( x = 0 \) in (4), we obtain
\[
 h'(y)h(0) - h(y)h'(0) \geq 0
\]
for all \( y \geq 0 \).
\[
h(0) = \ln b - \ln a
\]
\[
h'(0) = \lim_{x \to 0} \frac{1}{x} \left[ \frac{b^y - a^y}{x} - (\ln b - \ln a) \right] = \frac{1}{2} ((\ln b)^2 - (\ln a)^2).
\]
Hence \( h'(0) = h(0) \ln \sqrt{ab}. \) Since \( \sqrt{ab} \geq e \), we obtain that \( h'(0) \geq h(0) \).

It follows from (5) that \( h'(y) \geq h(y) \) for \( y \geq 0 \). Therefore, by (3) and (5),
\[
h(x)h''(x) \geq [h'(x)]^2 \geq h(x)h'(x)
\]
for \( x \geq 0 \). Thus \( h''(x) \geq h'(x) \) for all \( x \geq 0 \). That proves the lemma.

Set \( \psi(x) = h(\ln x) \) for \( x \geq 1 \). Then \( \psi'(x) = h'(\ln x)(1/x) > 0 \), \( \psi''(x) = (1/x^2)[h''(\ln x) - h'(\ln x)] \geq 0 \). Hence \( \psi(x) \) is increasing and convex. In addition,
\[
\frac{\psi(xy)}{\psi(x)} = \frac{h(\ln(xy))}{h(\ln x)} = \frac{h(\ln x + \ln y)}{h(\ln x)}.
\]
By Proposition 5 in [3], we have that for \( x, y \geq 0 \),
\[
\frac{h(x + y)}{h(x)} \geq (\sqrt{ab})^y.
\]
Therefore, for \( x, y \geq 1 \),
\[
\frac{\psi(xy)}{\psi(x)} \geq (\sqrt{ab})^{\ln y}.
\]
Set \( g(x) = (\sqrt{ab})^{\ln x} \). Then \( g^{-1}(x) = x^{1/\ln \sqrt{ab}} \).

By Theorem 1 \((\varphi, \tau)\), where
\[
\varphi(x) = x\psi(x)
\]
\[
\varphi(x) = \begin{cases} 
  x(b^{\ln x} - a^{\ln x})/\ln x = (x^{1+\ln b} - x^{1+\ln a})/\ln x, & \text{if } x > 1, \\
  \ln b - \ln a, & \text{if } x = 1,
\end{cases}
\]
\[
\tau(x) = x^{1/\ln \sqrt{ab}},
\]
is a Steffensen pair.
References

